COURSE NOTES

FOR

MAT 125 A

CALCULUS I

BY

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DEDICATION

To the ONE who makes it plain. I did my part. I wrote it on tablets.

"Write the vision and make it plain on tablets that he may run who reads it."

(From Habakkuk 2:2)

SPECIAL THANKS TO:
My Wife Belinda
Philip Derber
Jack Shoup
Fantley &Elizabeth Smither
Steve Butler
John & Patty Chamness
Abby DeHaan

EXPLANATION

These notes are the notes that I would generally write on the board or on an overhead projector to teach the material in the course. Since I generally know what I plan to write beforehand, I have written it down beforehand so that you do not have to take the notes. You can follow along in class with focus on what is to be learned with the notes taken for you.

WHY PENCIL AND BIG PRINT?

To "make it plain" (Hab 2:2). It is my desire to make the things taught to be easily mentally digestible. There are some wonderful meals fixed with love for me by my wife that are so blessed and digestible that I joke that the stomach can be by-passed and the food just be put into me intravenously! This book is intended to be like that for the mind...immediately absorbed by the mind.

This all began when I was teaching a class with computer generated notes. I then switched to pencil and big print. The response was unanimous; they liked the pencil and big print notes much better. It was said that when they did their homework, they had to recopy the computer generated notes to understand better, but with the pencil and big print notes they did not have to recopy them to understand.

A secondary reason for pencil and big print is that many texts are encyclopedic...containing far more information than can and needs to be consumed to know calculus excellently. So I go for the jugular and put in no more and no less than is needed to thrive mathematically.

You are seeing the note-taking style that served me well in getting a math Ph.D. and beyond.

Another reason I use pencil and big print is that I believe there is an anointing of clarity that comes with these notes and it is known that the "anointing teaches you" (1 Jn 2:27).

Rather than this being a second rate, antiquated learning system, I am giving you absolutely the best I know for you to learn with wisdom and joy. Drink it in.

Austin French

A

MIGHTY MICROPEDIA

FROM THE

GENRE OF

MUSTARDSEEDAPEDIAS

CALCULATORS AND COMPUTERS

If technology helps you, use it; if it does not help you, then use something that will help. That sentence could free many a folk from being duty bound to a computer or calculator. It can free them from a sense of shame for not being interested or excited about spending a great deal of time using a calculator or computer. It has been my experience that those students that feel like they just have to have a calculator are not the clearest math seers. They have a nervous dependency about them that works against their seeing math clearly and learning math peacefully in wisdom.

It is fine for you to use a calculator for this course. However, you must realize that I am teaching you mathematics, not computer science or calculator science. I am bringing forth a pure, clear math realm that excites me. I do not fret whether I am making it interesting or not. I do my part to make it clear, to make it plain. The beauty of the subject matter will draw and interest those hungry to see math clearly.

I teach what excites me in a way that excites me. I have found that the material I teach, in the way I teach it, prepares people excellently for other math courses even to Ph.D. work in mathematics.

Do not fret that your education is being slighted by the calculator not being magnified. In fact, I have found that the time spent in calculator magnification automatically causes some topics to be slighted or left out that are very important for calculus understanding.

New and newer models of calculators will come and go. What I am teaching is calculator model independent. Due to its nature, this course abides in a state of freshness and clarity. Discern carefully what you drink in educationally. Avoid image with little or no reality. Embrace wise, diligent study of substance rather than fluff.

Austin French

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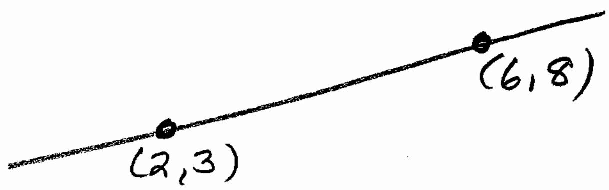
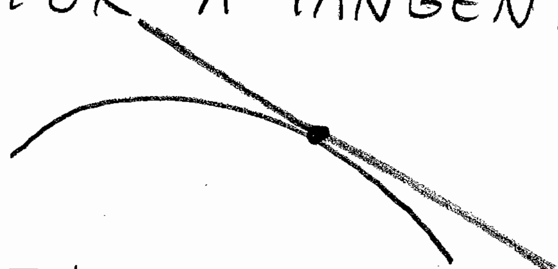
WELCOME TO CALCULUS!

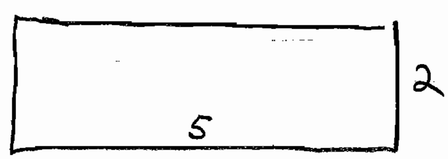
A. FORMER MATH LIFE:
EQUATION FOR A LINE BETWEEN TWO POINTS



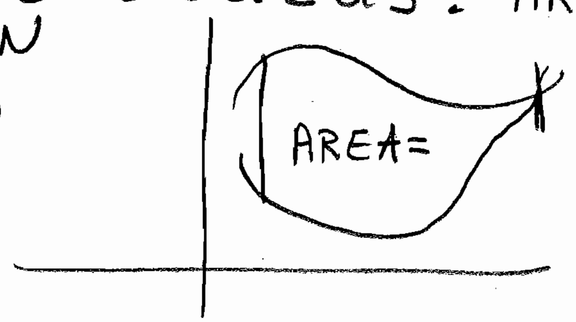
BUT IN CALCULUS:
EQUATION FOR A TANGENT LINE TO A GRAPH



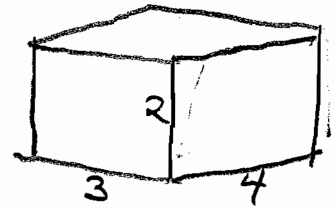
B. FORMER MATH LIFE: AREA OF A RECTANGLE



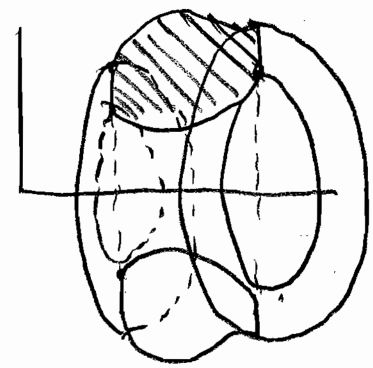
BUT IN CALCULUS: AREA BETWEEN CURVED LINES



C. FORMER MATH LIFE: VOLUME OF A BOX.



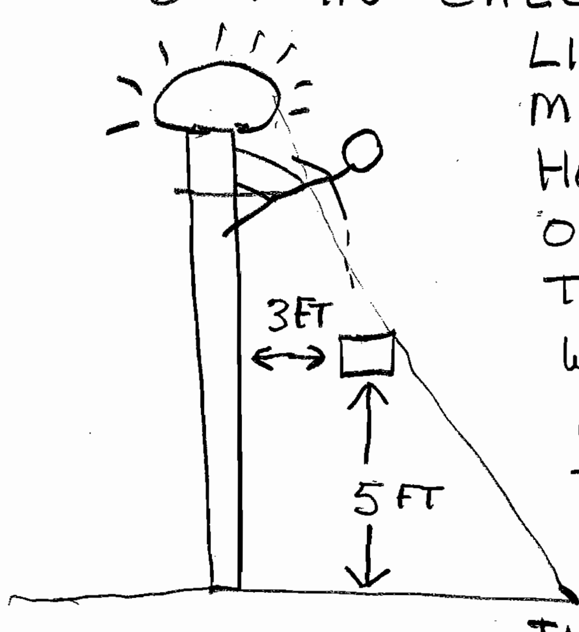
BUT IN CALCULUS:
SPIN SHADED REGION ABOUT x-AXIS. FIND VOLUME OF THAT SOLID.



D. FORMER MATH LIFE:

AVERAGE VELOCITY: 100 MILES IN 2 HOURS $\rightarrow \frac{100}{2} = 50$ MPH

BUT IN CALCULUS



LIGHT POLE,
MAN DROPS TOOL BOX.
HOW FAST IS THE TIP OF THE SHADOW OF THE TOOL BOX MOVING WHEN THE TOOL BOX IS 5 FEET FROM THE GROUND? (THE TOOL BOX IS FALLING 3 FEET FROM THE LIGHT POLE.)

UNDERSTANDING DEFINITIONS

A. DEFINITIONS: CLEARY, EXACTLY DESCRIBE THE CONCEPT, NO MORE NO LESS

B. ALL DEFINITIONS "IF AND ONLY IF"

12 IF AND ONLY IF DOZEN

C. A DEFINITION WRITES\SPEAKS THE CONCEPT INTO EXISTENCE IN THE COURSE

D. EXAMPLE (DEFINITION): p IS A QUALOM IF AND ONLY IF p IS A POSITIVE INTEGER GREATER THAN 7

E. THEOREM: (FOLLOWS FROM A DEFINITION) EVERY QUALOM IS GREATER THAN 5

F. THERE IS ONLY ONE DEFINITION FOR A CONCEPT IN A COURSE ... THE EXACT WORDING WHEN ORIGINALLY DEFINED

G. THINGS EQUIVALENT TO A DEFINITION FALL UNDER THE THEOREM NAME

H. SO ON TESTS WHEN ASKED FOR A DEFINITION, GIVE THE EXACT WORDING, NOT SOMETHING EQUIVALENT (LIKE A POSITIVE INTEGER GREATER THAN $7\frac{1}{2}$).

MERCIFUL ALGEBRA REVIEW

A. <u>DEFINITION</u>: A FUNCTION IS A SET OF ORDERED PAIRS SUCH THAT NO TWO ORDERED PAIRS HAVE THE SAME FIRST TERM

$f = \{(1,2),(5,3),(4,2)\}$ FUNCTION

$n = \{(1,2),(5,3),(1,7)\}$ NOT A FUNCTION

B. <u>DEFINITION</u>: f IS A FUNCTION

$f(x)$, READ "f OF x" IS THE SECOND TERM OF THE ORDERED PAIR IN f WHOSE FIRST TERM IS x

$f = \{(1,2),(5,3),(4,2)\}$

$f(1) = 2 \qquad f(5) = 3$

NOTE $f(1) = 2$ IFF $(1,2) \in f$

(IFF MEANS IF AND ONLY IF)

C. EQUATIONS <u>DEFINE</u> FUNCTIONS

$y = x^2 + 2x + 1$ DEFINES f

x INDEPENDENT VARIABLE
1ST TERMS

y DEPENDENT VARIABLE
2nd TERMS

$f = \{(x, x^2+2x+1) \mid x \text{ IS A REAL}\}$

$f(x) = x^2+2x+1$

EVALUATING f

$f(3) = 3^2+2(3)+1 = 16$

$f(x+p) = (x+p)^2+2(x+p)+1$
$= x^2+2xp+p^2+2x+2p+1$

D. <u>DEFINITIONS</u>. THE DOMAIN OF THE FUNCTION f IS THE SET OF ALL FIRST TERMS OF f. THE RANGE OF THE FUNCTION f IS THE SET OF ALL SECOND TERMS OF f

E. FINDING DOMAINS AND RANGE

$f = \{(1,2), (5,3), (4,2)\}$

DOMAIN OF $f = \mathrm{dom}(f) = \{1, 5, 4\}$

RANGE OF $f = \mathrm{ran}(f) = \{2, 3\}$

THE DOMAIN FOR A FUNCTION DEFINED BY AN EQUATION IS THE SET OF ALL VALUES FOR THE INDEPENDENT VARIABLE FOR WHICH THE EQUATION IS DEFINED.

$$f(x) = \frac{1}{\sqrt{2-3x}}$$

UNDER RADICAL ≥ 0 <u>AND</u> BOTTOM NOT 0

$$2-3x \geq 0 \quad \text{AND} \quad 2-3x \neq 0$$

$$2-3x > 0$$

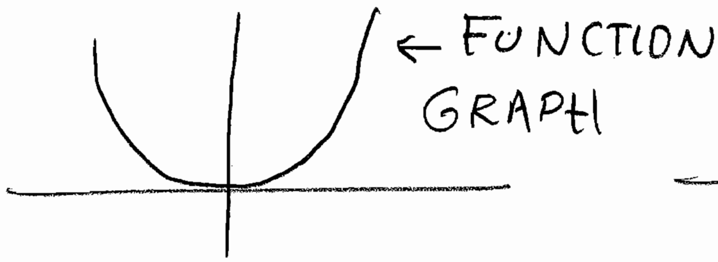
$$2 > 3x$$

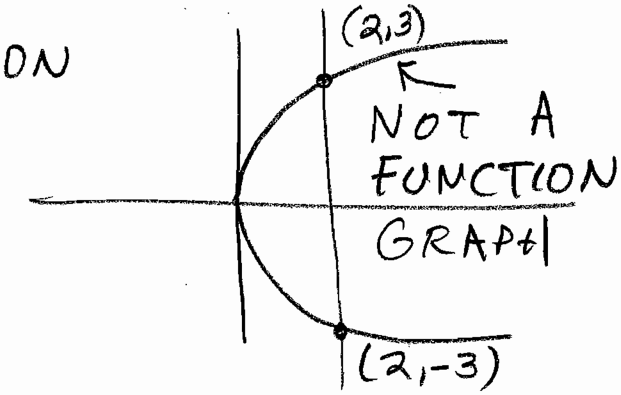
$$\frac{2}{3} > x$$

$$\mathrm{dom}(f) = \left(-\infty, \frac{2}{3}\right) = \left\{x \mid x < \frac{2}{3}\right\}$$

G. LOOKING AT A GRAPH AND DECIDING IF IT IS A GRAPH OF A FUNCTION

<u>FUNCTION</u> GR<u>APH</u>: NO VERTICAL LINE INTERSECTS TWICE

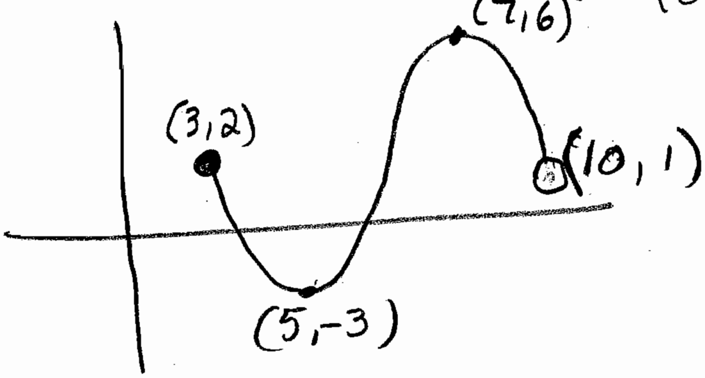




H. LOOK AT GRAPH AND TELL DOMAIN AND RANGE

$$\mathrm{dom}(f) = \{x \mid \text{VERTICAL LINE THROUGH } (x,0) \text{ INTERSECTS GRAPH}\}$$

$$\mathrm{ran}(f) = \{y \mid \text{HORIZONTAL LINE THROUGH } (0,y) \text{ INTERSECTS GRAPH}\}$$



$\mathrm{dom}(f) = [3, 10)$

$\mathrm{ran}(f) = [-3, 6]$

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HOMEWORK

1, 5, 6, 7, 8, 21, 23, 24, 25, 27, 29, 30

p. 20-22

ABSOLUTE VALUE

A. DEFINITION $|x|$ IS READ "THE ABSOLUTE VALUE OF x"

DEF. $|x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x < 0 \end{cases}$

B. STEPS FOR FINDING $|x|$

1. LOOK INSIDE THE ABSOLUTE VALUE SIGNS
2. IF THE THING INSIDE THE ABSOLUTE VALUE SIGNS IS GREATER THAN OR EQUAL TO 0,... WRITE IT DOWN.*
3. IF THE THING INSIDE THE ABSOLUTE VALUE SIGNS IS NEGATIVE,... WRITE IT DOWN AND PUT A MINUS IN FRONT OF IT*.

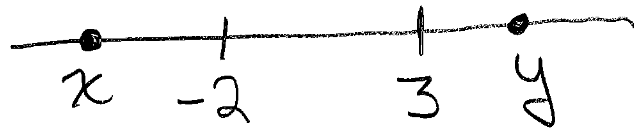
---

* EVEN IF YOU HAVE TO FORCE YOUR HAND TO DO IT.

C. $|5| = 5 \qquad |-5| = -(-5) = 5$

D. LET $x < -2$ AND $y > 3$

$$|\underbrace{y}_{\text{POS}}| = y$$



$$|\underbrace{x}_{\text{NEG}}| = -x$$

$$|\underbrace{x\,y}_{\substack{\text{NEG}\cdot\text{POS}\\ \text{NEG.}}}| = -xy$$

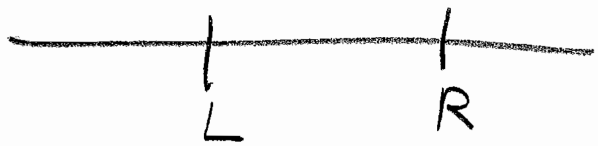
$$|\underbrace{-2\,x\,y}_{\substack{\text{NEG}\cdot\text{NEG}\cdot\text{POS}\\ \text{POS}}}| = -2xy$$

$$|\underbrace{5\,y}_{\substack{\text{POS}\cdot\text{POS}\\ \text{POS}}}| = 5y$$

$$|y - x| = |\underbrace{y + (-x)}_{\substack{\text{POS} + \text{POS}\\ \text{POS}}}| = y - x$$

$$|\underbrace{\underbrace{-2y}_{\substack{\text{NEG}\cdot\text{POS}\\ \text{NEG}}} + \underbrace{x}_{\text{NEG}}}_{\text{NEG}}| = -(-2y + x) = +2y - x$$

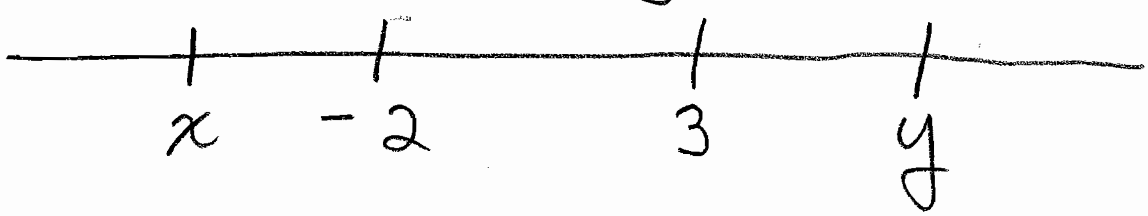
E. $\boxed{\begin{array}{l} \text{RIGHT} - \text{LEFT} > 0 \\ \text{LEFT} - \text{RIGHT} < 0 \end{array}}$



$R > L$, so $R - L > 0$

$L < R$, so $L - R < 0$

$x < -2$ AND $y > 3$



1. $\underbrace{3 - x}_{\substack{R - L \\ \text{POS}}}$ so $3 - x > 0$

2. $\left| \underbrace{y - x}_{\substack{R - L \\ \text{POS}}} \right| = y - x$

3. $\left| \underbrace{x - y}_{\substack{L - R \\ \text{NEG}}} \right| = -(x - y) = -x + y = y - x$

4 $|y + 2| = |\underbrace{y - (-2)}_{\substack{R - L \\ \text{POS}}}| = y + 2$

HOMEWORK: REMOVE ABSOLUTE VALUE SIGNS ACCORDING TO THE DEFINITION, SHOWING YOUR WORK

FOR ALL PROBLEMS ON THIS PAGE

$p > 7$ AND $m < -5$

1. $|m| =$
2. $|3m| =$
3. $|pm| =$
4. $|-5pm| =$
5. $|m-p| =$
6. $|m+5| =$
7. $|2p+5| =$
8. $|m+3| =$
9. $|7-p| =$
10. $|m-7| =$

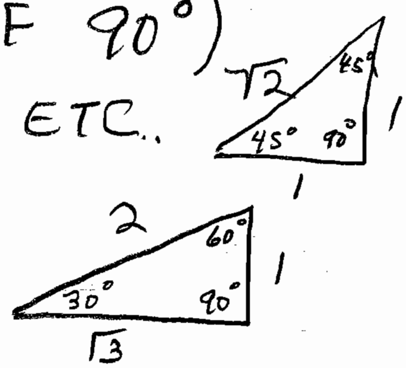
TRIG REVIEW (EXACT ANSWERS)

A KNOW DEFINITIONS ACCORDING TO TRIANGLES

$$\left.\begin{aligned} \sin\theta &= \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}} \\ \tan\theta &= \frac{\text{OPPOSITE}}{\text{ADJACENT}} \end{aligned}\right\} \text{ ETC.}$$

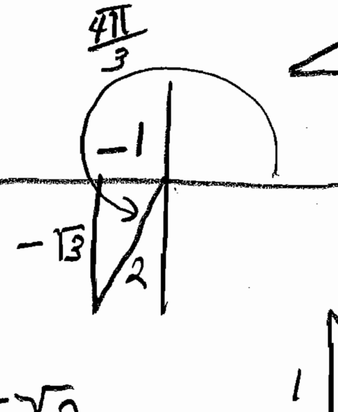
B. FAMOUS TRIANGLES (FOR ANGLES THAT ARE MULTIPLES OF $30°$, $60°$, $45°$ BUT NOT MULTIPLES OF $90°$)

$30° \longleftrightarrow \frac{\pi}{6}$ RADIANS ETC.

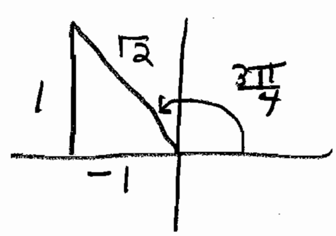


$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\cot\frac{4\pi}{3} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

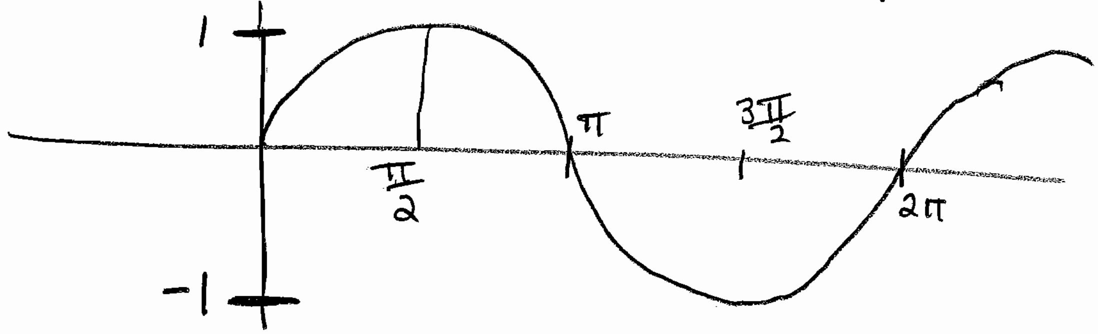


$$\sec\frac{3\pi}{4} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$$



C. KNOW GRAPHS OF TRIG FUNCTIONS AND USE TO EVALUATE MULTIPLES OF 90°

$\sin 270° = -1$

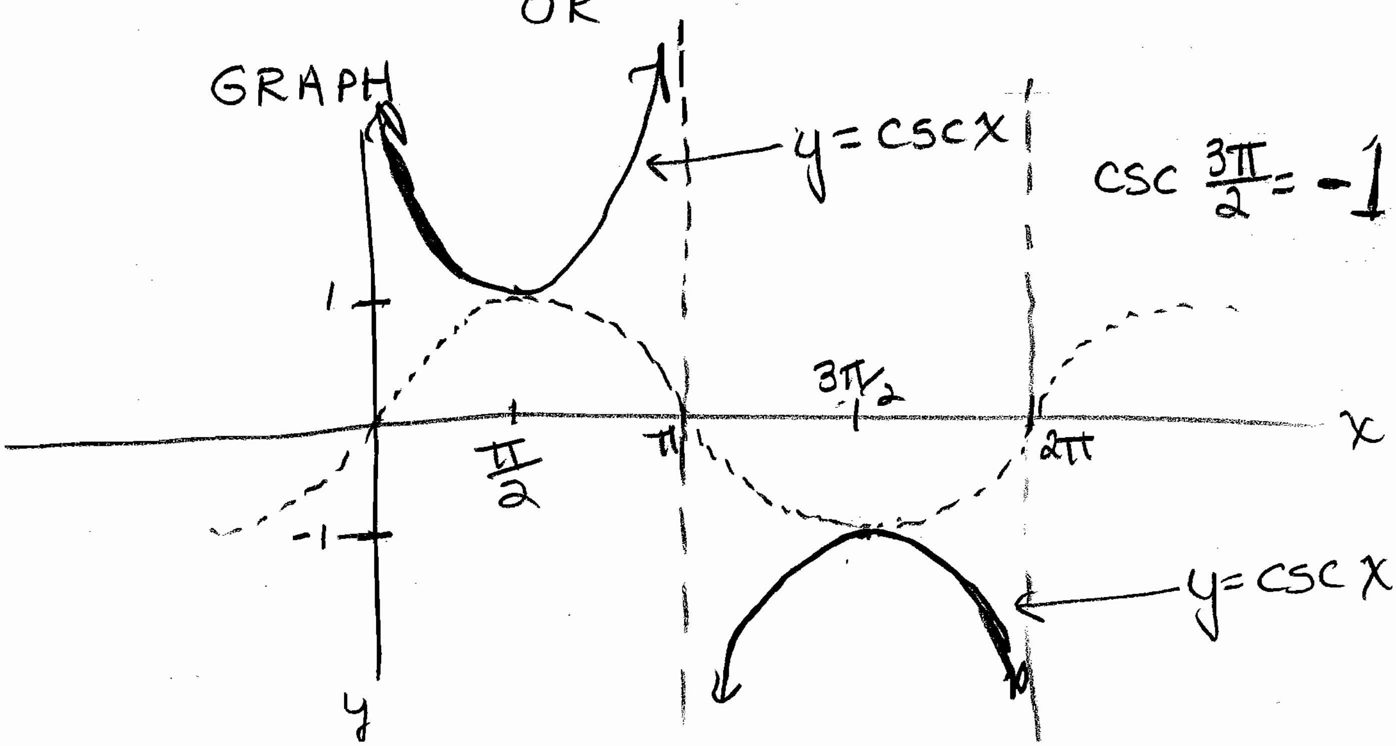


FIND $\csc 270°$ (i.e. $\frac{1}{\sin 270°}$)

EITHER DO $\csc 270° = \frac{1}{\sin 270°} = \frac{1}{-1} = -1$

OR

GRAPH



$\csc \frac{3\pi}{2} = -1$

D. KNOW TRIG IDENTITIES

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

HOMEWORK

1. DRAW THE PROPER TRIANGLE TO EVALUATE, THEN GIVE ANSWER <u>EXACTLY</u>.

$\sin 60^\circ =$

$\tan \frac{5\pi}{6} =$

$\sec \frac{7\pi}{4} =$

$\csc \frac{7\pi}{6} =$

2. DRAW THE PROPER GRAPH TO EVALUATE, THEN GIVE ANSWER

$\cos \pi =$

$\tan 270^\circ =$

$\sec -\pi =$

$\sin -270^\circ =$

$\csc \frac{3\pi}{2} =$

PAGES 18-25 OMITTED

WE NOW BEGIN CALCULUS, <u>INTUITIVELY</u>
(RIGOR IS COMING)

A. LIMIT OF A FUNCTION f AT a:

DENOTED $\lim_{x \to a} f(x)$

READ "THE LIMIT AS x APPROACHES a OF $f(x)$."

INTUITIVE IDEA: $\lim_{x \to a} f(x) = L$

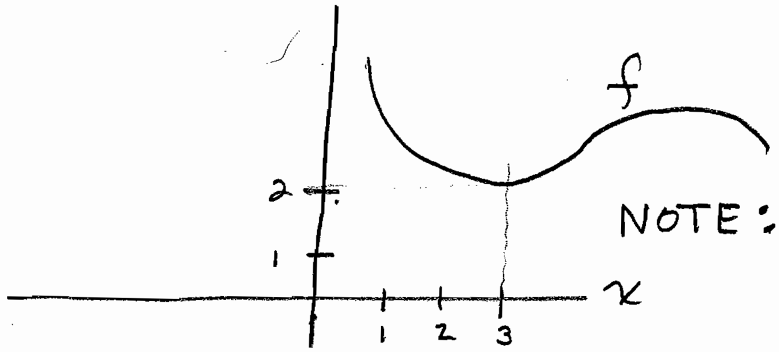
$f(x)$ GETS AS CLOSE AS YOU WANT TO L AND STAYS AS CLOSE AS YOU WANT TO L BY CHOOSING x SUFFICIENTLY CLOSE TO a (BUT $x \neq a$) FROM EITHER SIDE OF a

[THIS IS <u>NOT</u> THE DEFINITION FOR $\lim_{x \to a} f(x) = L$]

B. PICTURE EXAMPLES: FOR EACH

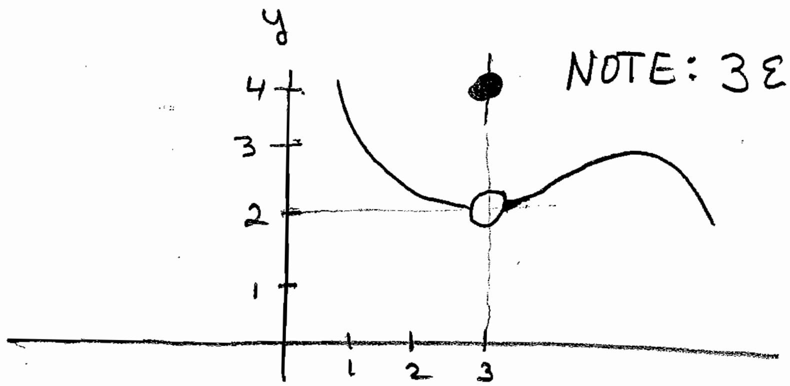
$$\lim_{x \to 3} f(x) = 2$$

1.



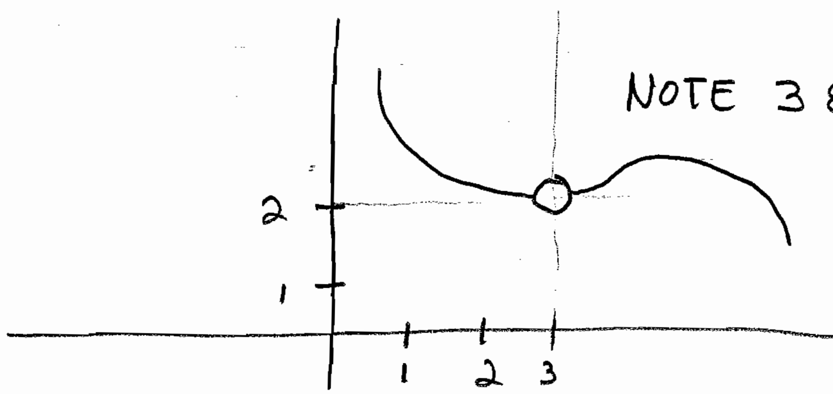
NOTE: $3 \in dom(f)$

2.



NOTE: $3 \in dom(f)$

3



NOTE $3 \notin dom(f)$

C. INTUITIVELY FIND THE LIMIT FOR FUNCTIONS DEFINED BY AN EQUATION

DIRECT SUBSTITUTION EXAMPLE

1. $\lim_{x \to 3} 2x-4 = 2$

x	$2x-4$
3.01	2.02
3.001	2.002
3.0001	2.0002

x	$2x-4$
2.99	1.98
2.999	1.998
2.9999	1.9998

NOTE: APPROACH 3 FROM EITHER SIDE YOU GET CLOSER AND CLOSER TO 2.

NOTE: THE LIMIT IN THIS CASE COULD HAVE BEEN CALCULATED BY SUBSTITUTING 3 IN FOR x

$f(3) = 2(3) - 4 = 6 - 4 = \boxed{2}$

THE SUBSITUTION JUST WORKED IN THIS CASE. WHAT DETERMINES THE VALUE OF THE LIMIT ARE THE VALUES OF $f(x)$ ON EITHER SIDE OF 3.

ALGEBRA, THEN DIRECT SUBSTITUTION EXAMPLE

2. FIND $\lim_{x \to 3} \frac{x^2-9}{x-3}$

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3}$$

KEEP THE LIMITS WHEN STRINGING OUT EQUALITIES UNTIL EVALUATED

$$= \lim_{x \to 3} x+3 \underset{\text{SUBSTITUTE 3 FOR } x}{=\!=\!=} 3+3 = 6$$

NOTE: YOU COULD NOT SUBSTITUTE IMMEDIATELY, BUT YOU COULD DO ALGEBRA AND SUBSTITUTE 3 FOR x. HOWEVER, THE SUBSTITUTION JUST WORKED IN THIS CASE. WHAT DETERMINES THE VALUE OF THE LIMIT ARE THE VALUES OF $f(x)$ ON EITHER SIDE OF 3.

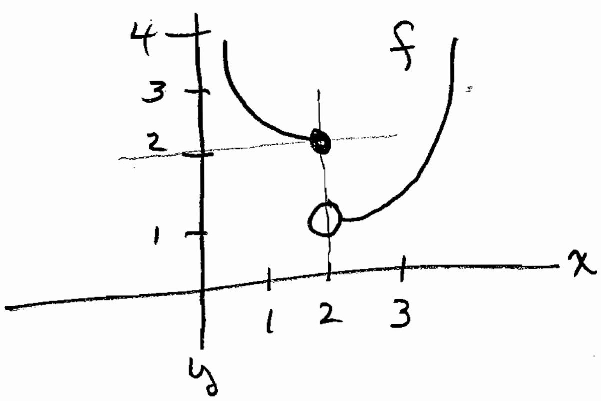
3. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

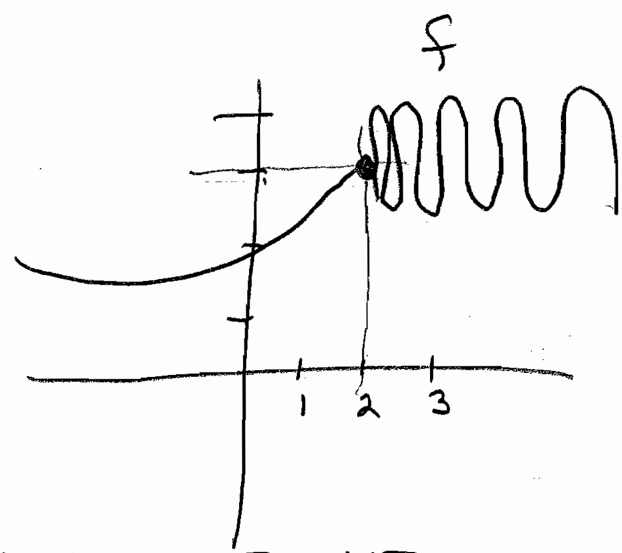
NO SUBSTITUTION EXAMPLE

YOU CANNOT SUBSTITUTE 0 FOR x. FOR INTUITION, SUBSTITUTE VALUES CLOSE TO 0 FROM EITHER SIDE, USE A CALCULATOR. THE RESULT GETS CLOSER AND CLOSER TO 1

D. HOWEVER, SOME FUNCTIONS $\lim\limits_{x \to a} f(x)$ DOES NOT EXIST

1. PICTURES





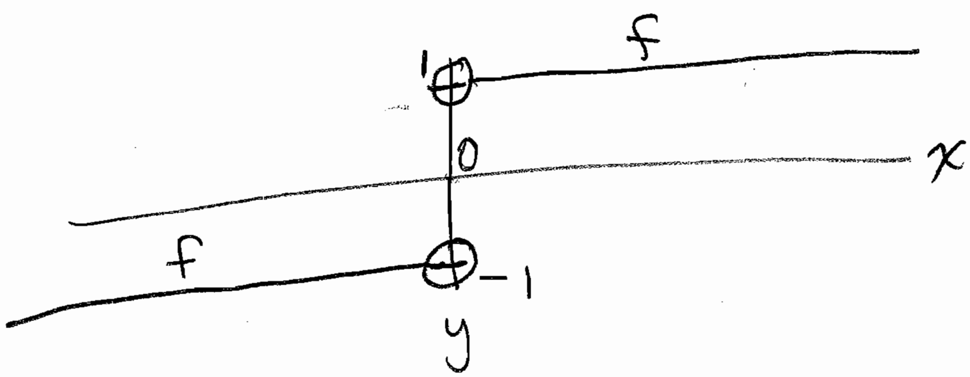
$\lim\limits_{x \to 2} f(x)$ DOES NOT EXIST

$\lim_{x \to a} f(x)$ DOES NOT EXIST (CONT.)

2. $\lim_{x \to 0} \frac{|x|}{x}$ DOES NOT EXIST

FOR $x > 0$ $\quad \frac{\overset{\text{POS}}{|x|}}{x} = \frac{x}{x} = 1$

FOR $x < 0$ $\quad \frac{\overset{\text{NEG}}{|x|}}{x} = \frac{-x}{x} = -1$



E. ADVANCED ALGEBRA EXAMPLE:

FIND $\lim_{x \to 0} \frac{\sqrt{x^2+25} - 5}{x^2}$

$$= \lim_{x \to 0} \frac{(\sqrt{x^2+25} - 5)(\sqrt{x^2+25} + 5)}{x^2 \, (\sqrt{x^2+25} + 5)}$$

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$$= \lim_{x \to 0} \frac{(x^2+25) - 25}{x^2\left(\sqrt{x^2+25} + 5\right)}$$

$$= \lim_{x \to 0} \frac{x^2}{x^2\left(\sqrt{x^2+25} + 5\right)} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+25} + 5}$$

$$= \frac{1}{\sqrt{0^2+25} + 5} = \frac{1}{10}$$

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HOMEWORK

PAGE 97 (4 a, d, e); (6 c, g, h, i); (7 c, f, g, h)

PAGE 112 11, 12, 14, 17, 18, 23, 27, 39, 40

LEFT AND RIGHT-HAND LIMITS
INTUITIVE DISCUSSION

A. LEFT-HAND LIMIT OF A FUNCTION f AT a:

DENOTED: $\lim\limits_{x \to a^-} f(x)$

READ: "THE LIMIT AS x APPROACHES a FROM THE LEFT OF $f(x)$"

OR

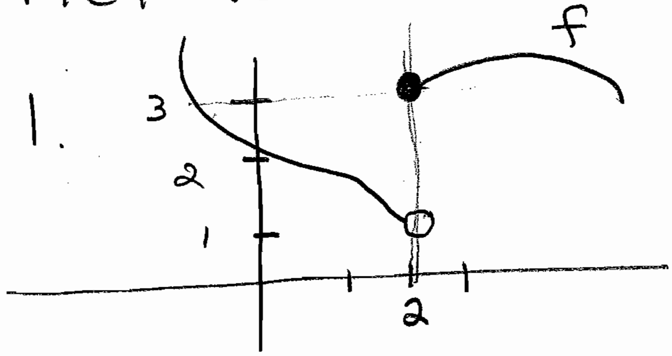
"THE LEFT-HAND LIMIT OF $f(x)$ AS x APPROACHES a"

INTUITIVE IDEA: $\lim\limits_{x \to a^-} f(x) = L$

$f(x)$ GETS, AND STAYS, AS CLOSE TO L AS YOU WANT BY CHOOSING x SUFFICIENTLY CLOSE TO a (BUT $x \neq a$) FROM THE LEFT SIDE OF a.

B. PICTURE EXAMPLES: $\lim_{x \to 2} f(x) = \text{DNE}^{*}$

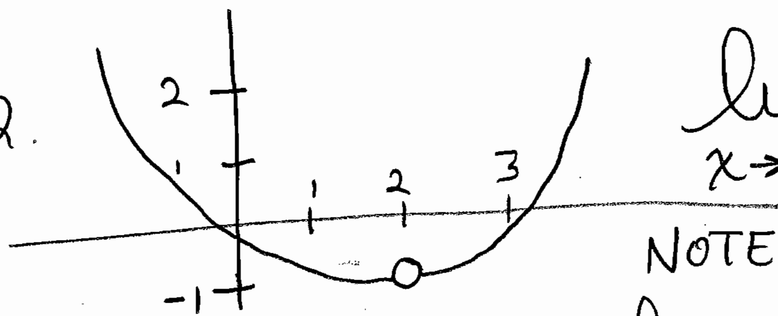
1.



$\lim_{x \to 2^-} f(x) = 1$

NOTE: $2 \in \text{dom}(f)$

2.



$\lim_{x \to 2^-} f(x) = -1$

NOTE $2 \notin \text{dom}(f)$

$\lim_{x \to 2} f(x) = -1$

C. SIMILARLY $\lim_{x \to a^+} f(x) = L$ IS <u>READ</u>

"THE LIMIT AS x APPROACHES a FROM THE RIGHT OF $f(x)$ EQUALS L".

SIMILAR INTUITIVE MEANING AS $\lim_{x \to a^-} f(x) = L$

FOR 1. ABOVE $\lim_{x \to 2^+} f(x) = 3$

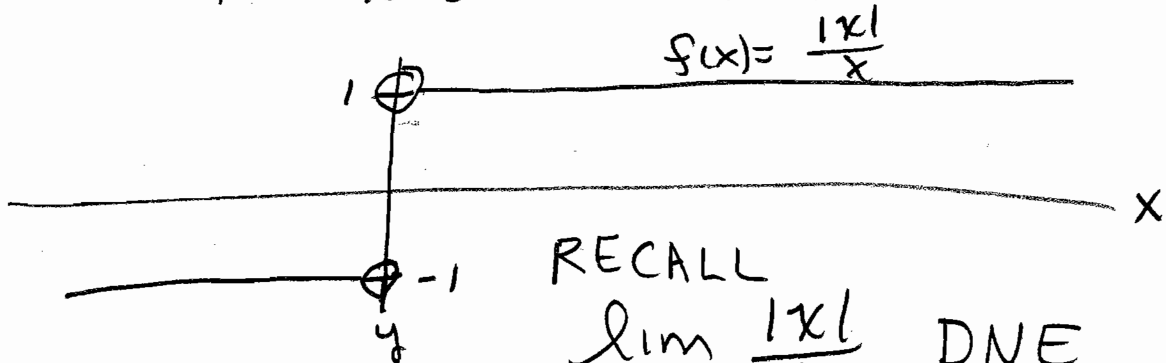
FOR 2. ABOVE $\lim_{x \to 2^+} f(x) = -1$

$*$DNE = DOES NOT EXIST

D. A LOOK AGAIN AT A PREVIOUS PROBLEM IN NEW LIMIT TERMINOLOGY

$$\lim_{x\to 0^-} \frac{\overset{NEG}{|x|}}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

$$\lim_{x\to 0^+} \frac{\overset{POS}{|x|}}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$



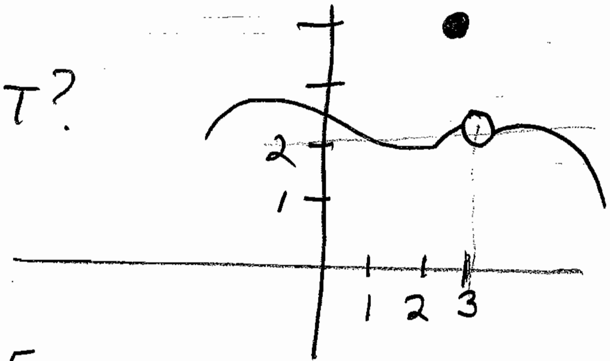
RECALL

$\lim_{x\to 0} \frac{|x|}{x}$ DNE

E. THEOREM: $\lim_{x\to a} f(x) = L$ IFF

$\lim_{x\to a^-} f(x) = L$ AND $\lim_{x\to a^+} f(x) = L$

F. DOES $\lim_{x \to 3} f(x)$ EXIST?



YES. $\lim_{x \to 3^-} f(x) = 2$

SAME

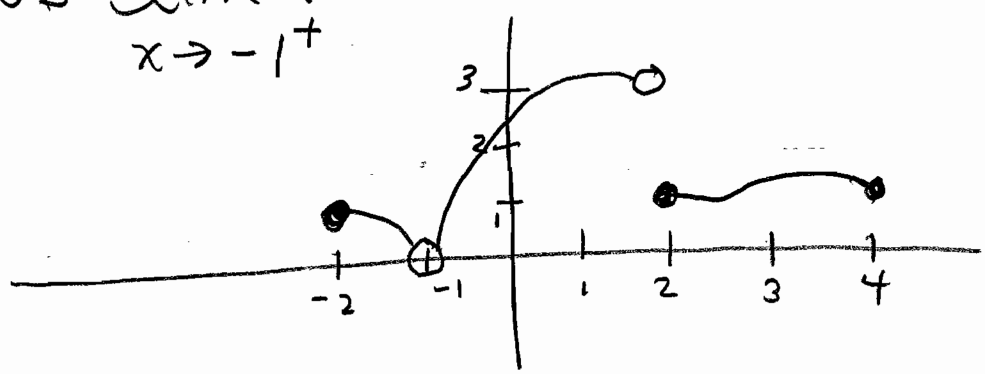
$\lim_{x \to 3^+} f(x) = 2$

YES. $\lim_{x \to 3} f(x)$ EXISTS AND EQUALS 2

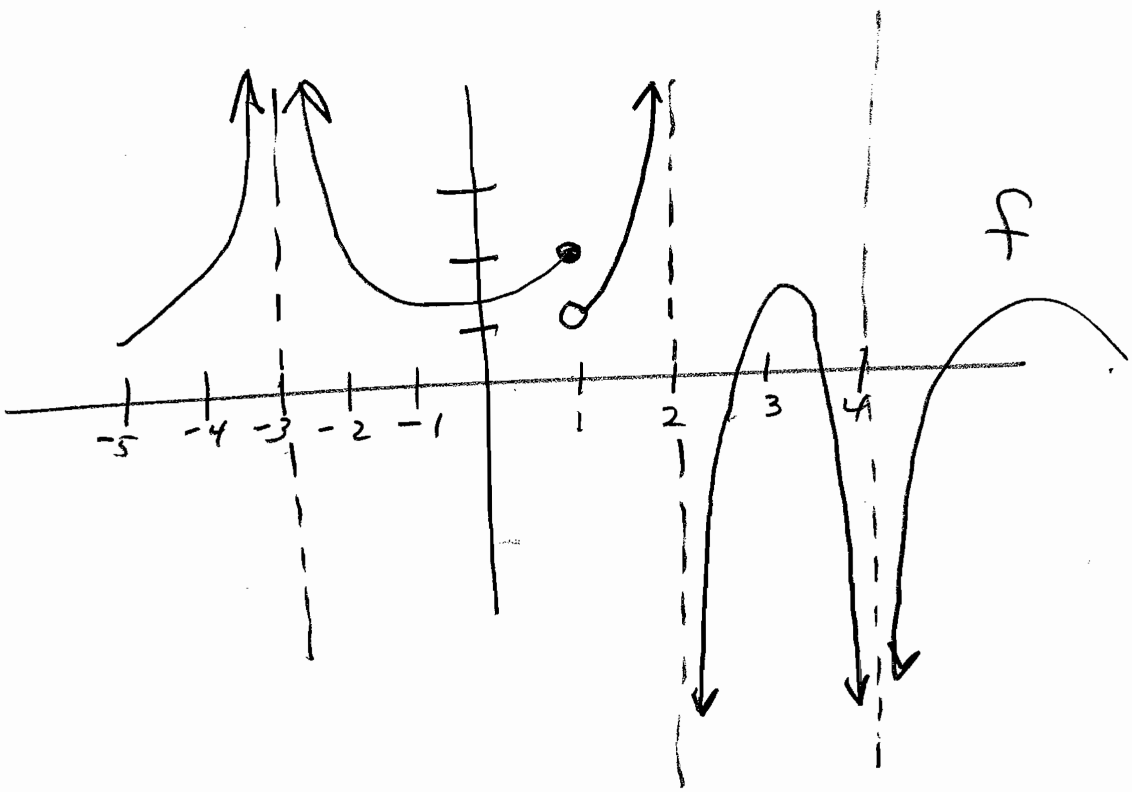
G. DRAW A GRAPH WHERE: $\lim_{x \to 2^-} f(x) = 3$

$\lim_{x \to 2^+} f(x) = 1$, $f(-1)$ IS UNDEFINED,

AND $\lim_{x \to -1^+} f(x) = 0$



H. $\lim_{x \to a^+} f(x) = +\infty$ OR $-\infty$ AND $\lim_{x \to a^-} f(x) = +\infty$ OR $-\infty$



$\lim_{x \to -3^-} f(x) = +\infty$

$\lim_{x \to -3^+} f(x) = +\infty$

$\lim_{x \to -3} f(x) = +\infty$

$\lim_{x \to 2^-} f(x) = +\infty$

$\lim_{x \to 2^+} f(x) = -\infty$

$\lim_{x \to 4} f(x) = -\infty$

$x = -3$ VERTICAL ASYMPTOTE

$x = 2$ VERTICAL ASYMPTOTE

$x = 4$ VERTICAL ASYMPTOTE

$x = K$ IS A VERTICAL ASYMPTOTE IFF $\lim_{x \to K^{\pm}} f(x) = \pm\infty$

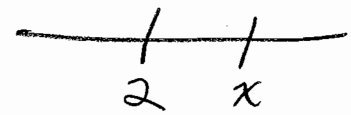
I. FINDING INFINITE LIMITS ($\lim_{x \to a^{\pm}} f(x) = \pm\infty$) BY LOOKING AT FUNCTION DEFINITION

1. $\dfrac{\overset{\text{AWAY 0}}{\text{POS.}}}{\underset{\text{LITTLE}}{\text{POS.}}} = \text{MIGHTY BIG}$ $\qquad \dfrac{6}{\frac{1}{1000}} = 6\left(\dfrac{1000}{1}\right) = \underset{6000}{\text{BIG}}$

2. $\dfrac{\overset{\text{AWAY 0}}{\text{POS}}}{\underset{\text{LITTLE}}{\text{NEG.}}} = \text{MIGHTY NEGATIVE}$ $\qquad \dfrac{6}{\frac{-1}{1000}} = 6\left(\dfrac{1000}{-1}\right) = -6000$

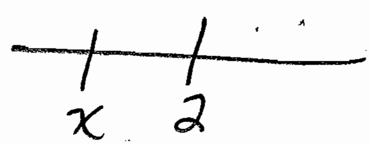
3. $\displaystyle\lim_{x \to 2^+} \frac{\overset{\text{AWAY 0 POS}}{6}}{x-2} = +\infty$

R-L
POS
LITTLE app. → 0



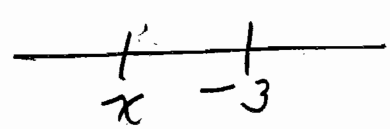
4. $\displaystyle\lim_{x \to 2^-} \frac{6 \leftarrow \overset{\text{AWAY 0}}{\text{POS}}}{x-2} = -\infty$

L-R
NEG
LITTLE app. → 0



5. $\displaystyle\lim_{x \to -3^-} \frac{-5}{x+3} = \lim_{x \to -3^-} \frac{(-5) \leftarrow \overset{\text{AWAY 0}}{\text{NEG}}}{x-(-3)} = +\infty$

L-R
NEG
LITTLE app. → 0



HOMEWORK

A. PAGES 106, 107 PROBLEMS 9, 13, 39, 40, 43, 44

EVALUATE EACH BELOW. SHOW REASONING.

B. $\displaystyle\lim_{x \to -2^-} \frac{4}{x+2}$

C. $\displaystyle\lim_{x \to 5^-} \frac{-7}{x-5}$

D. $\displaystyle\lim_{x \to 0^-} \frac{|-3x|}{x}$

E. $\displaystyle\lim_{x \to 5^-} \frac{|x-5|}{x-5}$

RIGOROUS DEFINITION OF LIMIT

A. OOZE FROM INTUITION TO RIGOR:

1. $\lim_{x \to a} f(x) = L$ INTUITIVELY MEANS

   $f(x)$ GETS, & STAYS, AS CLOSE TO L AS YOU LIKE BY CHOOSING x SUFFICIENTLY CLOSE TO a (BUT $x \neq a$) FROM EITHER SIDE OF a.

2. A METHOD SHOWN BY SWARTZ WILL BE USED TO GO FROM INTUITION TO RIGOR.

3. $f(x)$ GETS, & STAYS, CLOSE TO L MEANS $|f(x) - L|$ GETS, & STAYS CLOSE TO 0.

4. BUT CLOSE MEANS WHAT? $\frac{1}{10}, \frac{1}{100}, \ldots$? IT MEANS WHEN CHALLENGED BY ANY $\varepsilon > 0$ $|f(x) - L| < \varepsilon$ BY CHOOSING x SUFFICIENTLY CLOSE TO a (BUT $x \neq a$) FROM EITHER SIDE OF a

5. HOW CLOSE IS SUFFICIENTLY CLOSE? $\frac{1}{1000}, \frac{1}{10000}, \ldots$? NO... IT MEANS

6. FOR EACH $\varepsilon > 0$, THERE IS A $\delta > 0$ SUCH THAT FOR EACH $x \in dom(f)$, IF x IS WITHIN δ OF a (BUT $x \neq a$) FROM EITHER SIDE OF a, THEN

$$|f(x) - L| < \varepsilon$$

7. <u>REWRITTEN</u>: FOR EACH $\varepsilon > 0$, THERE IS A $\delta > 0$ SUCH THAT FOR EACH $x \in dom(f)$, IF $0 < |x - a| < \delta$, THEN $|f(x) - L| < \varepsilon$

8. RIGOROUS DEFINITION OF LIMIT WILL FOLLOW. MORE WILL BE ADDED TO INSURE x CAN BE PICKED ON EITHER SIDE OF a.

B. RIGOROUS DEFINITION OF $\lim_{x \to a} f(x) = L$

f IS DEFINED ON SOME OPEN INTERVAL CONTAINING a, EXCEPT AT POSSIBLY a ITSELF

AND

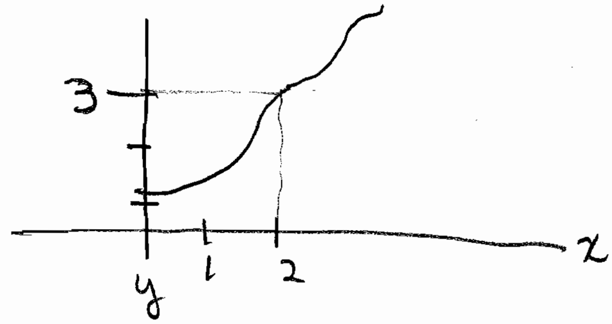
FOR EACH $\varepsilon > 0$, THERE IS A $\delta > 0$ SUCH THAT FOR ALL $x \in dom(f)$ IF $0 < |x - a| < \delta$, THEN $|f(x) - L| < \varepsilon$.

KNOW THIS DEFINITION

C. PICTURES TO SEE THE DEFINITION:

1. CLAIM

$$\lim_{x \to 2} f(x) = 3$$

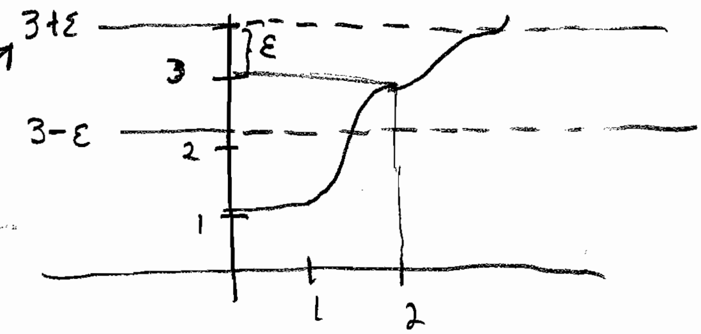


2. THE OPPONENT CHOOSES $\varepsilon > 0$ AND SAYS "SEE IF YOU CAN GET $|f(x) - 3| < \varepsilon$".

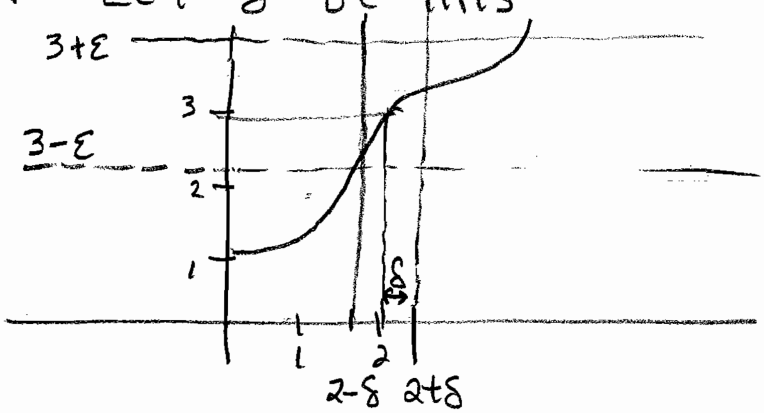
$$|f(x) - 3| < \varepsilon$$

$$-\varepsilon < f(x) - 3 < \varepsilon$$

$$3 - \varepsilon < f(x) < 3 + \varepsilon$$



3. YOU MEDITATE, DO SCRATCH WORK, RECEIVE WISDOM AND RESPOND WITH "LET δ BE THIS"



4. THEN IF $2 - \delta < x < 2 + \delta$

$$3 - \varepsilon < f(x) < 3 + \varepsilon$$

5. RECALL $2-\delta < x < 2+\delta$

$$-\delta < x-2 < \delta$$

$$|x-2| < \delta$$

$$3-\varepsilon < f(x) < 3+\varepsilon$$

$$-\varepsilon < f(x)-3 < \varepsilon$$

$$|f(x)-3| < \varepsilon$$

SO IF $0 < |x-2| < \delta$, THEN $|f(x)-3| < \varepsilon$

D. RIGOROUS PROOF STEPS TO PROVE

$$\lim_{x \to a} f(x) = L$$

1. ASSUME $\varepsilon > 0$ (SHOW THERE IS A $\delta > 0$ SUCH THAT FOR ALL $x \in \mathrm{dom}(f)$, IF $0 < |x-a| < \delta$, THEN $|f(x)-L| < \varepsilon$)

   AFTER MEDITATION, SCRATCH WORK, AND RECEIVING ILLUMINATION, NAME δ

2. LET $\delta =$ ____ (SHOW FOR ALL $x \in \mathrm{dom}(f)$, IF $0 < |x-a| < \delta$, THEN $|f(x)-L| < \varepsilon$)

3. ASSUME $x \in \mathrm{dom}(f)$ AND $0 < |x-a| < \delta$ (SHOW $|f(x)-L| < \varepsilon$)

E. GIVE A RIGOROUS PROOF THAT $\lim_{x \to 5} 3x-7=8$

IT IS GIVEN THAT $f(x)=3x-7$ IS DEFINED ON AN OPEN INTERVAL CONTAINING 5, EXCEPT POSSIBLY AT 5 ITSELF.

1. ASSUME $\varepsilon>0$ (SHOW THERE IS A $\delta>0$ SUCH THAT FOR ALL $x \in dom(f)$, IF $0<|x-5|<\delta$, THEN $|(3x-7)-8|<\varepsilon$)

> MEDITATION, SCRATCH WORK, ILLUMINATION,... NOT PART OF THE PROOF
>
> $$|(3x-7)-8|<\varepsilon$$
> $$|(3x-15)|<\varepsilon$$
> $$|3(x-5)|<\varepsilon$$
> $$|3||x-5|<\varepsilon$$
> $$3|x-5|<\varepsilon$$
> $$|x-5|<\frac{\varepsilon}{3} \text{ AHA } \delta$$

2. LET $\delta=\frac{\varepsilon}{3}>0$ (SHOW FOR ALL $x \in dom(f)$, IF $0<|x-5|<\delta$, THEN $|(3x-7)-8|<\varepsilon$)

3. ASSUME $0 < |x-5| < \delta$ AND $x \in \text{dom}(f)$ (SHOW $|(3x-7)-8| < \varepsilon$)

4. $|x-5| < \frac{\varepsilon}{3}$ — 3, 2, SUBSTITUTE $\frac{\varepsilon}{3}$ FOR δ.

5. $3|x-5| < \varepsilon$ — 4, MULT. BY 3

6. $|3||x-5| < \varepsilon$ — 5, $3 = |3|$

7. $|3(x-5)| < \varepsilon$ — 6, $|a||b| = |ab|$

8. $|3x-15| < \varepsilon$ — 7

9. $|(3x-7)-8| < \varepsilon$ — 8

THE RIGOROUS PROOF STEPS TO PROVE $\lim_{x \to 5} 3x-7 = 8$ WERE MET. BE SURE TO FOLLOW THIS FORMAT.

DO NOT GET DISCOURAGED. IT IS NOT UNUSUAL TO HAVE A HARD TIME DOING HARD THINGS. YOU CAN DO IT

F. DRILL ON FIRST 3 STEPS

PROVE $\lim_{x \to 5} 3x-7 = 8$

1. ASSUME $\varepsilon > 0$

* 2. LET $\delta =$ ______

3. ASSUME $0 < |x-5| < \delta$ AND $x \in dom(f)$. (SHOW $|(3x-7)-8| < \varepsilon$)

PROVE $\lim_{x \to 2} 4-5x = -6$

1. ASSUME $\varepsilon > 0$

* 2. LET $\delta =$ ______

3. ASSUME $0 < |x-2| < \delta$ AND $x \in dom(f)$. (SHOW $|(4-5x)-(-6)| < \varepsilon$)

PROVE $\lim_{x \to -4} 3x+1 = -11$

1. ASSUME $\varepsilon > 0$

* 2. LET $\delta =$ ______

3. ASSUME $0 < |x-(-4)| < \delta$ AND $x \in dom(f)$. (SHOW $|(3x+1)-(-11)| < \varepsilon$)

---

* FILL IN THIS LINE AFTER MEDITATION, SCRATCH WORK AND RECEIVING ILLUMINATION.

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Homework

20, 21 PAGE 118

Also prove by ε, δ definition

$$\lim_{x \to 3} 4x - 2 = 10$$

$$\lim_{x \to -7} 5 - 3x = 26$$

QUADRATIC ε, δ RIGOROUS LIMIT PROOFS

A. PROVE $\lim_{x \to 3} x^2 - x + 2 = 8$

THIS IS JUGULAR PROBLEM #1 TYPE

RECALL OUR PROOF START.

1. ASSUME $\varepsilon > 0$ (SHOW THERE IS A $\delta > 0$ SUCH THAT FOR ALL $x \in \mathrm{dom}(f)$ IF $0 < |x-3| < \delta$, THEN $|(x^2 - x + 2) - 8| < \varepsilon$)
2. LET $\delta =$ _____ ← MUCH MEDITATION, SCRATCH WORK, ILLUMINATION NEEDED HERE... SO LET IT BEGIN..

SCRATCH WORK

BE LOOKING FOR $|x-3|$

$$|(x^2 - x + 2) - 8| < \varepsilon$$

$$|x^2 - x - 6| < \varepsilon$$

$$|(x-3)(x+2)| < \varepsilon$$

$$|x-3||x+2| < \varepsilon$$

$$|x-3| < \frac{\varepsilon}{|x+2|}$$

WE WILL EVENTUALLY CHOOSE δ
SO THAT $\delta \le 1$

$$|x-3| < \delta \le 1$$

$$|x-3| < 1$$

$$-1 < x-3 < 1$$

ADD 5

$$4 < x+2 < 6$$

$|x+2| = x+2$ SINCE $x+2>0$

$$\frac{1}{6} < \frac{1}{x+2} < \frac{1}{4}$$

$$\frac{1}{6} < \frac{1}{|x+2|} < \frac{1}{4}$$

$\frac{\varepsilon}{6} < \frac{\varepsilon}{|x+2|}$ ← THIS HAPPENS

WHEN δ IS PICKED ≤ 1
LET $\delta = \min\{1, \varepsilon/6\}$
LET'S NOW DO A BEAUTIFUL PROOF. THIS **IS** NOT PART OF THE PROOF, THIS SCRATCH WORK

PROOF OF $\lim\limits_{x \to 3} x^2 - x + 2 = 8$

1. ASSUME $\varepsilon > 0$
2. LET $\delta =$ MINIMUM OF 1 AND $\varepsilon/6$, SO $\delta > 0$.
3. ASSUME $x \in \mathrm{dom}(f)$ AND $0 < |x-3| < \delta$
   (SHOW $|(x^2 - x + 2) - 8| < \varepsilon$)
4. $|x-3| \overset{3}{<} \delta \overset{2}{\leq} 1$
5. $|x-3| < 1$
6. $-1 < x-3 < 1$ 5
7. $4 < x+2 < 6$ 6, ADD 5
8. $|x+2| = x+2$ SINCE $x+2 \overset{7}{>} 0$
9. $\frac{1}{6} < \frac{1}{x+2} < \frac{1}{4}$ 7
10. $\frac{1}{6} < \frac{1}{|x+2|}$ 8, 9
11. $\frac{\varepsilon}{6} < \frac{\varepsilon}{|x+2|}$ 10, MULT. BY ε
12. $|x-3| \overset{3}{<} \delta \overset{2}{\leq} \varepsilon/6$
13. $|x-3| \overset{12}{<} \frac{\varepsilon}{6} \overset{11}{<} \frac{\varepsilon}{|x+2|}$

14.	$\lvert x-3 \rvert < \dfrac{\varepsilon}{\lvert x+2 \rvert}$	13
15.	$\lvert x-3 \rvert \lvert x+2 \rvert < \varepsilon$	14
16.	$\lvert (x-3)(x+2) \rvert < \varepsilon$	15
17.	$\lvert x^2-x-6 \rvert < \varepsilon$	16
18.	$\lvert (x^2-x+2)-8 \rvert < \varepsilon$	17

NOTE: x DOES NOT COME INTO EXISTENCE IN THE PROOF UNTIL AFTER δ IS NAMED. SO DO NOT SEE THE LINE

$$\lvert x-3 \rvert < \frac{\varepsilon}{\lvert x+2 \rvert}$$

IN YOUR SCRATCH WORK AND NAME

$$\delta = \frac{\varepsilon}{\lvert x+2 \rvert}$$

THIS WOULD MEAN x EXISTED BEFORE δ... AGAINST THE RULES!

B. PROVE $\lim_{x \to 2} 3x^2 - x + 1 = 11$

SCRATCH WORK

BE LOOKING FOR $|x-2|$

$$|(3x^2 - x + 1) - 11| < \varepsilon$$

$$|3x^2 - x - 10| < \varepsilon$$

$$|(x-2)(3x+5)| < \varepsilon$$

$$|x-2||3x+5| < \varepsilon$$

$$|x-2| < \frac{\varepsilon}{|3x+5|}$$

MAKE $|x-2| < 1$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

$$3 < 3x < 9$$

$$8 < 3x+5 < 14 \qquad 3x+5 = |3x+5|$$

$$8 < |3x+5| < 14$$

$$\frac{1}{14} < \frac{1}{|3x+5|} \; ; \; \frac{\varepsilon}{14} < \frac{\varepsilon}{|3x+5|}$$

LET δ BE MIN. OF 1 AND $\frac{\varepsilon}{14}$

PROOF OF $\lim_{x \to 2} 3x^2 - x + 1 = 11$

1. ASSUME $\varepsilon > 0$
2. LET $\delta =$ MINIMUM OF 1 AND $\frac{\varepsilon}{14}$, SO $\delta > 0$
3. ASSUME $x \in dom(f)$ AND $0 < |x-2| < \delta$

   (SHOW $|(3x^2 - x + 1) - 11| < \varepsilon$)

4. $|x-2| \overset{3}{<} \delta \overset{2}{\leq} 1$
5. $|x-2| \overset{4}{<} 1$
6. $-1 < x-2 < 1$ — 5
7. $1 < x < 3$ — 6
8. $3 < 3x < 9$ — 7
9. $8 < 3x+5 < 14$ — 8
10. $|3x+5| = 3x+5$ — 9
11. $8 < |3x+5| < 14$ — 10
12. $\frac{\varepsilon}{14} < \frac{\varepsilon}{|3x+5|}$ — 11

13. $|x-2| \overset{3}{<} \delta \overset{2}{\leq} \frac{\varepsilon}{14}$

14 $|x-2| \overset{13}{<} \frac{\varepsilon}{14} \overset{12}{<} \frac{\varepsilon}{|3x+5|}$

15. $|x-2||3x+5| < \varepsilon$

16. $|(x-2)(3x+5)| < \varepsilon$

17 $|3x^2 - x - 10| < \varepsilon$

18. $|(3x^2 - x + 1) - 11| < \varepsilon$

HOMEWORK

Do ε, δ proof by using the definition for
30 Page 118

AND

$$\lim_{x \to 5} x^2 - 2x - 3 = 12$$

LIMIT THEOREMS

A. ε, δ PROOFS COULD BE GIVEN OF THE FOLLOWING:

IF c IS A CONSTANT $\lim_{x \to a} f(x)$ EXISTS AND $\lim_{x \to a} g(x)$ EXISTS, THEN

1. $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

2. $\lim_{x \to a} (c f(x)) = c \left[\lim_{x \to a} f(x) \right]$

3. $\lim_{x \to a} (f(x) \cdot g(x)) = \left[\lim_{x \to a} f(x) \right] \left[\lim_{x \to a} g(x) \right]$

4. IF $\lim_{x \to a} g(x) \neq 0$, Then

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

5. $\lim_{x \to a} c = c$

B. IF IT IS SAID $\lim_{x\to a} f(x)$ <u>EXISTS</u>, THAT MEANS IT IS SOME SPECIFIC <u>NUMBER</u>. IF $\lim_{x\to a} f(x) = \pm\infty$, THEN $\lim_{x\to a} f(x)$ DOES <u>NOT</u> EXIST, BECAUSE $\pm\infty$ IS NOT A NUMBER.

C. SUPPOSE $\lim_{x\to 7} f(x) = 5$ AND $\lim_{x\to 7} g(x) = 3$

$$\lim_{x\to 7}\left(f(x)+g(x)\right) \overset{1}{=} \lim_{x\to 7} f(x) + \lim_{x\to 7} g(x)$$

$$= 5 + 3 = 8$$

$$\lim_{x\to 7}\left[3f(x)+\frac{19}{g(x)}\right] \overset{1}{=} \lim_{x\to 7} 3f(x) + \lim_{x\to 7}\frac{19}{g(x)}$$

$$\overset{2,4}{=} 3\lim_{x\to 7} f(x) + \frac{\lim_{x\to 7} 19}{\lim_{x\to 7} g(x)} \overset{5}{=} 3(5) + \frac{19}{3}$$

$$= 15 + \frac{19}{3} = \frac{64}{3}$$ RIGOROUSLY: READ BACKWARDS

D. RECALL, YOU MUST FIRST KNOW $\lim_{x \to a} f(x)$ EXISTS AND $\lim_{x \to a} g(x)$ EXISTS BEFORE YOU CAN SAY

$$\lim_{x \to a}[f(x)+g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

NOTE. LET $f(x) = \frac{1}{x^2}$ $\quad g(x) = \frac{-1}{x^2}$

$\lim_{x \to 0} f(x) = +\infty$, $\lim_{x \to 0} g(x) = -\infty$ BOTH DO NOT EXIST

$$\lim_{x \to 0} f(x)+g(x) = \lim_{x \to 0} \frac{1}{x^2} + \frac{-1}{x^2} = \lim_{x \to 0} 0 \overset{5}{=} 0$$

THE LIMIT OF THE SUM EXISTS

BUT

$$\lim_{x \to 0} \frac{1}{x^2} + \frac{-1}{x^2} \neq \underbrace{\lim_{x \to 0} \frac{1}{x^2} + \lim_{x \to 0} \frac{-1}{x^2}}_{\text{THESE DO NOT EXIST}}$$

E. THEOREM: IF $\lim_{x \to a} f(x)$ EXIST, THEN

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$, PROVIDED

n IS A POSITIVE INTEGER AND WHEN n IS EVEN, $\lim_{x \to a} f(x) \geq 0$

F. RECALL POLYNOMIALS ARE LIKE

$$p(x) = \frac{2}{3}x^5 - \pi x^2 + 4x - \frac{1}{2}$$

EXPONENTS ARE POSITIVE INTEGERS OR ZERO.

G. THEOREM: IF $p(x)$ IS A POLYNOMIAL, THEN $\lim_{x \to a} p(x) = p(a)$

[POLYNOMIALS ARE TYPES OF FUNCTIONS YOU CAN SUBSTITUTE a FOR x AND GET THE ANSWER.]

H. THE SQUEEZE THEOREM:

IF $l(x) \leq m(x) \leq r(x)$ FOR ALL $x \in (c, a) \cup (a, d)$ AND $\lim_{x \to a} l(x) = K$ AND $\lim_{x \to a} r(x) = K$, THEN $\lim_{x \to a} m(x) = K$

I. FIND $\lim_{x \to 0} x^2 \sin \frac{1}{x}$

1. $-1 \leq \sin \frac{1}{x} \leq 1$

2. $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$ $\quad$ 1, $x^2 \geq 0$

3. $\lim_{x \to 0} (-x^2) = -0^2 = 0$ $\quad$ G

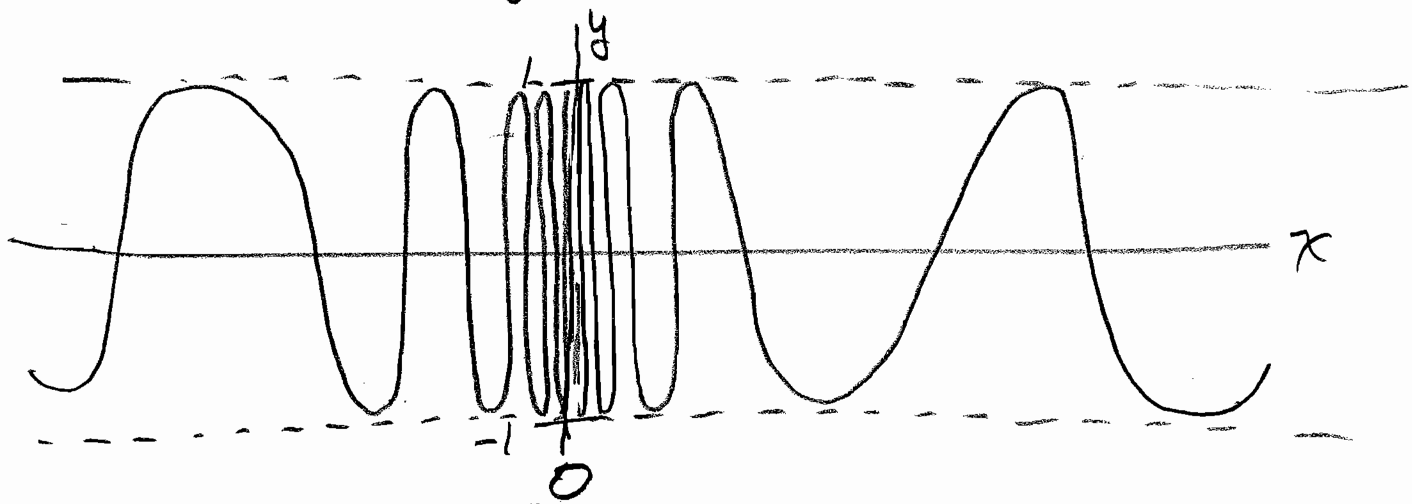
4. $\lim_{x \to 0} x^2 = 0^2 = 0$ $\quad$ G

5. $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$ $\quad$ 2, 3, 4 SQUEEZE THM.

NOTE: $\lim_{x\to 0} x^2 \sin\frac{1}{x} \neq \lim_{x\to 0} x^2 \lim_{x\to 0} \sin\frac{1}{x}$

LIMIT DOES NOT EXIST

GRAPH OF $y = \sin\frac{1}{x}$



J. IF $f(x)$ IS EITHER A RATIONAL, EXP., LOG, ALGEBRAIC, OR TRIG FUNCTION, THEN

$\lim_{x\to a} f(x) = f(a)$ JUST SUBSTITUTE IN a FOR x

RATIONAL $\equiv \dfrac{\text{POLYNOMIAL}}{\text{POLYNOMIAL}}$ $\qquad \dfrac{3x^2-6x}{2x^3-4x+3}$

ALGEBRAIC $\equiv$ EVEN INVOLVES ROOTS $\qquad \sqrt{\dfrac{2x-1}{x+3}} - 5x$

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HOMEWORK

PAGES 106/107 : PROBLEM 1
PROBLEM 37

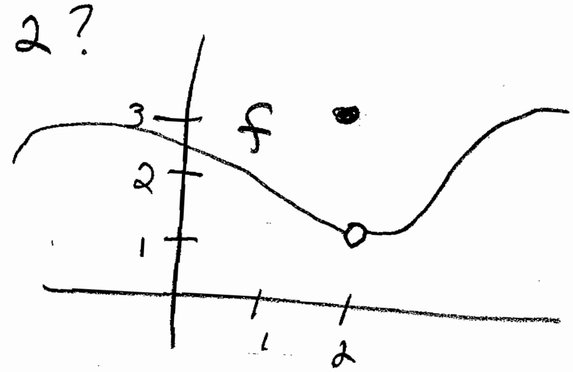
CONTINUITY

A. DEFINITION

FUNCTION f IS CONTINUOUS AT a IF AND ONLY IF

1. $a \in \mathrm{dom}(f)$
2. $\lim_{x \to a} f(x)$ EXISTS (i.e. IS A NUMBER)
3. $\lim_{x \to a} f(x) = f(a)$

B. IS f CONTINUOUS AT 2?

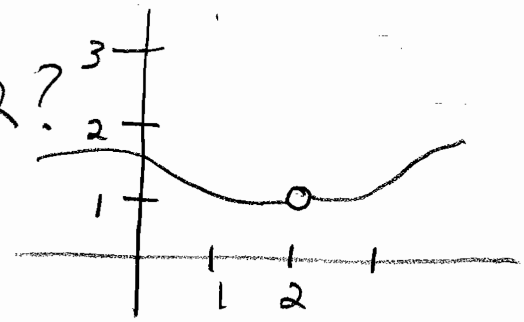


1. $2 \in \mathrm{dom}\, f$
2. $\lim_{x \to 2} f(x)$ exists (IT IS 1)
3. BUT $\lim_{x \to 2} f(x) \neq f(2)$

$\lim_{x \to 2} f(x) = 1 \qquad f(2) = 3$

ANSWER: NOT CONTINUOUS AT 2

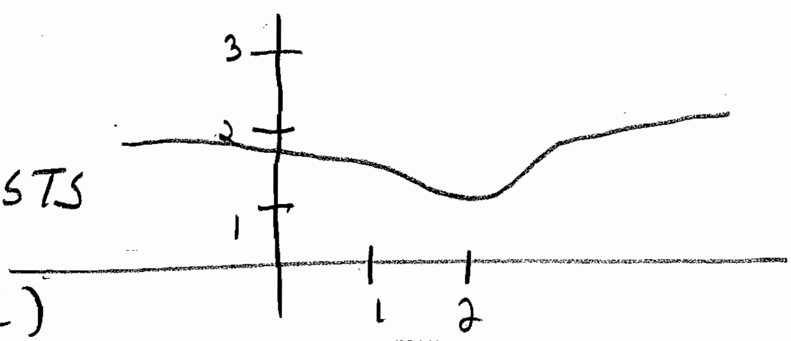
C. IS f CONTINUOUS AT 2?



1. $2 \notin \mathrm{dom}(f)$
2. $\lim_{x \to 2} f(x)$ EXISTS (IT IS 1)
3. $\lim_{x \to 2} f(x) \neq f(2)$ SINCE $f(2)$ DOES NOT EXIST

ANSWER: NOT CONTINUOUS AT 2

D. IS f CONTINUOUS AT 2?

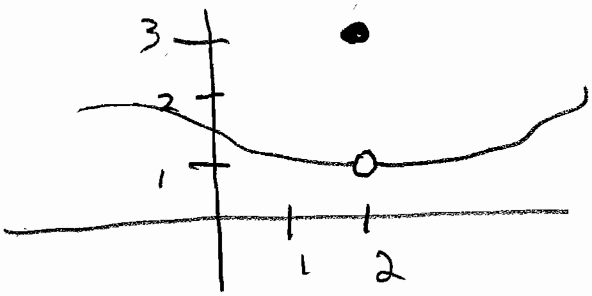


1. $2 \in \mathrm{dom}(f)$
2. $\lim_{x \to 2} f(x)$ EXISTS (IT IS 1)
3. $\lim_{x \to 2} f(x) = f(2)$

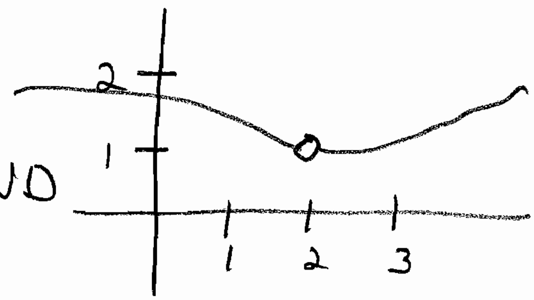
NOTE: $\lim_{x \to 2} f(x) = 1$ $\quad f(2) = 1$

ANSWER: YES, CONTINUOUS AT 2

E. REMOVABLE DISCONTINUITIES



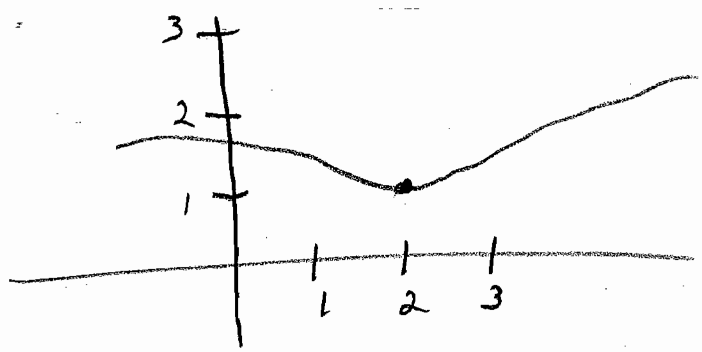
AND



ARE DISCONTINUOUS AT 2 BUT SINCE BOTH HAVE $\lim_{x \to 2} f(x) = 1$,

f COULD BE REDEFINED SO THAT $f(2) = 1 = \lim_{x \to 2} f(x)$

AND f WOULD BE CONTINUOUS AT 2.

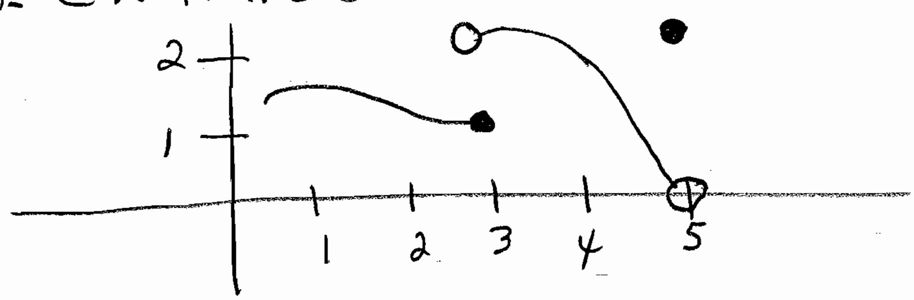


DEF: f IN NOT CONTINUOUS AT a BUT CAN BE REDEFINED AT a TO BE CONTINUOUS AT a ~~IFF~~ f HAS A REMOVABLE DISCONTINUITY AT a

F. **DEFINITION:** f IS CONTINOUS FROM THE LEFT AT a IFF

1) $a \in \mathrm{dom}(f)$
2) $\lim\limits_{x \to a^-} f(x)$ EXISTS (i.e. IS A NUMBER)
3) $\lim\limits_{x \to a^-} f(x) = f(a)$

G. EXAMPLE



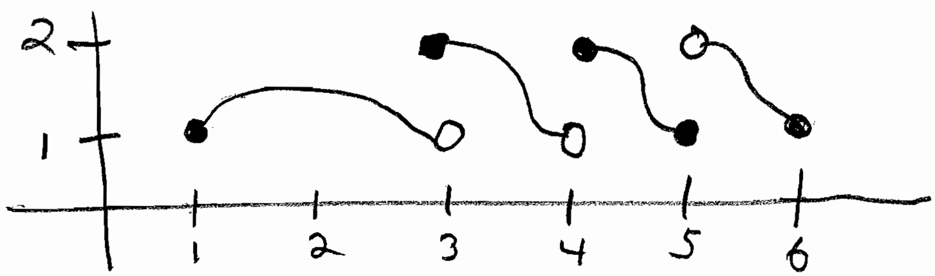
f IS CONTINUOUS FROM THE LEFT AT 3.

f IS NOT CONTINUOUS FROM THE LEFT AT 5.

f IS NOT CONTINUOUS AT 3.

G. DEFINITION f IS CONTINUOUS FROM THE RIGHT AT a IFF

1) $a \in \text{dom}(f)$
2) $\lim\limits_{x \to a^+} f(x)$ EXISTS (i.e. IS A NUMBER)
3) $\lim\limits_{x \to a^+} f(x) = f(a)$



f IS CONTINUOUS FROM THE RIGHT AT 1, 3, AND 4.

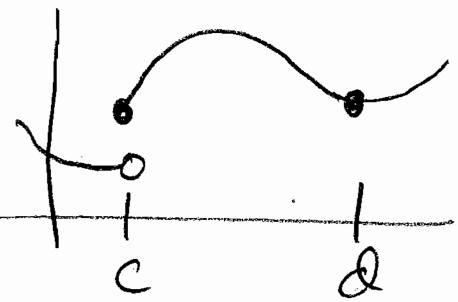
f IS NOT CONTINUOUS FROM THE RIGHT AT 5

f IS NOT CONTINUOUS AT 1, 3, 4, 5, 6

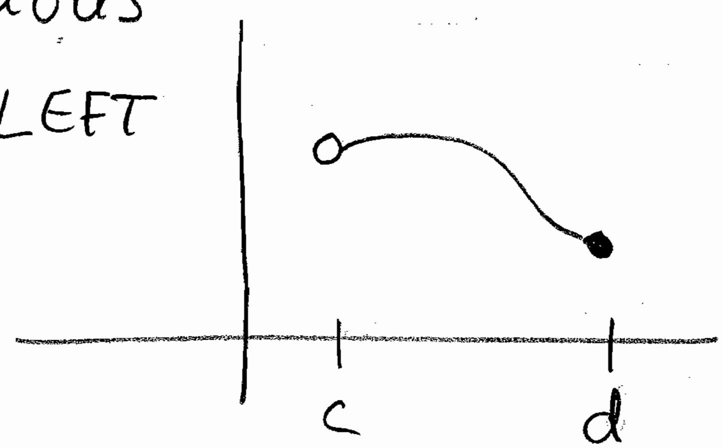
H. DEF. f IS DISCONTINUOUS AT a IFF f IS NOT CONTINUOUS AT a.

I. <u>CONTINUOUS</u> <u>ON</u> <u>INTERVALS</u>

1. f IS CONTINUOUS ON $[c,d]$ IFF FOR EACH $a \in (c,d)$ f IS CONTINUOUS AT a AND f IS CONTINUOUS FROM THE RIGHT AT c AND f IS CONTINUOUS FROM THE LEFT AT d.



2. f IS CONTINUOUS ON $(c,d]$ IFF FOR EACH $a \in (c,d)$, f IS CONTINUOUS AT a AND f IS CONTINUOUS FROM THE LEFT AT d.



J. <u>RECALL</u>: SOME FUNCTIONS WE JUST SUBSTITUTED IN TO EVALUATE THE LIMIT... THAT CLASS OF FUNCTIONS IS THE CLASS OF CONTINUOUS FUNCTIONS

POLYNOMIALS, RATIONAL, ALGEBRAIC, TRIG, EXPONENTIAL, AND LOG FUNCTIONS ARE ALL CONTINUOUS AT EACH POINT OF THEIR DOMAINS THAT ARE NOT END POINTS.

K. <u>THEOREM</u> IF $\lim_{x \to a} g(x)$ EXISTS AND $\lim_{x \to a} g(x)$ IS A NUMBER IN $\operatorname{dom}(f)$ AND f IS CONTINUOUS AT THAT NUMBER, THEN

$$\lim_{x \to a} (f \circ g)(x) = \lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

SO ... (CONT. NEXT PAGE)

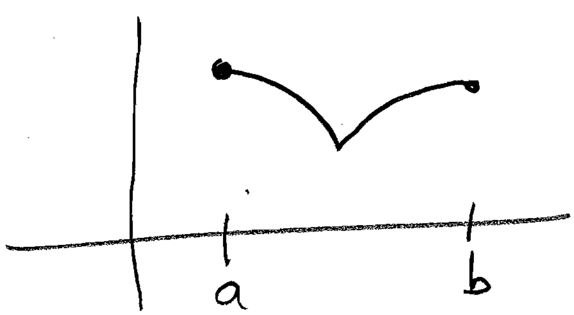
$$\lim_{x\to 2}\sqrt{3x-1} = \sqrt{\lim_{x\to 2} 3x-1} = \sqrt{3(2)-1} = \sqrt{5}$$

$$\lim_{x\to 2} e^{3x-1} = e^{\lim_{x\to 2} 3x-1} = e^{3(2)-1} = e^{5}$$

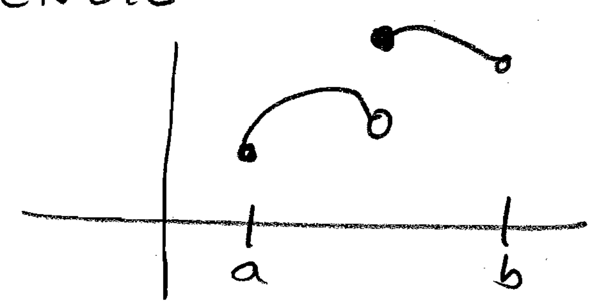
$$\lim_{x\to 2} \ln(3x-1) = \ln\left(\lim_{x\to 2}(3x-1)\right) = \ln(3(2)-1) = \ln 5$$

L. INTUITION ABOUT CONTINUITY:

IF f IS CONTINUOUS ON $[a,b]$, IT IS CONNECTED. YOU CAN DRAW FROM $(a, f(a))$ TO $(b, f(b))$ WITHOUT PICKING UP YOUR PENCIL



CONTINUOUS ON $[a,b]$



NOT CONTINUOUS ON $[a,b]$

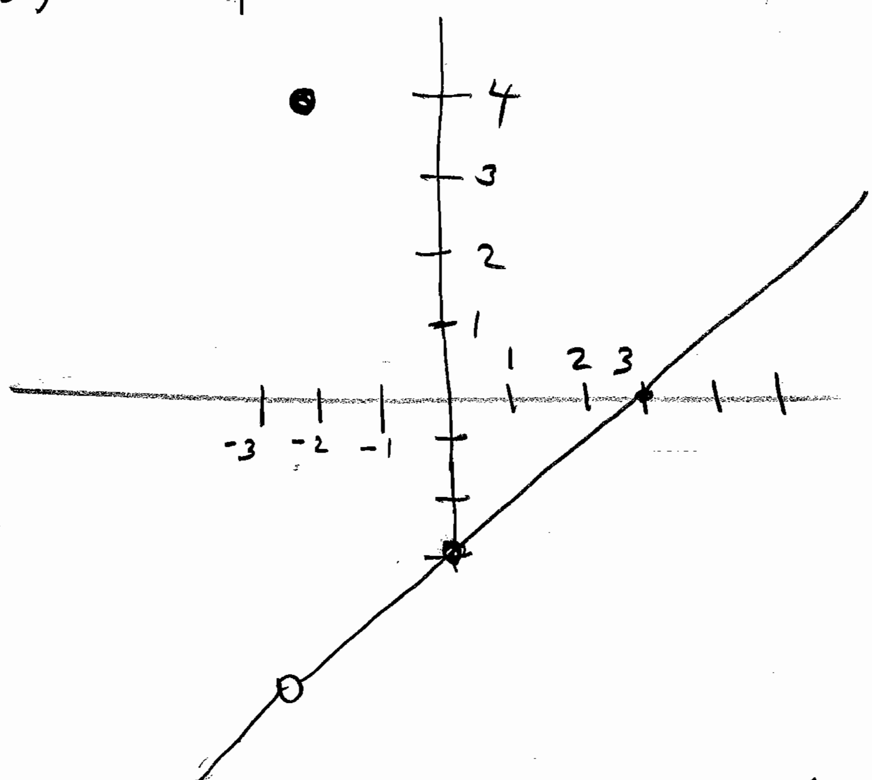
M. CONTINUITY LOOKING AT FUNCTIONS DEFINED BY EXPRESSIONS (NOT PICTURES i.e. GRAPHS)

1. $f(x) = \sqrt{x-2}$ $\qquad x-2 \geq 0$, $x \geq 2$

CONTINUOUS ON $[2, \infty)$

2. $f(x) = \dfrac{x^2 - x - 6}{x+2} \left(= \dfrac{\cancel{(x+2)}(x-3)}{\cancel{(x+2)}} = x-3 \right)$

$f(-2) = +4$



f IS CONTINUOUS ON $(-\infty, -2) \cup (-2, \infty)$

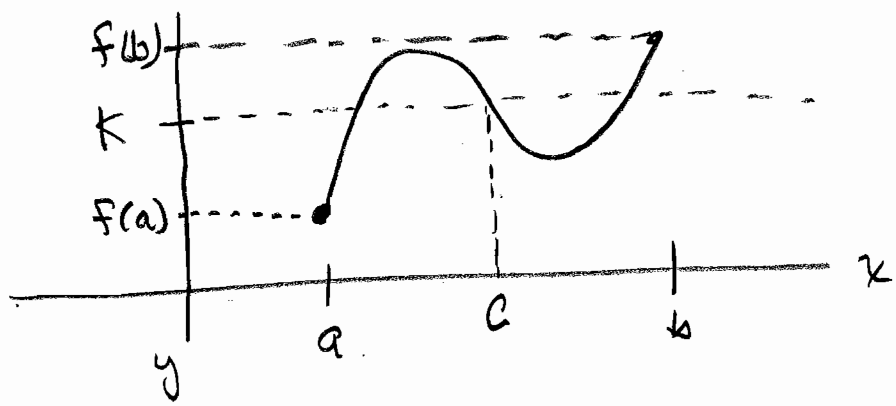
-2 IS A REMOVABLE DISCONTINUITY

HOMEWORK

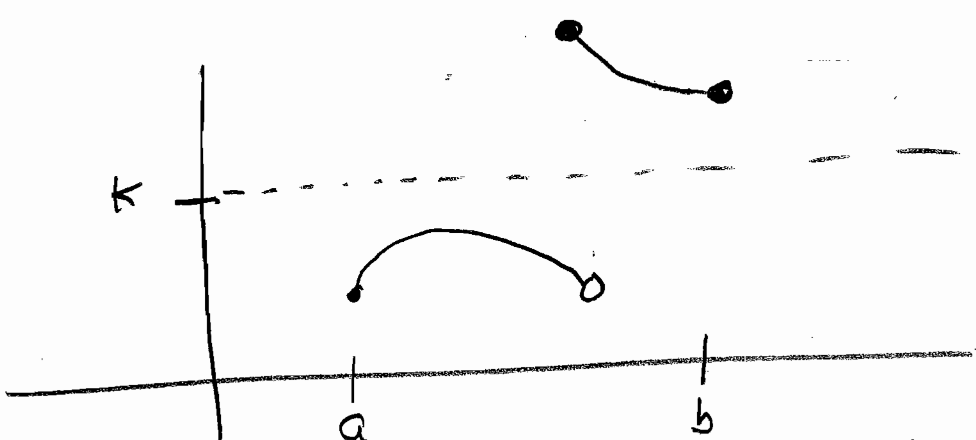
PAGES 128-129: 3, 4, 5, 17, 18, 43 a, b

INTERMEDIATE VALUE THEOREM

A. <u>THM</u>: IF f IS CONTINUOUS ON $[a,b]$ AND K IS BETWEEN $f(a)$ AND $f(b)$, THEN THERE IS A $c \in [a,b]$ SUCH THAT $f(c)=K$



<u>NOTE</u>: CONTINUITY IS NEEDED.



NO $c \in [a,b]$ SUCH THAT $f(c)=K$
f IS <u>NOT</u> CONTINUOUS ON $[a,b]$

B. <u>EXAMPLE</u>: LET $f(x) = x^2 + 3x$.
$f(1) = 4$ $f(2) = 10$ 7 IS BETWEEN $f(1)$ AND $f(2)$. FIND $c \in [1,2]$ SUCH THAT $f(c) = 7$. THE EXISTENCE OF c IS ASSURED SINCE f IS CONTINUOUS ON $[1,2]$ AND THE INTERMEDIATE VALUE THEOREM APPLIES.

$$f(c) = c^2 + 3c = 7$$

$$c^2 + 3c - 7 = 0$$

$$c = \frac{-3 \pm \sqrt{3^2 - 4(1)(-7)}}{2(1)} = \frac{-3 \pm \sqrt{37}}{2}$$

$\dfrac{-3 - \sqrt{37}}{2} \notin [1,2]$ SO IT IS NOT c.

$\dfrac{-3 + \sqrt{37}}{2} \in [1,2]$. IT IS A DESIRED c.

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HOMEWORK

LET $f(x) = 2x^2 - 3x$

$f(-1) = 5 \qquad f(5) = 35$

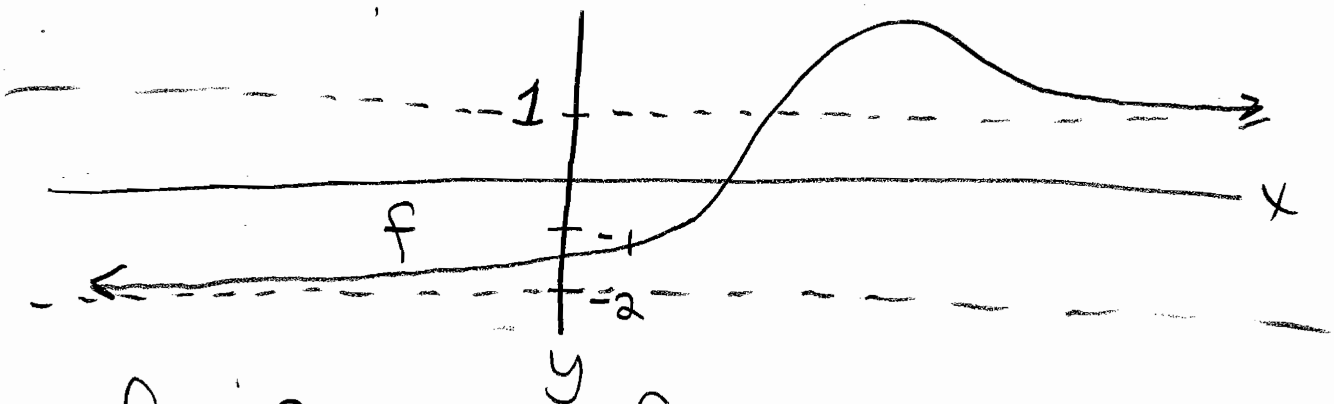
17 IS BETWEEN $f(-1)$ AND $f(5)$.

FIND $c \in [-1, 5]$ SUCH THAT $f(c) = 17$.

THE EXISTENCE OF c IS ASSURED SINCE f IS CONTINUOUS ON $[-1, 2]$ AND THE INTERMEDIATE VALUE THEOREM APPLIES

$\lim\limits_{x \to \pm\infty}$ AND HORIZONTAL ASYMPTOTES

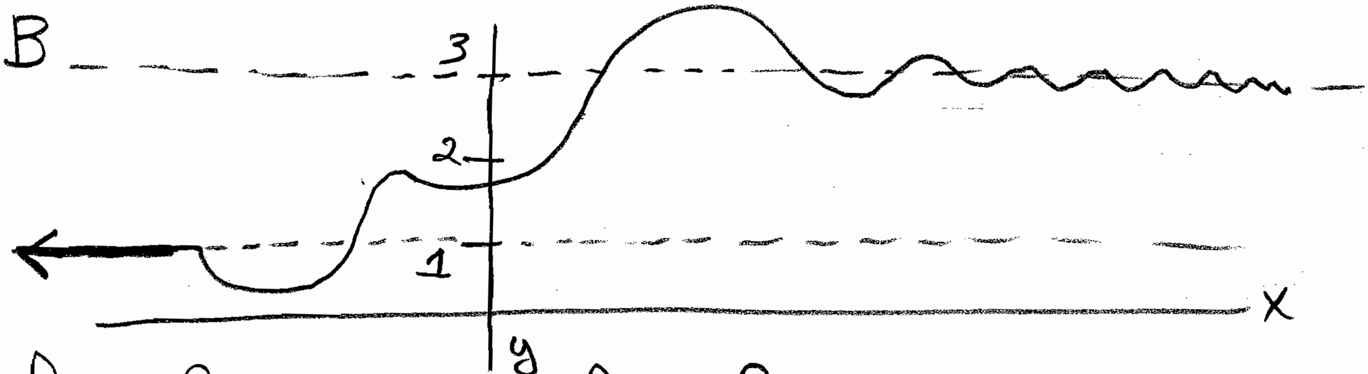
A. PICTURES OF $\lim\limits_{x \to \pm\infty} f(x) = L$



$$\lim_{x \to +\infty} f(x) = 1 \qquad \lim_{x \to -\infty} f(x) = -2$$

$y = 1$; $y = -2$ ARE <u>HORIZONTAL</u> ASYMPTOTES

B



$$\lim_{x \to +\infty} f(x) = 3 \qquad \lim_{x \to -\infty} f(x) = 1$$

$y = 3$; $y = 1$ ARE <u>HORIZONTAL</u> ASYMPTOTES

C. **DEFINITION:** $y = K$ IS A HORIZONTAL ASYMPTOTE IFF **EITHER**

$$\lim_{x \to \infty} f(x) = K \quad \text{OR} \quad \lim_{x \to -\infty} f(x) = K$$

D. **INTUITIVE IDEA** FOR $\lim_{x \to +\infty} f(x) = K$

$f(x)$ GETS AS CLOSE TO K AS YOU LIKE AND STAYS AS CLOSE TO K AS YOU LIKE BY CHOOSING x SUFFICIENTLY LARGE.

E. **DEFINITION** FOR $\lim_{x \to +\infty} f(x) = K$

f IS DEFINED ON SOME INTERVAL (a, ∞) AND

FOR EVERY $\varepsilon > 0$, THERE IS A POSITIVE NUMBER N SUCH THAT FOR ALL $x > N$,

$$|f(x) - K| < \varepsilon$$

F. THE SUM, DIFFERENCE, PRODUCT, QUOTENT, COMPOSITION LIMIT THEOREMS FOR $\lim_{x \to \pm\infty} f(x)$ ARE SIMILAR TO $\lim_{x \to a} f(x)$

G. $\boxed{\text{\underline{THEOREM}: FOR } r > 0, \quad \lim_{x \to \pm\infty} \frac{1}{x^r} = 0}$

H. <u>EXAMPLES</u>: $\lim_{x \to \infty} \frac{1}{x} = 0$

$$\lim_{x \to \infty} \frac{1}{\sqrt{x}} = \lim_{x \to \infty} \frac{1}{x^{\frac{1}{2}}} = 0$$

$$\lim_{x \to -\infty} \frac{3 + 5x}{x^6} = \lim_{x \to -\infty} \frac{3}{x^6} + \frac{5x}{x^6} =$$

$$\lim_{x \to -\infty} \frac{3}{x^6} + \lim_{x \to -\infty} \frac{5x}{x^6} = \lim_{x \to -\infty} \frac{3}{x^6} + \lim_{x \to -\infty} \frac{5}{x^5}$$

$$= 3 \lim_{x \to -\infty} \frac{1}{x^6} + 5 \lim_{x \to -\infty} \frac{1}{x^5} \overset{G}{=} 3(0) + 5(0) = 0$$

H. (CONTINUED) IN THE FUTURE WE CAN JUST WRITE

$$\lim_{x \to -\infty} \frac{3+5x}{x^6} = \lim_{x \to -\infty} \frac{3}{x^6} + \frac{5x}{x^6}$$

$$= \lim_{x \to -\infty} \frac{3}{x^6} + \frac{5}{x^5} = 3(0) + 5(0) = 0$$

I EVALUATE $\displaystyle\lim_{x \to \infty} \frac{2x^3+7}{3x-5x^3+2}$

$$= \lim_{x \to \infty} \frac{\frac{1}{x^3}(2x^3+7)}{\frac{1}{x^3}(3x-5x^3+2)}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{7}{x^3}}{\frac{3}{x^2} - 5 + \frac{2}{x^3}} = \frac{2+0}{0-5+0} = -\frac{2}{5}$$

NOTE : MULTIPLY TOP AND BOTTOM BY

$$\frac{1}{x^{(\text{HIGHEST POWER OF THE DENOMINATOR})}}$$

2. EVALUATE $\displaystyle\lim_{x\to-\infty} \frac{3-4x}{\sqrt{5+6x^2}}$

NEG

$$= \lim_{x\to-\infty} \frac{\frac{1}{|x|}(3-4x)}{\frac{1}{|x|}\sqrt{5+6x^2}}$$

$$= \lim_{x\to-\infty} \frac{\frac{1}{-x}(3-4x)}{\frac{1}{\sqrt{x^2}}\sqrt{5+6x^2}}$$

$$= \lim_{x\to-\infty} \frac{-\frac{3}{x}+4}{\sqrt{\frac{5+6x^2}{x^2}}} = \lim_{x\to-\infty} \frac{-\frac{3}{x}+4}{\sqrt{\frac{5}{x^2}+6}}$$

$$= \frac{-3(0)+4}{\sqrt{5(0)+6}} = \frac{4}{\sqrt{6}}$$

$\boxed{\text{HOMEWORK}}$

PAGES 140-141

3, 4, 13, 14, 17, 20, 24, 26, 32

CURVE SKETCHING WITH ASYMPTOTE INFORMATION

A. FIND ALL HORIZONTAL AND VERTICAL ASYMPTOTES, FOR EACH VERTICAL ASYMPTOTE $x=a$ FIND $\lim_{x \to a^{\pm}}$, FIND x AND y INTERCEPTS, PLOT 6 KEY POINTS, ANALYZE, AND SKETCH

$$f(x) = \frac{x^2-25}{x^2-x-6}$$

1. FIND ALL HORIZONTAL ASYMPTOTES

$$\lim_{x \to +\infty} \frac{x^2-25}{x^2-x-6} = \lim_{x \to +\infty} \frac{\frac{1}{x^2}(x^2-25)}{\frac{1}{x^2}(x^2-x-6)}$$

$$= \lim_{x \to \infty} \frac{1-\frac{25}{x^2}}{1-\frac{1}{x}-\frac{6}{x^2}} = \frac{1-0}{1-0-0} = 1$$

SO $y=1$ IS A HORIZONTAL ASYMPTOTE.

$$\lim_{x \to -\infty} \frac{x^2-25}{x^2-x-6} = \lim_{x \to -\infty} \frac{\frac{1}{x^2}(x^2-25)}{\frac{1}{x^2}(x^2-x-6)}$$

$$= \lim_{x\to-\infty} \frac{1-\frac{25}{x^2}}{1-\frac{1}{x}-\frac{6}{x^2}} = \frac{1-0}{1-0-0} = 1$$

AS ALREADY MENTIONED, $y=1$ IS A HORIZONTAL ASYMPTOTE.

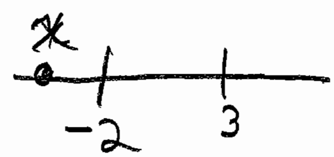
2. FIND ALL VERTICAL ASYMPTOTES

$$f(x) = \frac{x^2-25}{x^2-x-6} = \frac{x^2-25}{(x-3)(x+2)}$$

APP. -21 → NEG. AWAY 0

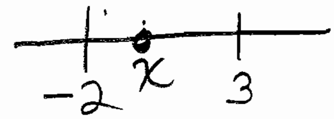
$$\lim_{x\to-2^-} \frac{x^2-25}{\underbrace{\underset{\text{NEG}}{\underset{\text{L-R}}{(x-3)}}\cdot\underset{\text{NEG}}{\underset{\text{L-R}}{(x-[-2])}}}_{\text{POS}}} = -\infty$$

POS $\xrightarrow{\text{APP.}}$ 0



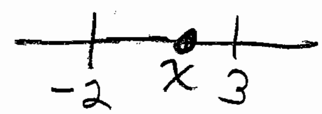
$x=-2$ IS A VERTICAL ASYMPTOTE

$$\lim_{x\to -2^+} \frac{\overbrace{x^2-25}^{\text{APP.}\to -21\ \text{NEG AWAY 0}}}{\underbrace{\underbrace{(x-3)}_{\substack{\text{L-R}\\ \text{NEG}}}\cdot\underbrace{(x-[-2])}_{\substack{\text{R-L}\\ \text{POS}}}}_{\text{NEG}\ \xrightarrow{\text{APP.}}\ 0}} = +\infty$$



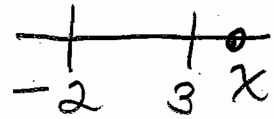
ALREADY MENTIONED $x = -2$ IS A VERTICAL ASYMPTOTE.

$$\lim_{x\to 3^-} \frac{\overbrace{x^2-25}^{\text{APP.}\to -16\ \text{NEG AWAY 0}}}{\underbrace{\underbrace{(x-3)}_{\substack{\text{L-R}\\ \text{NEG}}}\cdot\underbrace{(x-[-2])}_{\substack{\text{R-L}\\ \text{POS}}}}_{\text{NEG}\ \xrightarrow{\text{APP.}}\ 0}} = +\infty$$



$x=3$ IS A VERTICAL ASYMPTOTE

$$\lim_{x\to 3^+} \frac{\overbrace{x^2-25}^{\text{APP.}\to -16\ \text{NEG AWAY 0}}}{\underbrace{\underbrace{(x-3)}_{\substack{\text{R-L}\\ \text{POS}}}\cdot\underbrace{(x-[-2])}_{\substack{\text{R-L}\\ \text{POS}}}}_{\text{POS}\ \xrightarrow{\text{APP.}}\ 0}} = -\infty$$



ALREADY MENTIONED THAT $x=3$ IS A VERTICAL ASYMPTOTE

3. FIND ALL x- INTERCEPTS

$$y = \frac{x^2-25}{x^2-x-6} = 0 \qquad \text{SET } y=0$$

$$x^2-25=0$$

$$(x-5)(x+5)=0$$

$$x=5, \quad x=-5$$

$(5,0), (-5,0)$ x INTERCEPTS

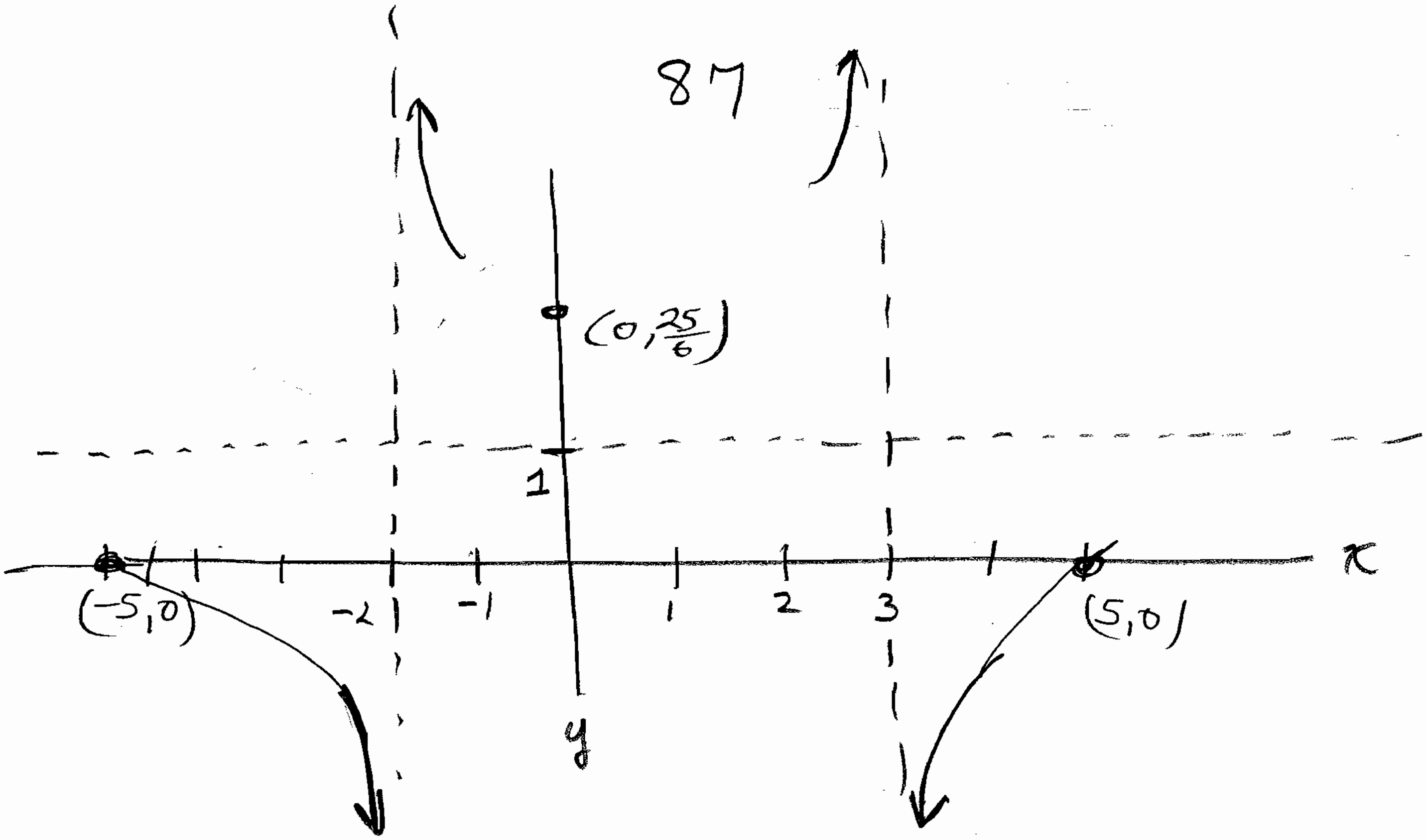
4. FIND ALL y - INTERCEPTS

SET $x=0$

$$y=\frac{x^2-25}{x^2-x-6} = \frac{0^2-25}{0^2-0-6} = \frac{25}{6}$$

$\left(0, \frac{25}{6}\right)$ y - INTERCEPT

LET'S NOW SEE WHAT INFO WE CAN APPLY TO THE GRAPHING NOW.



5. ANALYZE: DOES THE GRAPH CROSS THE LINE $y=1$?

$$y = \frac{x^2-25}{x^2-x-6} = 1$$

$$x^2 - 25 = x^2 - x - 6$$

$$-25 = -x - 6$$

$x = 19$ YES AT $(19,1)$

6. PLOTTING INFO

$x=1$ $\quad y = \frac{1^2-25}{1^2-1-6} = \frac{-24}{-6} = 4$ $\quad (1,4)$

$x=-1$ $\quad y = \frac{(-1)^2-25}{(-1)^2-(-1)-6} = \frac{-24}{-4} = 6$ $\quad (-1,6)$

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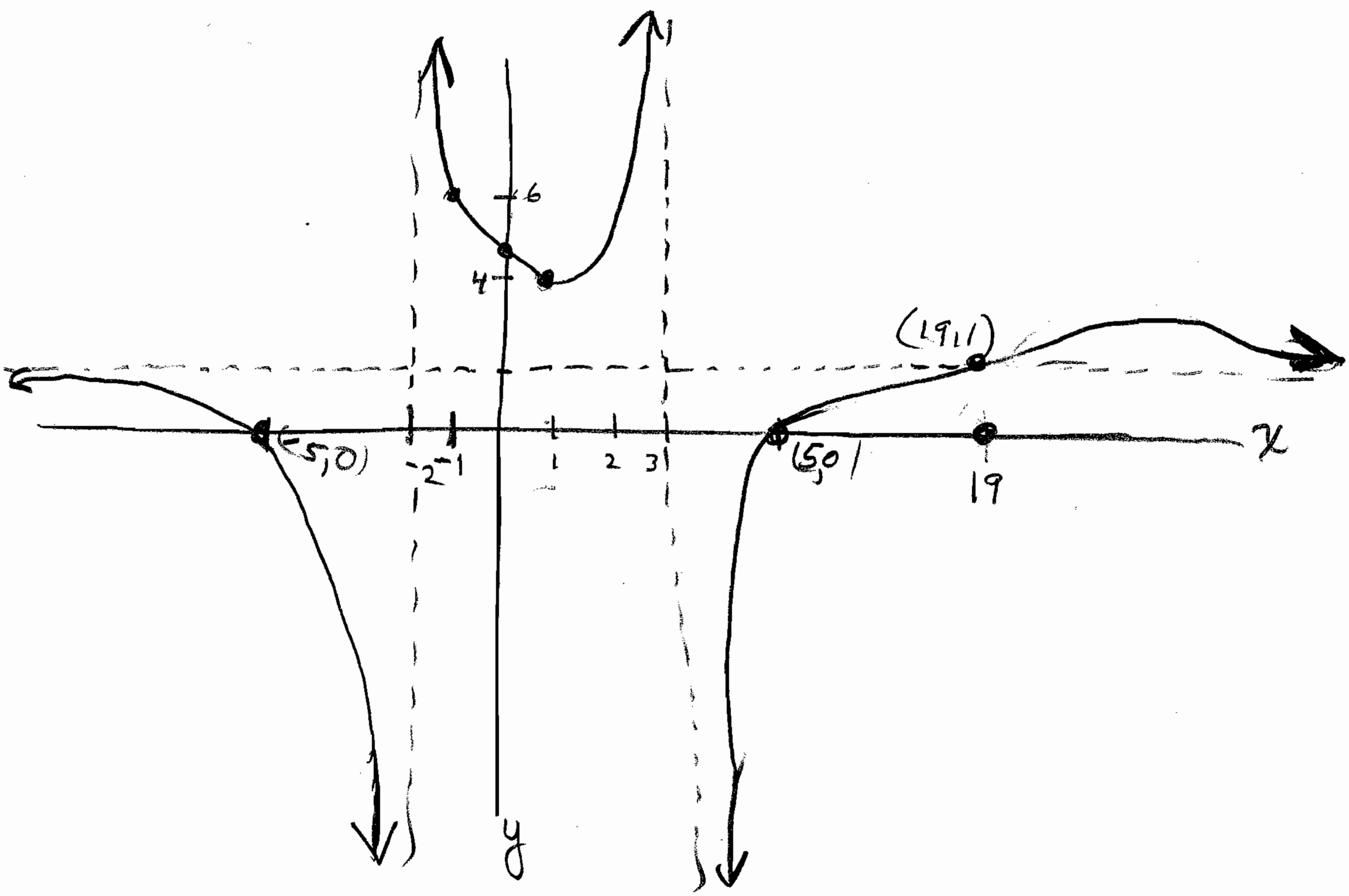
NOW PUT IT ALL TOGETHER



HOMEWORK

FOR EACH OF THESE PROBLEMS, FIND ALL HORIZONTAL AND VERTICAL ASYMPTOTES, FOR EACH VERTICAL ASYMPTOTE $x = a$, FIND $\lim_{x \to a^{\pm}}$, FIND x AND y INTERCEPTS, PLOT 6 KEY POINTS, ANALYZE, AND SKETCH

PAGE 142: 39, 41, 43

HINT FOR HOMEWORK

TOP APPROACHES $-\infty$
HENCE IS NEGATIVE

$$\lim_{x \to -\infty} \frac{x+3}{\underbrace{2 - \frac{1}{x}}_{\text{APPROACHING } 2 \text{ POSITIVE}}} = -\infty$$

DERIVATIVES, TANGENT LINES, VELOCITY

A. **DEFINITION:** THE DERIVATIVE OF FUNCTION f AT a, DENOTED $f'(a)$, IS DEFINED BY

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

B. EXAMPLE: FOR $f(x) = 2 - 3x + 4x^2$, FIND $f'(a)$, BY DEFINITION.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} =$$

$$\lim_{h \to 0} \frac{2 - 3(a+h) + 4(a+h)^2 - [2 - 3a + 4a^2]}{h}$$

$$= \lim_{h \to 0} \frac{2 - 3a - 3h + 4(a^2 + 2ah + h^2) - 2 + 3a - 4a^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2}-\cancel{3a}-3h+\cancel{4a^2}+8ah+4h^2-\cancel{2}+\cancel{3a}-\cancel{4a^2}}{h}$$

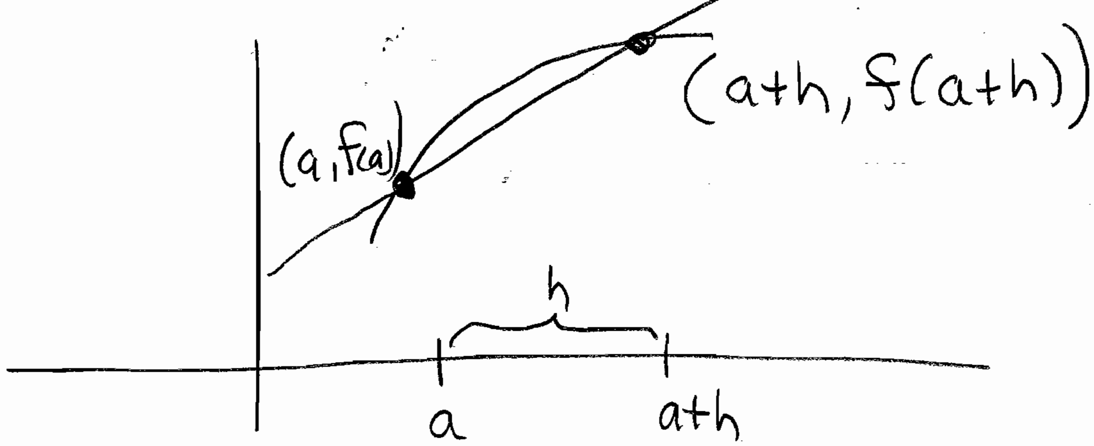
$$= \lim_{h\to 0} \frac{-3h+8ah+4h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-3+8a+4h)}{\cancel{h}} = \lim_{h\to 0} -3+8a+4h$$

$$= -3+8a = f'(a)$$

C. AN APPLICATION OF DERIVATIVES

SLOPES OF TANGENT LINES



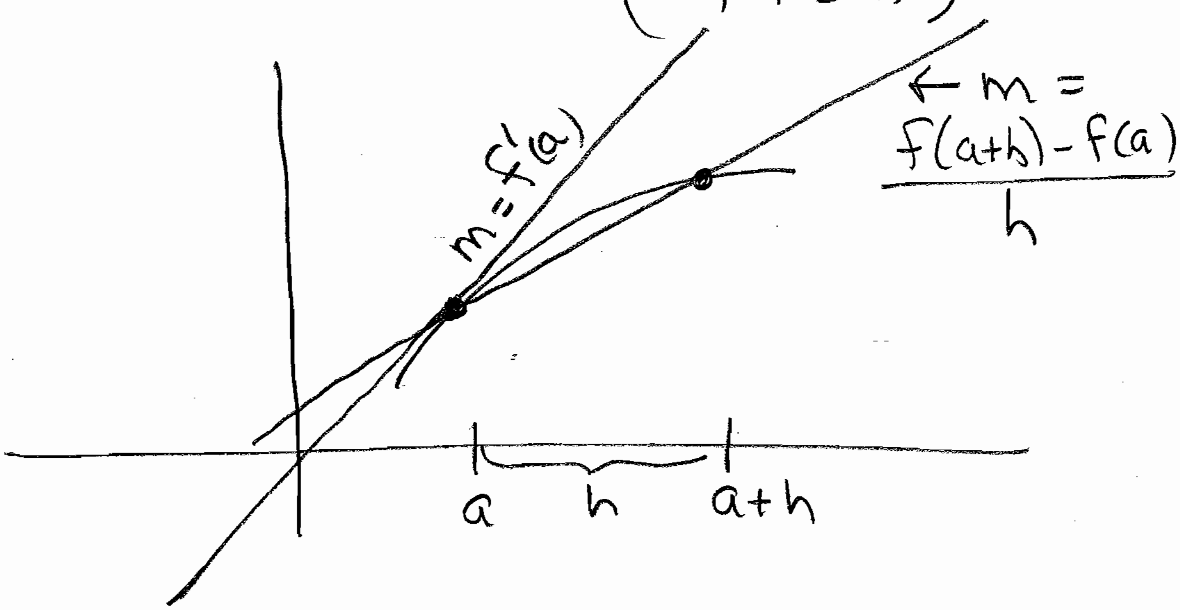
THE SLOPE OF THE LINE BETWEEN

$(a, f(a))$ AND $(a+h, f(a+h))$ IS

$$m = \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$m = \frac{f(a+h) - f(a)}{h}$$

NOW AS h APPROACHES 0 THIS SLOPE APPROACHES THE SLOPE OF THE TANGENT LINE AT THE POINT $(a, f(a))$



D. GRAPHICAL INTERPRETATION OF

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} :$$

THE SLOPE OF THE LINE TANGENT TO THE GRAPH OF f AT THE POINT $(a, f(a))$.

E. FIND AN EQUATION FOR THE LINE TANGENT TO THE GRAPH OF $f(x) = 2 - 3x + 4x^2$ AT THE POINT $(1,3)$.

A "FIND AN EQUATION FOR A LINE" TYPE OF PROBLEM.

NEED: A POINT AND A SLOPE.

<u>POINT</u>: $(1,3)$

<u>SLOPE</u>: $f'(1)$ THE SLOPE OF THE LINE TANGENT TO THE GRAPH OF f AT THE POINT $(1, f(1)) = (1,3)$

WE PREVIOUSLY FOUND $f'(a) = -3+8a$, so $f'(1) = -3+8(1) = 5$

EQUATION $y-3 = 5(x-1)$

E. FOR $f(x) = \sqrt{x+1}$, FIND $f'(x)$, BY DEFINITION.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$$

$$\lim_{h\to 0} \frac{\sqrt{(x+h)+1} - \sqrt{x+1}}{h} =$$

$$\lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} =$$

$$\lim_{h\to 0} \frac{(x+h+1)-(x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}} = \frac{1}{\sqrt{x+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

F. AN APPLICATION OF THE DERIVATIVE TO INSTANTANEOUS VELOCITY

SUPPOSE $f(t)$ IS THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME (t).

THE AVERAGE VELOCITY OF THE OBJECT OVER TIME INTERVAL $[t, t+h]$ IS $\dfrac{f(t+h)-f(t)}{(t+h)-t} =$

$$\frac{f(t+h)-f(t)}{h}$$

SUPPOSE $f(t)$ IS THE MILEMARKER AT TIME t. SUPPOSE AT 1 PM YOU ARE AT THE 100 MILE MARKER AND AT 3 PM YOU ARE AT THE 230 MILE MARKER. THE AVERAGE VELOCITY OVER TIME INTERVAL

$[1,3]=[1,1+2]$ (i.e. $h=2$) IS

$$\frac{f(t+h)-f(t)}{h}=\frac{f(1+2)-f(1)}{2}$$

$$=\frac{230-100}{2}=\frac{130}{2}=65 \text{ MPH}.$$

TO FIND THE INSTANTANEOUS VELOCITY AT TIME t IN GENERAL, TAKE THE $\lim_{h \to 0}$. SO,

INSTANTANEOUS VELOCITY AT TIME t IS $\lim_{h \to 0} \frac{f(t+h)-f(t)}{h} = f'(t)$.

IF $f(t)$ IS THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME t, THEN $f'(t)$ IS THE <u>INSTANTANEOUS VELOCITY</u> OF THE OBJECT AT TIME t.

6. THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME t IS $f(t)=\sqrt{t+1}$ (t IN SECONDS. $f(t)$ IN FEET)

FIND THE INSTANTANEOUS VELOCITY OF THE OBJECT AT TIME $t=5$.

WE HAVE PREVIOUSLY SEEN (PART E) FOR $f(x)=\sqrt{x+1}$ THAT $f'(x)=\dfrac{1}{2\sqrt{x+1}}$. SO FOR

$f(t)=\sqrt{t+1}$, $f'(t)=\dfrac{1}{2\sqrt{t+1}}$.

THE ANSWER IS $f'(5)=\dfrac{1}{2\sqrt{5+1}}$

$f'(5)=\dfrac{1}{2\sqrt{6}}\ \dfrac{\text{ft}}{\text{sec}}$.

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HOMEWORK

PAGES 150, 151 : FIND ANY DERIVATIVE BY DEFINITION

3 a-ii, b

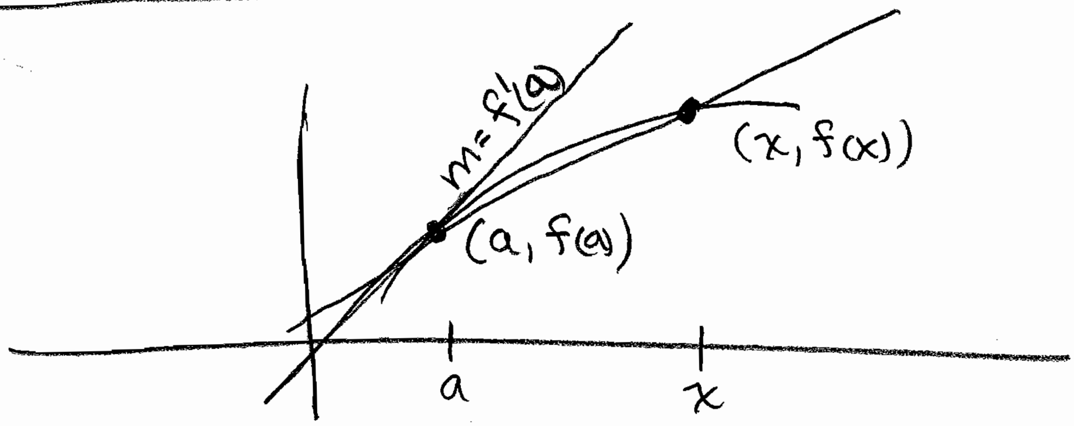
13

25

29

EQUIVALENT FORMULATION OF DERIVATIVE

A.



1. SLOPE OF THE LINE BETWEEN THE POINTS $(a, f(a))$ AND $(x, f(x))$

$$\text{slope} = \frac{f(x) - f(a)}{x - a}$$

2. SLOPE OF THE LINE TANGENT TO THE GRAPH $y = f(x)$ AT $(a, f(a))$

$$m = f'(a) \overset{\text{E.F.}^*}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

ALSO, RECALL

$$m = f'(a) \overset{\text{DEF.}}{=} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

E.F. ≡ EQUIVALENT FORMULATION

B. <u>EXAMPLE</u>: LET $f(x) = 2 - 3x + 4x^2$.
FIND $f'(a)$ BY THE <u>EQUIVALENT FORMULATION</u> TO THE DEFINITION OF $f'(a)$.

$$f'(a) \overset{E.F.}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{(2 - 3x + 4x^2) - (2 - 3a + 4a^2)}{x - a}$$

$$= \lim_{x \to a} \frac{\cancel{2} - 3x + 4x^2 - \cancel{2} + 3a - 4a^2}{x - a}$$

$$= \lim_{x \to a} \frac{-3x + 3a + 4x^2 - 4a^2}{x - a}$$

$$= \lim_{x \to a} \frac{-3(x - a) + 4(x^2 - a^2)}{x - a}$$

$$= \lim_{x \to a} \frac{-3(x - a) + 4(x - a)(x + a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}[-3+4(x+a)]}{\cancel{(x-a)}}$$

$$= \lim_{x \to a} -3+4(x+a) = -3+4(a+a)$$

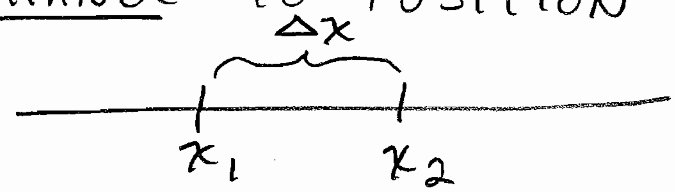
$$= -3+4(2a) = -3+8a = f'(a)$$

<u>NOTE:</u> THIS IS THE SAME ANSWER WE GOT FOR $f'(a)$ WHEN $f'(a)$ WAS FOUND BY THE <u>DEFINITION</u> IN THE PREVIOUS LESSON.

C. IT IS GOOD TO KNOW BOTH THE <u>DEFINITION</u> FOR $f'(a)$ AS WELL AS THE <u>EQUIVALENT FORMULATION</u> FOR $f'(a)$. SOMETIMES IT IS CLEARER TO USE ONE RATHER THAN THE OTHER.

D. Δx, Δy NOTATION (THIS IS NOTATION USED IN SOME BOOKS TO DESCRIBE DERIVATIVES.)

1. START AT POSITION x_1 AND CHANGE TO POSITION x_2



THE AMOUNT OF CHANGE IS Δx.

$$\boxed{\Delta x = x_2 - x_1 \qquad \text{DEFINITION}}$$

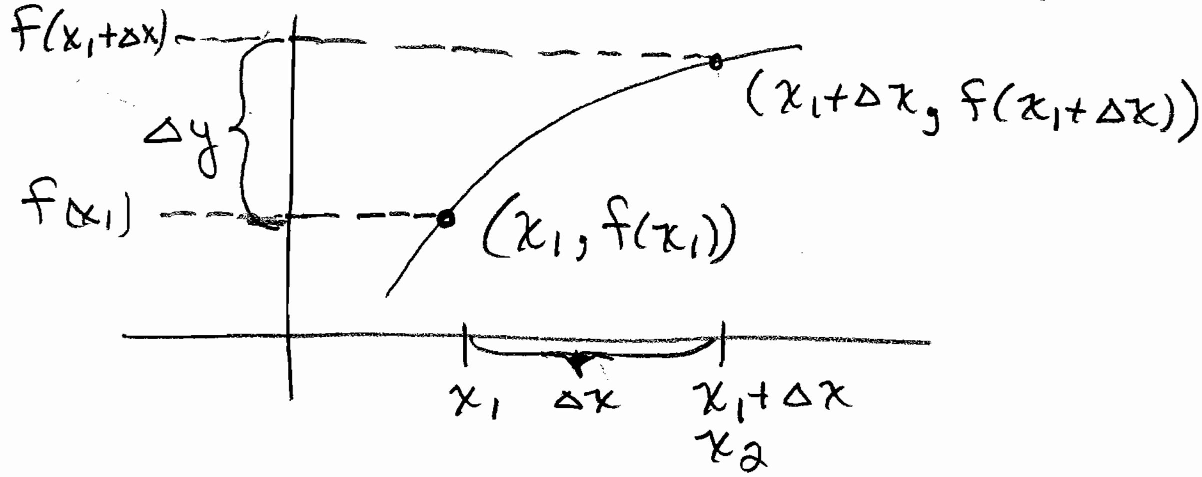
2. NOTE $x_2 = x_1 + \Delta x$

3. SO WHEN YOU SEE Δx THINK THERE IS A STARTING NUMBER x_1, CHANGE TO x_2, $\Delta x = x_2 - x_1$

4. LET $y = f(x)$

$$\begin{aligned} \Delta y &= f(x_1 + \Delta x) - f(x_1) \\ &= f(x_2) - f(x_1) \end{aligned}$$

5. PICTURES OF Δx AND Δy



E. NOTATION SOME USE FOR DERIVATIVES

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

SUBSTITUTE x FOR a

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \begin{array}{l} \leftarrow \text{CHANGE IN } y \\ \leftarrow \text{CHANGE IN } x \end{array}$$

F. DERIVATIVE AS AN INSTANTANEOUS RATE OF CHANGE.

1. $y = f(x)$

$f'(x)$ IS THE INSTANTANEOUS RATE OF CHANGE OF y WITH RESPECT TO x.

2. $y = f(t)$

t : TIME (SECONDS)

y : POSITION AT TIME t (FEET)

$f'(t)$ IS THE INSTANTANEOUS RATE OF CHANGE OF POSITION WITH RESPECT TO TIME (I.E. VELOCITY)

UNITS: FEET PER SECOND (I.E. ft/sec)

3. $V = f(r)$

r : radius (FEET)

V : Volume (CUBIC FEET)

$f'(r)$ IS THE INSTANTANEOUS RATE OF CHANGE OF VOLUME WITH RESPECT TO RADIUS

UNITS: CUBIC FEET PER FOOT (I.E. $\frac{ft^3}{ft}$)

G. RECOGNIZING DERIVATIVES BY LOOKING AT LIMITS. EACH IS $f'(a)$. NAME $f(x)$ AND a.

1. $\lim\limits_{h \to 0} \dfrac{(2+h)^2 - 4}{h} = \lim\limits_{h \to 0} \dfrac{(2+h)^2 - 2^2}{h}$

$f(x) = x^2 \qquad a = 2$

<u>NOTE</u>: $f'(2) = \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(2+h)^2 - 2^2}{h} = \lim\limits_{h \to 0} \dfrac{(2+h)^2 - 4}{h}$

2. $\lim\limits_{x \to 3} \dfrac{x^3 - x - 3^3 + 3}{x - 3} = \lim\limits_{x \to 3} \dfrac{(x^3 - x) - (3^3 - 3)}{x - 3}$

$f(x) = x^3 - x \qquad a = 3$

<u>NOTE</u>: $f'(3) = \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$

$= \lim\limits_{x \to 3} \dfrac{x^3 - x - (3^3 - 3)}{x - 3} = \lim\limits_{x \to 3} \dfrac{x^3 - x - 3^3 + 3}{x - 3}$

HOMEWORK

A. PAGE 151: 31, 32, 33, 34, 35, 36
45 ab

B. FIND $f'(a)$ FOR PROBLEM 25, PAGE 151 USING THE EQUIVALENT FORMULATION.

OTHER DERIVATIVE NOTATION AND DIFFERENTIABILITY IMPLIES CONTINUITY

A. LEIBNIZ NOTATION FOR $f'(x)$

$$f'(x) = \frac{dy}{dx} \quad \text{WHERE } y = f(x)$$

$\frac{dy}{dx}$ IS READ "THE DERIVATIVE OF y WITH RESPECT TO x."

B. $\frac{dy}{dx}$ IS ONE SYMBOL NOT A FRACTION.

INTUITIVELY THE NOTATION COMES FROM $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

C. RECALL: FROM PREVIOUS WORK
IF $f(x) = 2 - 3x + 4x^2$, $f'(x) = -3 + 8x$.

SO IF $y = 2 - 3x + 4x^2$, $\frac{dy}{dx} = -3 + 8x$

D. EVALUATING A DERIVATIVE AT A SPECIFIC VALUE USING LEIBNIZ NOTATION

<u>PRIME</u> <u>NOTATION</u>: $f(x) = 2-3x+4x^2$

$f'(x) = -3+8x$. $f'(5) = -3+8(5) = 37$

<u>LEIBNIZ</u> <u>NOTATION</u>: $y = 2-3x+4x^2$

$\frac{dy}{dx} = -3+8x$. $\left.\frac{dy}{dx}\right|_{x=5} = -3+8(5) = 37$

E. $\boxed{\text{DIFFERENTIABILITY} \rightarrow \text{CONTINUITY}}$

<u>THEOREM</u>: IF f IS DIFFERENTIABLE AT a THEN f IS CONTINUOUS AT a.

<u>REASONS</u>: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ SO $f(a)$ EXISTS

1. $a \in \operatorname{dom}(f)$

SINCE BOTH $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

AND $\lim_{x \to a} x-a$ EXIST, SO

$$0 = f'(a) \cdot 0 = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} x - a$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot x - a$$

$$= \lim_{x \to a} f(x) - f(a).$$

HENCE, $\lim_{x \to a} f(x) - f(a) = 0$.

SO $\lim_{x \to a} f(x) = \lim_{x \to a} [f(x) - f(a)] + f(a)$

$$= \lim_{x \to a} f(x) - f(a) + \lim_{x \to a} f(a) = 0 + f(a) = f(a)$$

THEREFORE

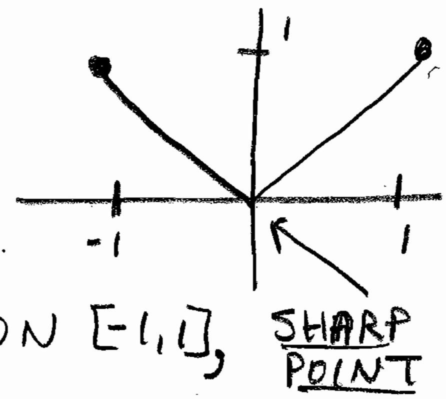
2) $\lim_{x \to a} f(x)$ EXISTS, AND

3) $\lim_{x \to a} f(x) = f(a)$

THUS, f IS CONTINUOUS AT a.

F. $\boxed{\text{IT IS NOT NECESSARILY TRUE THAT CONTINUOUS} \rightarrow \text{DIFFERENTIABILITY}}$

FOR $f(x)=|x|$ ON $[-1,1]$ ITS GRAPH IS →



f IS CONTINUOUS ON $[-1,1]$, BUT f IS NOT DIFFERENTIABLE AT 0 SINCE THE GRAPH HAS A SHARP POINT THERE, INDICATING THERE IS <u>NOT</u> ONE AND ONLY ONE TANGENT TO THE GRAPH AT 0 WITH $f'(0)$ BEING ITS SLOPE

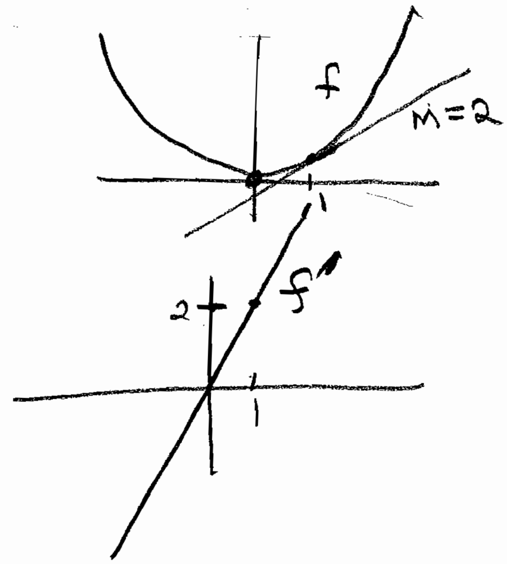
G. $\boxed{\text{THE GRAPH OF THE FUNCTION } f \text{ AND THE GRAPH OF ITS DERIVATIVE } f'}$

WHEN f HAS HORIZONTAL TANGENT ON ITS GRAPH, f' WILL HAVE VALUE 0.

LET $f(x) = x^2$

WE COULD DERIVE

$f'(x) = 2x$

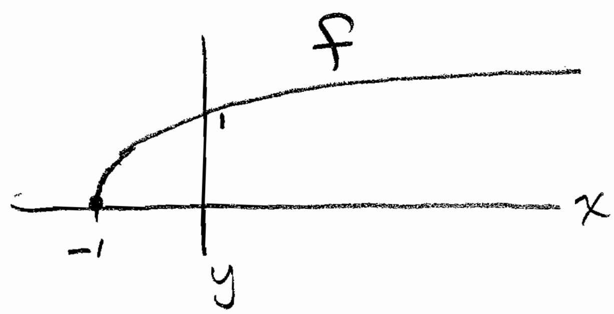


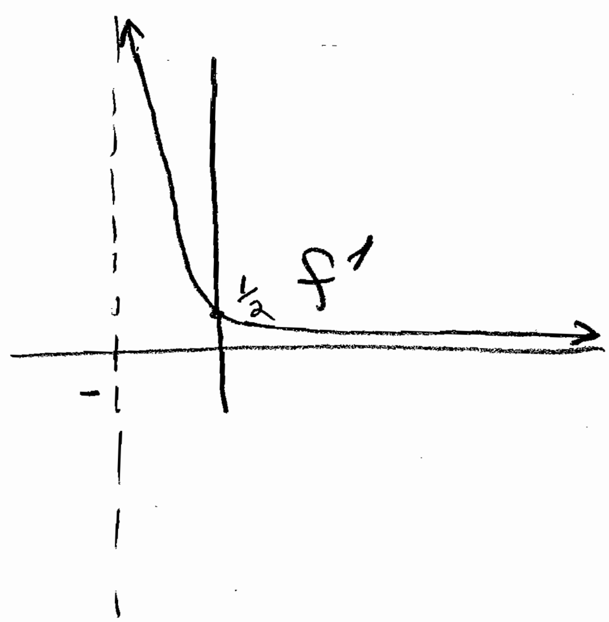
IF THE GRAPH OF f HAS A VERTICAL TANGENT, THEN f' DOES NOT EXIST AT THAT POINT.

LET $f(x) = \sqrt{x+1}$

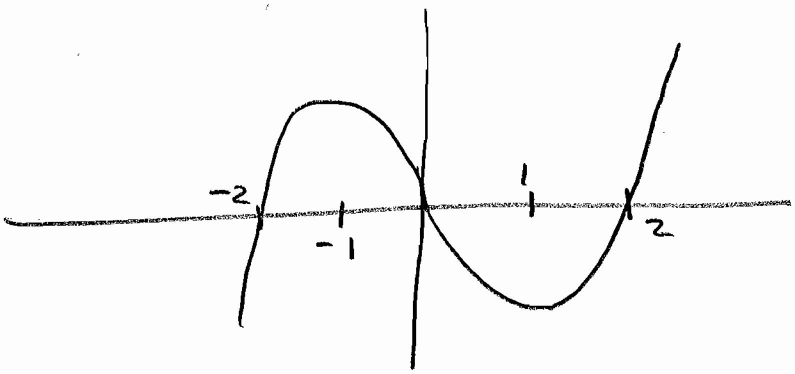
WE HAVE SEEN

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$



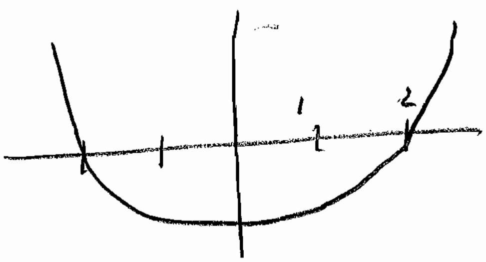


LOOK AT THE GRAPH OF f AND PICK OUT THE GRAPH OF f'

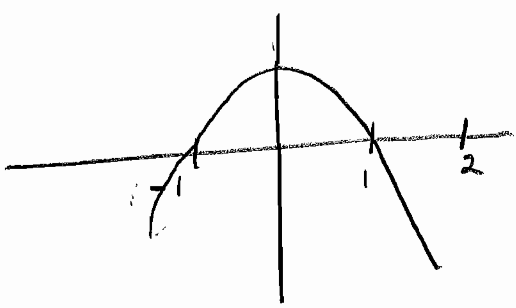


← GRAPH OF f

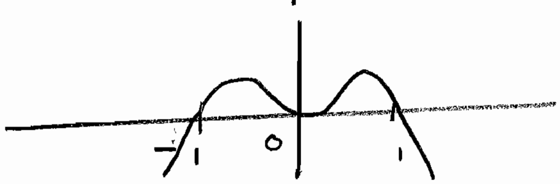
PICK BELOW THE GRAPH OF f'



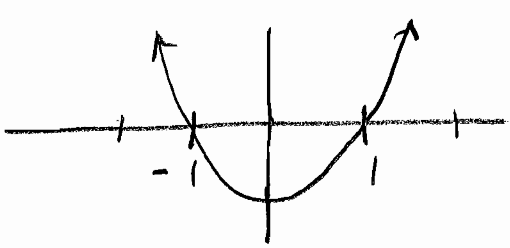
NO $f'(1)$ AND $f'(-1)$ MUST BE 0



NO $f'(2)$ MUST BE POSITIVE



NO $f'(0)$ MUST BE NEGATIVE



← GRAPH OF f'

$f'(1) = f'(-1) = 0$

$f(0)$ IS NEGATIVE

$f(2)$ IS POSITIVE

H. $\boxed{\text{MORE NOTATIONS FOR DERIVATIVES}}$

$\frac{d}{dx}$ IS AN OPERATOR THAT TAKES THE DERIVATIVE WITH RESPECT TO x. LET $y = f(x)$

SIMILARY D IS A DERIVATIVE OPERATOR

$$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

I. $\boxed{\text{HIGHER DERIVATIVES}}$

1ST DERIVATIVE $\equiv$ THE DERIVATIVE

2ND DERIVATIVE $\equiv$ THE DERIVATIVE OF THE FIRST DERIVATIVE.

3RD DERIVATIVE $\equiv$ THE DERIVATIVE OF THE SECOND DERIVATIVE.

$\vdots$

J. $\boxed{\text{LEIBNIZ NOTATION FOR HIGHER DERIVATIVES}}$

2nd DERIVATIVE $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ *

3rd DERIVATIVE $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$

$\vdots$

nth DERIVATIVE $\frac{d^ny}{dx^n}$

*READ "THE 2ND DERIVATIVE OF y WITH RESPECT TO x."

K. $\boxed{\text{PRIME AND D NOTATION FOR HIGHER DERIVATIVES}}$

2^{nd} DERIVATIVE $f''(x) = D^2 f(x) = D_x^2 f(x)$

3^{rd} DERIVATIVE $f'''(x) = f^{(3)}(x) = D^3 f(x) = D_x^3 f(x)$

⋮

n^{th} DERIVATIVE $f^{(n)}(x) = D^n f(x) = D_x^n f(x)$.

$$\boxed{\begin{array}{l} \text{NOTE: } f^5(x) \neq f^{(5)}(x) \\ f^5(x) = [f(x)]^5. \quad f^{(5)}(x) = D^5 f(x) = 5^{TH} \text{ DERIVATIVE} \end{array}}$$

L. EXAMPLE: LET $y = f(x) = 2 - 3x + 4x^2$

WE HAVE SEEN $y' = \frac{dy}{dx} = Df(x) = -3 + 8x$

SINCE $y = 8x - 3 = mx + b$, SLOPE

THE SLOPE (= THE DERIVATIVE) IS 8, SO

$y'' = \frac{d^2y}{dx^2} = D^2 y = D_x^2(2 - 3x + 4x^2) = 8 = f''(x)$

SINCE $f''(x) = 8 = 0x + 8 = mx + b$, SLOPE

THE SLOPE (= THE DERIVATIVE) IS 0. SO

$y''' = f'''(x) = f^{(3)}(x) = \frac{d^3y}{dx^3} = D^3 y = D_x^3 y = 0$

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HOMEWORK

A. 3 PAGE 162

B. BE ABLE TO NAME A FUNCTION CONTINUOUS ON $(-1,1)$ BUT NOT DIFFERENTIABLE ON $(-1,1)$

C. Let $f(x) = |x|$

1. Find $\lim_{h \to 0^-} \dfrac{f(0+h) - f(0)}{h}$
2. Find $\lim_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h}$
3. What do steps 1. and 2 tell you about $\lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$?
4. What does step 3 tell you about $f'(0)$?

D. 45 PAGE 164

DERIVATIVE FORMULAS

A. $\boxed{\text{THE DERIVATIVE OF A CONSTANT IS } 0}$

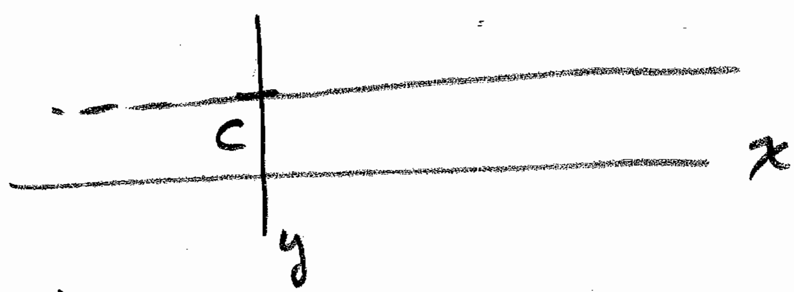
$$\frac{d}{dx}(c) = 0$$

IF $f(x) = c$, THEN $f'(x) = 0$

PROOF: ASSUME $f(x) = c$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = 0$$

RECALL $f'(x)$ IS THE SLOPE OF THE LINE TANGENT TO THE GRAPH OF f AT $(x, f(x))$. FOR $f(x) = c$



ANY POINT ON THE GRAPH OF $y = f(x) = c$ HAS A HORIZONTAL TANGENT, HENCE THE SLOPE IS 0.

$\frac{d}{dx}(5)=0$. FOR $y=3$, $\frac{dy}{dx}=0$

FOR $f(x)=7$, $f'(x)=0$

B. $\boxed{\text{FOR } f(x)=x,\ f'(x)=1}$

$$\frac{d}{dx}(x)=1$$

<u>PROOF</u>: ASSUME $f(x)=x$. (SHOW $f'(x)=1$)

$$f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}=\lim_{t\to x}\frac{t-x}{t-x}=\lim_{t\to x}1=1$$

C. $\boxed{\text{FOR } f(x)=x^n,\ f'(x)=nx^{n-1}}$

<u>EXAMPLES</u>: $\frac{d}{dx}(x^5)=5x^4$

FOR $f(x)=\sqrt{x}=x^{\frac{1}{2}}$, $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$

LET $y=\frac{1}{\sqrt{x}}=\frac{1}{x^{1/2}}=x^{-1/2}$

$y'=-\frac{1}{2}x^{-3/2}$

PROOF FOR $n=3$ THAT $\frac{d}{dx}(x^n) = nx^{n-1}$

LET $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{- x^3}}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2 + 0 + 0 = 3x^2$$

SO FOR $f(x) = x^3$, $f'(x) = 3x^2$

D. IF c IS A CONSTANT AND $f'(x)$ EXISTS, THE $\dfrac{d}{dx}[c\,f(x)] = c\dfrac{d}{dx}f(x)$

EXAMPLES: $\dfrac{d}{dx}(5x^3) = 5\dfrac{d}{dx}(x^3) = 5(3x^2)$
$= 15x^2$

LET $h(x) = \dfrac{4}{\sqrt{x}} = \dfrac{4}{x^{1/2}} = 4x^{-1/2}$

$h'(x) = 4\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = -2x^{-3/2} = \dfrac{-2}{x^{3/2}}$

PROOF: LET $g(x) = c\,f(x)$ AND ASSUME $f'(x)$ EXISTS. (SHOW $g'(x) = c f'(x)$)

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} =$$

$$\lim_{h\to 0}\frac{c\,f(x+h) - c\,f(x)}{h} = \lim_{h\to 0} c\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = c\,f'(x) = g'(x)$$

E $\boxed{\begin{array}{l}\text{IF BOTH } f'(x) \text{ AND } g'(x) \text{ EXIST,} \\ \text{THEN } \dfrac{d}{dx}\left[f(x) \pm g(x)\right] = \dfrac{d}{dx}(f(x)) \pm \dfrac{d}{dx}(g(x))\end{array}}$

i.e. $\dfrac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$

<u>EXAMPLES</u>: $\dfrac{d}{dx}(x^2+x^3) = \dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(x^3)$

$= 2x^1 + 3x^2 = 2x + 3x^2$

WE CAN LEAVE OUT THE MIDDLE STEPS AND DO THIS:

FOR $g(x) = x^2 + x^3$, $g'(x) = 2x + 3x^2$

LET $y = 5x^4 - 2\sqrt{x} - 3$. FIND y'.

$y = 5x^4 - 2x^{1/2} - 3$

$y' = 20x^3 - x^{-1/2} - 0$

$y' = 20x^3 - \dfrac{1}{x^{1/2}} = 20x^3 - \dfrac{1}{\sqrt{x}}$

FOR $f(x) = \dfrac{(2x)^3 - x^4 + 1}{\sqrt[3]{x}}$, FIND $f'(x)$

$$f(x) = \frac{8x^3 - x^4 + 1}{x^{1/3}} = \frac{8x^3}{x^{1/3}} - \frac{x^4}{x^{1/3}} + \frac{1}{x^{1/3}}$$

$$f(x) = 8x^{\frac{8}{3}} - x^{\frac{11}{3}} + x^{-1/3}$$

$$f'(x) = 8\left(\tfrac{8}{3}\right)x^{5/3} - \frac{11}{3}x^{8/3} - \frac{1}{3}x^{-4/3}$$

$$f'(x) = \frac{64}{3}x^{\frac{5}{3}} - \frac{11}{3}x^{8/3} - \frac{1}{3}x^{-4/3}$$

F. $\dfrac{d}{dx}(e^x) = e^x$ GIVEN WITHOUT PROOF

$$\frac{d}{dx}(5e^x) = 5\frac{d}{dx}(e^x) = 5e^x$$

CONSTANT

LET $y = e^{x+2} = e^x e^2 = e^2 e^x$

$$y' = e^2 \frac{d}{dx}e^x = e^2 e^x = e^{2+x} = e^{x+2}$$

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HOMEWORK

PAGES 180-181:

ODDS 3 THROUGH 31,
32, 34, 45, 46, 52

PRODUCT AND QUOTIENT DERIVATIVE FORMULAS

A. PRODUCT FORMULA FOR DERIVATIVES

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

B. PROOF INDICATION IN THE NOTATION

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a)$$

$$(fg)'(a) = \lim_{x \to a} \frac{(fg)(x) - (fg)(a)}{x - a} =$$

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} =$$

$$\lim_{x \to a} \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(x)\left[g(x) - g(a)\right] + g(a)\left[f(x) - f(a)\right]}{x - a}$$

$$= \lim_{x \to a} f(x)\left[\frac{g(x)-g(a)}{x-a}\right] + \frac{g(a)[f(x)-f(a)]}{x-a}$$

$$= \lim_{x \to a} f(x) \lim_{x \to a} \frac{g(x)-g(a)}{x-a} +$$

$$\lim_{x \to a} g(a) \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$= f(a)\,g'(a) + g(a)\,f'(a) = (fg)'(a)$$

f is diff. at a so f is cont. at a.

C. EXAMPLE OF THE DERIVATIVE OF A PRODUCT

LET $h(x) = (\sqrt{x})(5x^4 - 2x^{-3})$

$h(x) = x^{\frac{1}{2}}(5x^4 - 2x^{-3})$

$h'(x) = x^{\frac{1}{2}}(20x^3 + 6x^{-4}) + (5x^4 - 2x^{-3})\frac{1}{2}x^{-\frac{1}{2}}$

D. FIND $\frac{d}{dx} e^{2x}$. $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x e^x) =$

$e^x \frac{d}{dx} e^x + e^x \frac{d}{dx} e^x = e^x e^x + e^x e^x = 2e^{2x}$

E. QUOTIENT FORMULA FOR DERIVATIVES

$$\frac{d}{dx}\left(\frac{T(x)}{B(x)}\right) = \left(\frac{T}{B}\right)'(x) = \frac{B(x)\,T'(x) - T(x)\,B'(x)}{[B(x)]^2}$$

PROOF INDICATION NOT GIVEN.

F. LET $f(x) = \dfrac{2-3x}{5x+4}$

$$f'(x) = \frac{(5x+4)(-3) - (2-3x)(5)}{(5x+4)^2}$$

$$f'(x) = \frac{-15x - 12 - 10 + 15x}{(5x+4)^2} = \frac{-22}{(5x+4)^2}$$

G. LET $f(x) = \dfrac{\sqrt{x}(3x^5-2)}{5x-7} = \dfrac{x^{\frac{1}{2}}(3x^5-2)}{5x-7}$

$$f'(x) = \frac{\left[(5x-7)\left[x^{\frac{1}{2}}(15x^4) + (3x^5-2)\frac{1}{2}x^{-\frac{1}{2}}\right] - x^{\frac{1}{2}}(3x^5-2)(5)\right]}{(5x-7)^2}$$

$$= \frac{\left[(5x-7)\left(x^{\frac{1}{2}}(15x^4) + \frac{(3x^5-2)}{2x^{1/2}}\right) - 5x^{\frac{1}{2}}(3x^5-2)\right]}{(5x-7)^2}$$

$$= \frac{\left[(5x-7)\left(\frac{2x(15x^4)+(3x^5-2)}{2x^{1/2}}\right) - 5x^{\frac{1}{2}}(3x^5-2)\right]}{(5x-7)^2}$$

$$= \frac{\dfrac{(5x-7)(33x^5-2) - 10x(3x^5-2))}{2x^{1/2}}}{\dfrac{(5x-7)^2}{1}}$$

$$= \frac{165x^6 - 10x - 231x^5 + 14 - 30x^6 + 20x}{2x^{1/2}(5x-7)^2}$$

$$= \frac{135x^6 - 231x^5 + 10x + 14}{2\sqrt{x}\,(5x-7)^2} = f'(x)$$

H. FIND AN EQUATION FOR THE TANGENT LINE TO THE GRAPH OF $f(x) = \dfrac{8x-3}{e^{2x}(2x-7)}$ AT THE POINT $\left(0, \frac{3}{7}\right)$

$$\boxed{\text{RECALL } \frac{d}{dx}(e^{2x}) = 2e^{2x}}$$

NEED A POINT AND A SLOPE $(f'(0))$

$$f'(x) = \frac{e^{2x}(2x-7)8 - (8x-3)\left[e^{2x}(2) + (2x-7)2e^{2x}\right]}{\left[e^{2x}(2x-7)\right]^2}$$

$$f'(0) = \frac{(-7)8 - (-3)\left[2 + (-7)2\right]}{(-7)^2}$$

$$= \frac{-56 + 3[-12]}{49} = \frac{-56-36}{49} = -\frac{92}{49} = m$$

EQUATION OF TANGENT LINE

$$y - y_1 = m(x - x_1)$$

$$y - \frac{3}{7} = \frac{-92}{49}(x - 0)$$

I THE DERIVATIVE OF THE PRODUCT OF 3 THINGS. FIND $h'(x)$ WHERE

$$h(x) = (2-3x)(5x+2)(7-8x)$$

GROUP TO WHERE IT IS A PRODUCT OF TWO THINGS

$$h(x) = [(2-3x)(5x+2)](7-8x)$$

$$h'(x) = [(2-3x)(5x+2)](-8) + (7-8x)[(2-3x)(5) + (5x+2)(-3)]$$

J. 2 WAYS TO FIND $\frac{d}{dx}\left(\frac{5}{x}\right)$

1) $$\frac{d}{dx}\left(\frac{5}{x}\right) = \frac{d}{dx}(5x^{-1}) = 5\frac{d}{dx}(x^{-1}) = 5(-x^{-2}) = \frac{-5}{x^2}$$

2) $$\frac{d}{dx}\left(\frac{5}{x}\right) \overset{\text{QUOT. FORMULA}}{=} \frac{x(0) - 5(1)}{x^2} = \frac{-5}{x^2}$$

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HOMEWORK

PAGES 187, 188, 189:

3, 4, 6, 7, 10, 11, 12, 13, 15, 17, 21,

27, 32, 55

PAGES 127-140 OMITTED

DERIVATIVES OF TRIG FUNCTIONS

A. GIVEN WITHOUT PROOF: $\lim_{t \to 0} \frac{\sin t}{t} = 1$

B. <u>PROVE</u>: $\lim_{t \to 0} \frac{(\cos t) - 1}{t} = 0$

$$\lim_{t \to 0} \frac{(\cos t) - 1}{t} = \lim_{t \to 0} \frac{[(\cos t) - 1]}{t} \cdot \frac{[(\cos t) + 1]}{[(\cos t) + 1]}$$

$$= \lim_{t \to 0} \frac{(\cos^2 t) - 1}{t\,[(\cos t) + 1]}$$

$$\boxed{\begin{array}{c}\text{RECALL} \\ \sin^2 t + \cos^2 t = 1\end{array}}$$

$$= \lim_{t \to 0} \frac{-\sin^2 t}{t\,[(\cos t) + 1]}$$

$$= \lim_{t \to 0} \frac{(\sin t)}{t} \frac{[-\sin t]}{[(\cos t) + 1]}$$

$$= \lim_{t \to 0} \frac{\sin t}{t} \lim_{t \to 0} \frac{-\sin t}{(\cos t) + 1} = 1 \cdot \left(\frac{-0}{1+1}\right) = 0$$

C. FIND $\lim_{t \to 0} \frac{5t}{\sin 8t} =$

$$\lim_{t \to 0} \frac{5}{8} \cdot \frac{8t}{\sin 8t} = \frac{5}{8} \lim_{t \to 0} \frac{8t}{\sin 8t}$$

$$= \frac{5}{8} \lim_{t \to 0} \frac{1}{\frac{\sin 8t}{8t}} = \frac{5}{8} \cdot \frac{1}{1} = \frac{5}{8}$$

SINCE $\lim_{t \to 0} \frac{\sin t}{t} = 1$, $\lim_{t \to 0} \frac{\sin 8t}{8t} = 1$

D. $\boxed{\frac{d}{dx}(\sin x) = \cos x}$

PROOF: LET $f(x) = \sin x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$\lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h} \right]$$

$$= \lim_{h \to 0} (\sin x)\left[\frac{(\cos h) - 1}{h}\right] + (\cos x)\left[\frac{\sin h}{h}\right]$$

(USING SUM & PRODUCT LIMIT THEOREMS)

$$= (\sin x)[0] + (\cos x)[1] = \cos x = f'(x)$$

E. $\boxed{\dfrac{d}{dx}(\csc x) = -\csc x \cot x}$

PROOF: LET $f(x) = \csc x = \dfrac{1}{\sin x}$

$$f'(x) = \frac{(\sin x)(0) - 1(\cos x)}{(\sin x)^2} =$$

$$\frac{-\cos x}{(\sin x)(\sin x)} = -\left[\frac{1}{\sin x}\right]\left[\frac{\cos x}{\sin x}\right] = -\csc x \cot x$$

F. $$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

FORMULA GIVEN WITHOUT PROOF

G. $$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

PROOF: $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

H. KNOW THESE TRIG DERIVATIVES

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cos x) = -\sin x$ $\qquad$ $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

I. $\frac{d}{dx}\left(\frac{\csc x}{1+\cot x}\right) =$

$$\frac{(1+\cot x)(-\csc x \cot x)-(\csc x)(-\csc^2 x)}{(1+\cot x)^2}$$

$$\frac{-(\csc x)[\cot x + \cot^2 x - \csc^2 x]}{(1+\cot x)^2}$$

$$= \frac{-\csc x\,[(\cot x) - 1]}{(1+\cot x)^2}$$

$$\boxed{\begin{array}{c}\text{RECALL}\\ 1+\cot^2 x = \csc^2 x\end{array}}$$

J. LET $y = \sec x\,(e^x + \cos x \csc x)$

$$\frac{dy}{dx} = (\sec x)\left[e^x + (\cos x)(-\csc x \cot x) + (\csc x)(-\sin x)\right] + (e^x + \cos x \csc x)\sec x \tan x$$

$$= \sec x\left[e^x - \frac{\cos x}{\sin x}\cot x - \frac{\sin x}{\sin x}\right] + \left(e^x + \frac{\cos x}{\sin x}\right)\sec x \tan x$$

$$= \sec x\left[e^x - \cot x \cot x - 1\right] + (e^x + \cot x)\sec x \tan x$$

$$= (\sec x)e^x - \sec x \cot^2 x - \sec x + e^x \sec x \tan x + \cot x \tan x \sec x$$

$$= (\sec x)\left[e^x - \cot^2 x - 1 + e^x \tan x + 1\right]$$

$$= (\sec x)\left[e^x - \cot^2 x + e^x \tan x\right]$$

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HOMEWORK

PAGES 195, 196

1 THROUGH 16, 18, 19, 23,

39, 40, 42

THE CHAIN RULE
DERIVATIVE OF THE COMPOSITION

A. IF $g'(x)$ EXISTS AND f IS DIFFERENTIABLE AT $g(x)$ THEN $(f \circ g)'(x)$ EXISTS AND

$$(f \circ g)'(x) = \underset{\uparrow \text{ OF}}{f'}(g(x))\underset{\uparrow \text{ TIMES}}{\,}g'(x)$$

CHAIN RULE

B. PROOF <u>INDICATION</u> OF CHAIN RULE

$$(f \circ g)'(a) = \lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x - a} =$$

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} =$$

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}$$

$$= f'(g(a))\, g'(a)$$

C. LET $h(x) = \sin x^2$. FIND $h'(x)$.

LET $f(x) = \sin x$ AND $g(x) = x^2$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = \sin x^2 = h(x)$$

$$f'(x) = \cos x \quad g'(x) = 2x$$

$$h'(x) = (f \circ g)'(x) = f'(g(x))\, g'(x)$$

$$= f'(x^2)\, 2x = (\cos x^2)\, 2x$$

NOW THAT YOU ARE TRAINED AT RECOGNIZING $h(x) = \sin x^2$ AS A COMPOSITION, IN THE FUTURE YOU CAN SIMPLY WRITE

$$h(x) = \sin x^2$$

$$h'(x) = (\cos x^2)\, 2x$$

DERIVATIVE FORMULAS CONSIDERING THE CHAIN RULE

$$\frac{d}{dx}\sin u = (\cos u)u'$$

$$\frac{d}{dx}\cos u = -(\sin u)u'$$

$$\frac{d}{dx}\tan u = (\sec^2 u)u'$$

$$\frac{d}{dx}\cot u = -(\csc^2 u)u'$$

$$\frac{d}{dx}\sec u = (\sec u \tan u)u'$$

$$\frac{d}{dx}\csc u = -(\csc u \cot u)u'$$

$$\frac{d}{dx}e^u = e^u u'$$

D. EXAMPLE: $h(x) = \cos x^3$

$h'(x) = -(\sin x^3)\, 3x^2$

E. EXAMPLE: $\frac{d}{dx} \csc x^5 =$

$-(\csc x^5 \cot x^5)\, 5x^4$

F. EXAMPLE: $\frac{d}{dx} e^{x^2} = e^{x^2} 2x$

G. EXAMPLE $y = \tan e^{x^2}$

$\frac{dy}{dx} = (\sec^2 e^{x^2})(e^{x^2} 2x)$

H. EXAMPLE: $y = \frac{e^{5x} - \sin x^2}{\sec e^{3x}}$

$$y' = \frac{(\sec e^{3x})[e^{5x}5 - (\cos x^2)2x] - [e^{5x} - \sin x^2](\sec e^{3x} \tan e^{3x})\, e^{3x}3}{[\sec e^{3x}]^2}$$

I. $$\boxed{\frac{d}{dx} u^n = n u^{n-1} u'}$$ POWER RULE

<u>ILLUSTRATION</u>: LET $h(x) = (1+x^2)^{\frac{5}{3}}$

LET $f(x) = x^{\frac{5}{3}}$ AND $g(x) = 1+x^2$

$$(f \circ g)(x) = f(g(x)) = f(1+x^2) = (1+x^2)^{\frac{5}{3}} = h(x)$$

$$h'(x) = (f \circ g)'(x) = f'(g(x))\,g'(x)$$

$$\boxed{\text{TIME OUT} \quad f'(x) = \tfrac{5}{3} x^{2/3} \quad g'(x) = 2x}$$

TIME IN $\quad = f'(1+x^2)\,2x = \frac{5}{3}(1+x^2)^{2/3}\,2x$

J. FOR $y = \sqrt{1+x^2} = (1+x^2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x) = \frac{x}{(1+x^2)^{1/2}}$$

$$= \frac{x}{\sqrt{1+x^2}}$$

K. FOR $y = \cos^3 x^2 = (\cos x^2)^3$

$$y' = 3(\cos x^2)^2(-\sin x^2)2x$$

$$y' = -6x \cos^2 x^2 \sin x^2$$

$$y' = -6x \cos^2(x^2)\sin(x^2)$$

L. FOR $y = e^{\cos x^2} - \tan\left(\frac{2-3x}{5x+2}\right)$

$$y' = e^{\cos x^2}[-\sin x^2]2x -$$

$$\left[\sec^2\left(\frac{2-3x}{5x+2}\right)\right]\left[\frac{(5x+2)(-3)-(2-3x)5}{(5x+2)^2}\right]$$

M. FOR $f(x) = a^x = e^{\ln a^x} = e^{x\ln a}$,

$$f'(x) = e^{x\ln a}\ln a = e^{\ln a^x}\ln a = a^x \ln a$$

N. $\frac{d}{dx} a^x = a^x \ln a$ $\boxed{\frac{d}{dx} a^u = (a^u \ln a)u'}$

O. FOR $y = 2^{\sin x}$

$y' = (2^{\sin x})(\ln 2)\cos x$

P. LEIBNIZ NOTATION FOR THE CHAIN RULE: FOR $y = f(u)$ AND $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Q FOR $y = \sin u$ AND $u = x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u)2x = (\cos x^2)2x$$

SAME PROBLEM: FOR $y = \sin x^2$

FIND $\frac{dy}{dx}$ BY LEIBNIZ NOTATION

FOR THE CHAIN RULE

LET $u = x^2$ $y = \sin u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u)2x = (\cos x^2)2x$$

R. FOR $f(x) = \sin(\tan(\cos e^{2x}))$

$$f'(x) = [\cos(\tan(\cos e^{2x}))] \cdot [\sec^2(\cos e^{2x})] \cdot$$
$$[-\sin e^{2x}]\, e^{2x}\, 2$$

S. LET $f(x) = (5 \cdot 3^{x^2} - \cos(x^2+x))^8$

$$f'(x) = 8\left(5 \cdot 3^{x^2} - \cos(x^2+x)\right)^7 \cdot$$
$$\left[5 \cdot (3^{x^2} \ln 3)\, 2x + [\sin(x^2+x)](2x+1)\right]$$

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$\boxed{\text{HOMEWORK}}$

PAGES 203, 204:

ODDS 7 THROUGH 49, 53

IMPLICIT DIFFERENTIATION

A. AN EXPLICITLY DEFINED FUNCTION

$$y = f(x) = x^2 + 7x + 3$$

B. AN EQUATION CAN IMPLICITLY DEFINE AT LEAST ONE FUNCTION $y = f(x)$ (I.E. YOU DO NOT EXPLICITLY HOW THE FUNCTION IS DEFINED NECESSARILY, BUT EACH $(x, y) \in f$ SATISFIES THE EQUATION)

EXAMPLE: $x^2 + y^2 = 4$ IMPLICITLY DEFINES AT LEAST ONE FUNCTION $y = f(x)$. TWO WILL NOW BE SHOWN.

$$y^2 = 4 - x^2$$

$$\sqrt{y^2} = |y| = \sqrt{4 - x^2}$$

$$y = \pm\sqrt{4 - x^2}$$

$y = f_1(x) = +\sqrt{4-x^2}$ SATISFIES

$x^2+y^2=4$ ALSO,

$y = f_2(x) = -\sqrt{4-x^2}$ SATISFIES

$x^2+y^2=4$

C. <u>IMPLICIT DIFFERENTIATION</u> IS

A MEANS OF TAKING THE DERIVATIVE OF A <u>IMPLICITLY DEFINED FUNCTION</u>.

(<u>NOTE</u>: YOU CAN KNOW ITS DERIVATIVE EVEN THOUGH YOU MAY NOT KNOW EXPLICITLY HOW THE FUNCTION IS DEFINED!!)

D. <u>BUILD-UP</u> TO IMPLICIT DIFFERENTIATION

SUPPOSE $y = f(x) = 2x + x^5$

THINK OF y AS AN EXPRESSION INVOLVING x (y IS A FUNCTION OF x)

NOW TAKE THE DERIVATIVE OF y^4

$$\frac{d}{dx}(y^4) = \frac{d}{dx}(2x+x^5)^4 = 4(2x+x^5)^3(2+5x^4)$$

$$\frac{d}{dx}(y^4) \begin{cases} = 4y^3\frac{dy}{dx} = 4y^3y' \\ = 4f^3(x)f'(x) \end{cases}$$

SO IF YOU KNOW y IS A DIFFERENTIABLE FUNCTION OF x, THEN

$$\frac{d}{dx}(y^4) = 4y^3y'$$

$$\frac{d}{dx}(y^8) = 8y^7y'$$

$$\frac{d}{dx}\left(y^{\frac{3}{2}}\right) = \frac{3}{2}y^{\frac{1}{2}}y'$$

AS YOU DO IMPLICIT DIFFERENTIATION THINK OF y AS AN EXPRESSION INVOLVING x BUT YOU MAY NEVER KNOW EXPLICITLY WHAT THAT EXPRESSION IS.

E. IN IMPLICIT DIFFERENTIATION YOU ARE ASSUMING y IS A DIFFERIENTIABLE FUNCTION OF x

F. DRILL PREPARING FOR IMPLICIT DIFFERENTIATION. GET ESTABLISHED IN THIS.

$$\frac{d}{dx}\left(x^5y^4\right) =$$

$$x^5 4y^3y' + y^4 5x^4$$

TREAT AS A PRODUCT OF EXPRESSIONS IN x; ONE EXPLICIT (x^5); ONE IMPLICIT (y^4)

$$\frac{d}{dx}\left(\sin x^5y^4\right) =$$

$$\left(\cos x^5y^4\right)\left(x^5 4y^3y' + y^4 5x^4\right)$$

$$\frac{d}{dx}\left(xy^3\right) = x3y^2y' + y^3(1)$$

$$= x3y^2\frac{dy}{dx} + y^3$$

$$\frac{d}{dx}\left(\sin^7(x^5y^4)\right) = \frac{d}{dx}\left(\sin(x^5y^4)\right)^7$$

$$\left[7\sin^6(x^5y^4)\right]\left[\cos(x^5y^4)\right]\left[x^5 4y^3y' + y^4 5x^4\right]$$

G. IMPLICIT DIFFERENTIATION

1. DIFFERENTIATE BOTH SIDES
2. SOLVE FOR y'

FIND y' BY IMPLICIT DIFFERENTIATION FOR $y^3 + x^5y^4 = x^7y^6$

$$\frac{d}{dx}\left(y^3+x^5y^4\right) = \frac{d}{dx}\left(x^7y^6\right)$$

$$3y^2y' + x^5 4y^3y' + y^4 5x^4 = x^7 6y^5y' + y^6 7x^6$$

$$3y^2y' + 4x^5y^3y' - 6x^7y^5y' = 7x^6y^6 - 5x^4y^4$$

$$y'\left(3y^2 + 4x^5y^3 - 6x^7y^5\right) = x^4y^4\left(7x^2y^2 - 5\right)$$

$$y' = \frac{x^4y^4\left(7x^2y^2-5\right)}{3y^2+4x^5y^3-6x^7y^5}$$

H. FIND AN EQUATION FOR THE LINE TANGENT TO THE GRAPH OF

$$y + \sin xy = \pi + 2x \quad \text{AT } \left(\tfrac{1}{2}, \pi\right)$$

TO GET SLOPE, FIND y' BY IMPLICIT DIFFERENTIATION

$$\frac{d}{dx}(y + \sin xy) = \frac{d}{dx}(\pi + 2x)$$

$$\frac{dy}{dx} + (\cos xy)\left(x\frac{dy}{dx} + y(1)\right) = 0 + 2$$

$$\frac{dy}{dx} + x(\cos xy)\frac{dy}{dx} + y\cos xy = 2$$

$$\frac{dy}{dx} + x(\cos xy)\frac{dy}{dx} = 2 - y\cos xy$$

$$\frac{dy}{dx}\left(1 + x(\cos xy)\right) = 2 - y\cos xy$$

$$\frac{dy}{dx} = \frac{2 - y\cos xy}{1 + x\cos xy}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(\frac{1}{2},\pi)} = \frac{2 - \pi\cos\frac{1}{2}\pi}{1+\frac{1}{2}\cos\frac{1}{2}\pi}$$

$$= \frac{2-0}{1+0} = 2 = \text{SLOPE } m$$

$$y - y_1 = m(x - x_1)$$
$$y - \pi = 2\left(x - \frac{1}{2}\right)$$

I. IMPLICIT DIFFERENTIATION USING $f(x)$ NOTATION

FIND $f'(x)$ WHERE
FIND y'

$$x^2 + f(x)^2 = 1 \quad (\text{i.e. } x^2+y^2=1)$$

DIFFERENTIATE BOTH SIDES w.r.t. x

$$2x + 2f(x)f'(x) = 0$$
$$2f(x)f'(x) = -2x$$
$$f'(x) = \frac{-2x}{2f(x)} = \frac{-x}{f(x)}$$

J. FIND y' BY IMPLICIT DIFFERENTIATION

$$x^3 + y^3 = 6$$

$$3x^2 + 3y^2 y' = 0$$

$$3y^2 y' = -3x^2$$

$$y' = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2} = y'$$

SOLVE FOR y EXPLICITLY

$$y^3 = 6 - x^3$$

$$y = (6-x^3)^{1/3}$$

FIND y' EXPLICITLY

$$y' = \frac{1}{3}(6-x^3)^{-\frac{2}{3}}(-3x^2) = -\frac{x^2}{(6-x^3)^{2/3}}$$

NOTE THIS SOLVES IMPLICIT DIFF. EQ.

$$y' = \frac{-x^2}{y^2} = \frac{-x^2}{[(6-x^3)^{1/3}]^2} = \frac{-x^2}{(6-x^3)^{2/3}}$$

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2ND DERIVATIVE: IMPLICIT DIFFERENTIATION

FOR $x^6+y^6=7$, FIND y''.

TAKE DERIVATIVE OF BOTH SIDES: $6x^5+6y^5y'=0$

$$6y^5y' = -6x^5$$

$$y' = \frac{-6x^5}{6y^5} = \boxed{\frac{-x^5}{y^5} = y'}$$

$$y'' = \frac{y^5(-5x^4)-(-x^5)5y^4y'}{(y^5)^2}$$

$$y'' = \frac{-5x^4y^5+5x^5y^4\left(\frac{-x^5}{y^5}\right)}{y^{10}}$$

$$y'' = \frac{-5x^4y^5 - \frac{5x^{10}}{y}}{y^{10}}$$

$$y'' = \frac{\frac{-5x^4y^6-5x^{10}}{y}}{\frac{y^{10}}{1}} = \frac{-5x^4y^6-5x^{10}}{y^{11}}$$

164 B

HOMEWORK

PAGES 213, 214:

1, 5, 8, 13, 16, 19, 21, 26, 33, 35

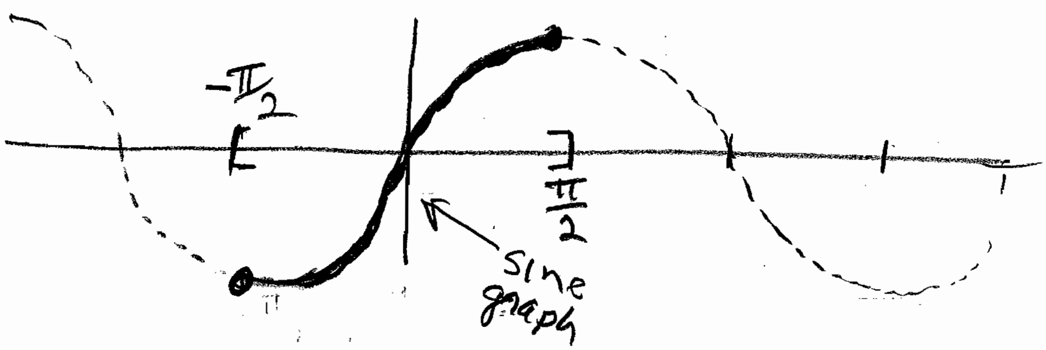
DERIVATIVES OF INVERSE TRIG FUNCTIONS

A. $\boxed{\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}}$

PROOF INDICATION (ASSUMING arcsin IS DIFFERENTIABLE).

$y = \arcsin x$ IFF $\sin y = x$ AND $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\arcsin = \sin^{-1}$ THE INVERSE OF



IMPLICITLY DIFFERENTIATING

$\sin y = x$

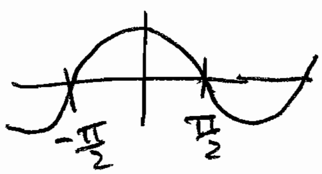
$(\cos y)\frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \sin^2 y$$

$$\sqrt{\cos^2 y} = |\cos y| = \sqrt{1-(\sin y)^2}$$



POS. OR 0 FOR $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

SINCE $\sin y = x$

$$\cos y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

USING THE CHAIN RULE

$$\boxed{\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot u'}$$

B. NOTE THE DIFFERENCE BETWEEN DERIVATIVE FORMULAS WITHOUT AND WITH THE CHAIN RULE

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \cdot u'$$

THE DIFFERENCE: REPLACE x WITH u AND MULTIPLY BY u'.

C. $$\frac{d}{dx} \sin^{-1}(x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$$

D. $$\frac{d}{dx} \arcsin e^{3x} = \frac{1}{\sqrt{1-(e^{3x})^2}} \cdot e^{3x} 3$$

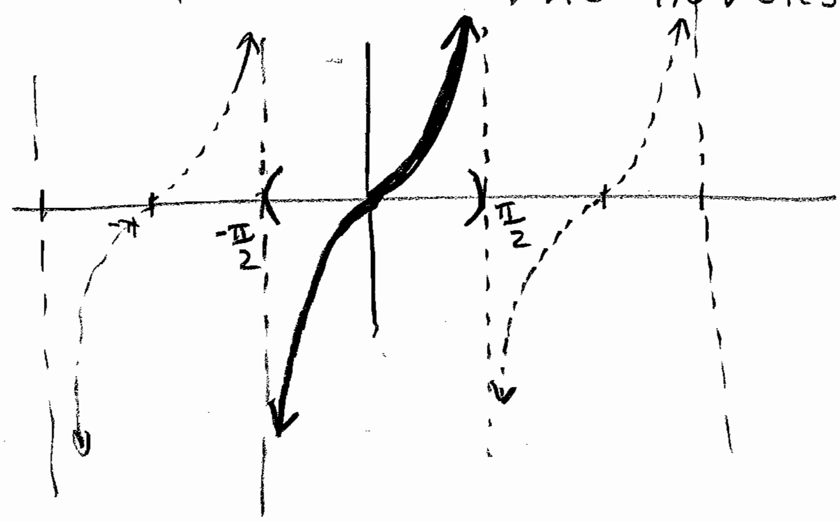
$$= \frac{3e^{3x}}{\sqrt{1-e^{6x}}}$$

E. $$\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}}$$

PROOF INDICATION (ASSUMING arctan IS DIFFERENTIABLE)

$y = \arctan x$ IFF $\tan y = x$ AND $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\arctan = \tan^{-1}$ THE INVERSE OF



IMPLICITLY DIFFERENTIATING

$$\tan y = x$$

$$(\sec^2 y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

USING THE CHAIN RULE

$$\frac{d}{dx}\arctan u = \frac{1}{1+u^2}\cdot u'$$

F. $$\frac{d}{dx}\left(\tan^{-1}(x^2)\right) = \frac{1}{1+(x^2)^2}\cdot 2x = \frac{2x}{1+x^4}$$

G. LET $f(x) = \arctan(e^{3x})$

$$f'(x) = \frac{1}{1+(e^{3x})^2}\cdot e^{3x}\cdot 3 = \frac{3e^{3x}}{1+e^{6x}}$$

H. **DERIVATIVES OF INVERSE TRIG FUNCTIONS**

* $$\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}}\cdot u' \qquad \frac{d}{dx}(\cos^{-1}u) = \frac{-1}{\sqrt{1-u^2}}\cdot u'$$

* $$\frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2}\cdot u' \qquad \frac{d}{dx}(\cot^{-1}u) = \frac{-1}{1+u^2}\cdot u'$$

* $$\frac{d}{dx}(\sec^{-1}u) = \frac{1}{u\sqrt{u^2-1}}\cdot u'$$

$$\frac{d}{dx}(\csc^{-1}u) = \frac{-1}{u\sqrt{u^2-1}}\cdot u'$$

* KNOW THESE

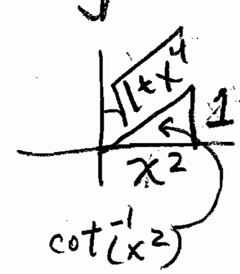
I. SOME FORMULAS YOU ARE TO KNOW; SOME YOU ARE TO LOOK UP AND USE.

J. LET $f(x) = \dfrac{x^2 + \tan^{-1} e^{5x}}{x + \sec^{-1} x^3}$

$$f'(x) = \frac{\left[(x+\sec^{-1}x^3)\left(2x + \frac{1}{1+(e^{5x})^2}\cdot e^{5x}(5)\right) - (x^2+\tan^{-1}e^{5x})\left(1 + \frac{1}{x^3\sqrt{(x^3)^2-1}}\cdot 3x^2\right)\right.}{(x+\sec^{-1}x^3)^2}$$

K. LET $f(x) = [\csc(\cot^{-1}(x^2))]e^{5x}$

$$f'(x) = [\csc(\cot^{-1}(x^2))]e^{5x}(5) +$$
$$(e^{5x})\left[-\csc(\cot^{-1}(x^2))\cot(\cot^{-1}(x^2))\right]\left[\frac{-1}{1+(x^2)^2}\cdot 2x\right]$$
$$= 5e^{5x}\Big[\underbrace{\csc(\cot^{-1}(x^2))}_{=\sqrt{1+x^4}}\Big] +$$
$$\frac{2x\,e^{5x}}{1+x^4}\left[\csc(\cot^{-1}(x^2))\;x^2\right]$$
$$= 5e^{5x}\sqrt{1+x^4} + \frac{2x^3 e^{5x}(\sqrt{1+x^4})}{1+x^4}$$



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HOMEWORK

PAGE 214: 45 THROUGH 54,
57

NO PAGES 172-178

LOG DERIVATIVES

A. $\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$

PROOF: LET $y = \ln x$ FIND y'

$e^y = x$

IMPLICIT DIFF. TO FIND y'

$e^y y' = 1$

$$y' = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

B. $\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\cdot u' = \frac{u'}{u}}$ KNOW THIS

C. $$\frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1}$$

D. $$\frac{d}{dx}(\cos x^2)(\ln(x^3+1)) =$$

$$(\cos x^2)\frac{3x^2}{x^3+1} + [\ln(x^3+1)][-\sin x^2](2x)$$

E. $$\boxed{\frac{d}{dx}\log_a x = \frac{1}{x}\cdot\frac{1}{\ln a}}$$

PROOF: LET $y = \log_a x$. FIND y'

$$y = \log_a x = \frac{\ln x}{\ln a} = \frac{1}{\ln a}\cdot \ln x$$

CONSTANT

$$y' = \frac{1}{\ln a}\cdot\frac{1}{x} = \frac{1}{x}\cdot\frac{1}{\ln a} = \frac{d}{dx}\log_a x$$

F. $$\boxed{\frac{d}{dx}\log_a u = \frac{1}{u}\cdot\frac{1}{\ln a}\cdot u'}$$ KNOW THIS

G. LET $f(x) = \log_7 x^2$

$$f'(x) = \frac{1}{x^2}\cdot\frac{1}{\ln 7}\cdot 2x = \frac{2}{x\ln 7}$$

NOTE: $F(x) = \log_7 x^2 = 2\log_7 x$

$$f'(x) = 2\left(\frac{1}{x}\right)\left(\frac{1}{\ln 7}\right)$$

H. Let $f(x) = \dfrac{\csc x^2 + \log_7(x^2+1)}{\ln(\cos x^4)}$

$$f'(x) = \frac{\ln(\cos x^4)\left[(-\csc x^2 \cot x^2)(2x) + \frac{1}{x^2+1}\cdot\frac{1}{\ln 7}\cdot 2x\right] - \left[(\csc x^2 + \log_7(x^2+1))\frac{1}{\cos x^4}(-\sin x^4)4x^3\right]}{[\ln(\cos x^4)]^2}$$

I. <u>NOTE:</u> FOR $y = \log_a x$, $a = e$ IS THE ONLY BASE WHERE

$\frac{d}{dx}\log_a x = \frac{1}{x}$ SINCE

$\frac{d}{dx}\log_a x = \frac{1}{x}\frac{1}{\ln a}$ AND $\ln a = 1$ WHEN $a = e$

J. $\boxed{\text{LOGARITHMIC DIFFERENTIATION}}$

FOR $y = (x^2+1)^{10}(x^5-3)^7$ FIND y' BY LOG DIFFERENTIATION

1. TAKE ln OF BOTH SIDES

$$\ln y = \ln\left[(x^2+1)^{10}(x^5-3)^7\right]$$

2. USE LOGARITHM PROPERTIES

$$\ln y = \ln(x^2+1)^{10} + \ln(x^5-3)^7$$

$$\ln y = 10\ln(x^2+1) + 7\ln(x^5-3)$$

3. FIND y' BY IMPLICIT DIFFERENTIATION

$$\frac{y'}{y} = 10\left(\frac{2x}{x^2+1}\right) + 7\left(\frac{5x^4}{x^5-3}\right)$$

$$y' = y\left[\frac{20x}{x^2+1} + \frac{35x^4}{x^5-3}\right]$$

$$y' = (x^2+1)^{10}(x^5-3)^7\left[\frac{20x}{x^2+1} + \frac{35x^4}{x^5-3}\right]$$

$$= \frac{d}{dx}(x^2+1)^{10}(x^5-3)^7$$

K. $\boxed{\frac{d}{dx}(\text{varying})^{\text{varying}} \text{ BY LOG. DIFF.}}$

FIND $\frac{d}{dx}\left[(\cos x)^{\sin x}\right]$

LET $y = (\cos x)^{\sin x}$

$$\ln y = \ln\left[(\cos x)^{\sin x}\right]$$

$$\ln y = \sin x \ln(\cos x)$$

$$\frac{y'}{y} = (\sin x)\frac{(-\sin x)}{\cos x} + \left[\ln(\cos x)\right]\cos x$$

$$y' = y\left[\frac{-\sin^2 x}{\cos x} + [\ln(\cos x)]\cos x\right]$$

$$y' = (\cos x)^{\sin x}\left[\frac{-\sin^2 x}{\cos x} + [\ln(\cos x)]\cos x\right]$$

$$= \frac{d}{dx}\left[(\cos x)^{\sin x}\right]$$

L. $\frac{d}{dx}\ln|x| = \frac{1}{x}$

PROOF: CASE 1 $x>0$

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$$

CASE 2 $x<0$

$$\frac{d}{dx}\ln|\overset{neg}{x}| = \frac{d}{dx}\ln(-x) = \frac{1}{(-x)}(-1) = \frac{1}{x}$$

SO $\boxed{\frac{d}{dx}\ln|u| = \frac{1}{u}\cdot u' = \frac{u'}{u}}$ KNOW THIS

M. $\frac{d}{dx}(\text{VARYING})^{\text{VARYING}}$ BY $\square = e^{\ln \square}$

Let $y = (\cos x)^{\sin x}$. FIND y'

$$y = (\cos x)^{\sin x} = e^{\ln (\cos x)^{\sin x}} = e^{\sin x \ln \cos x}$$

$$y' = e^{\sin x \ln \cos x}\left[(\sin x)\frac{1}{\cos x}(-\sin x) + (\ln \cos x)\cos x\right]$$

$$y' = (\cos x)^{\sin x}\left[\frac{-\sin^2 x}{\cos x} + \cos x \ln \cos x\right]$$

NOTE THIS IS THE SAME ANSWER AS WE GOT BEFORE WHEN WE WORKED THE PROBLEM BY LOG DIFFERENTIATION.

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HOMEWORK

PAGE 220: 2, 3, 4, 5, 7, 9, 10, 11, 12, 14, 17, 19, 23, 30, 37, 41, 43, 48

ALSO $\frac{d}{dx} \ln|x^3+x^5|$

DERIVATIVE APPLICATION TO PHYSICS

A. RECALL, IF $f(t)$ IS THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME t, THEN $f'(t)$ IS THE INSTANTANEOUS VELOCITY OF THE OBJECT AT TIME t.

B. VELOCITY BASICS (HORIZONTAL COORDINATE LINE)

1. VELOCITY = 0 (I.E. $f'(t)=0$) THE OBJECT IS STOPPED.
2. VELOCITY POSITIVE (I.E. $f'(t)>0$) THE OBJECT IS MOVING TO THE RIGHT.
3. VELOCITY NEGATIVE (I.E. $f'(t)<0$) THE OBJECT IS MOVING TO THE LEFT

C. ANALYSIS OF OBJECT MOTION: SUPPOSE $f(t) = 2t^3 - 15t^2 + 36t$ IS THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME t.

t: SECONDS $f(t)$: FEET $t \in [0, 4]$

1. WHAT IS THE VELOCITY AT TIME t?

$f'(t) = 6t^2 - 30t + 36$ ← VELOCITY

2. WHEN (AT WHAT TIME) IS THE OBJECT STOPPED (AT REST)?

FIND TIME WHEN $f'(t) = 0$

$6t^2 - 30t + 36 = 0$

$6(t^2 - 5t + 6) = 0$

$6(t-2)(t-3) = 0$

$t = 2$ OR $t = 3$

3. WHAT ARE THE POSITIONS OF THE OBJECT WHEN THE OBJECT IS AT REST?

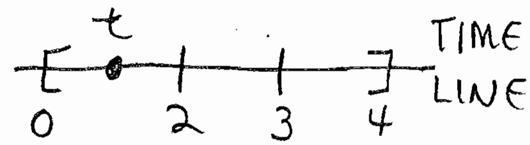
$f(2) = 2(2^3) - 15(2^2) + 36(2) = 16 - 60 + 72 = 28$

$f(3) = 2(3^3) - 15(3^2) + 36(3) = 54 - 135 + 108$
$= 27$

4. FIND THE TIME INTERVALS WHEN THE OBJECT IS MOVING TO THE <u>RIGHT</u>. FIND THE TIME INTERVALS WHEN THE OBJECT IS MOVING TO THE <u>LEFT</u>.

$$f'(t) = 6t^2 - 30t + 36 = 6(t^2 - 5t + 6)$$
$$= 6(t-2)(t-3)$$

$\boxed{\text{CASE 1}}$ $t \in [0,2)$

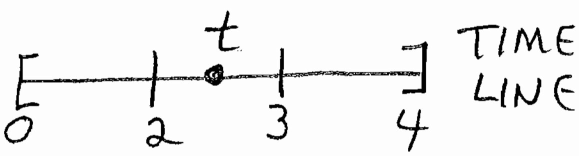


$$f'(t) = 6(\underset{L-R}{t-2})(\underset{L-R}{t-3}) > 0$$ VELOCITY POSITIVE

$$\underbrace{pos \cdot neg \cdot neg}_{pos}$$

OBJECT MOVES RIGHT ON $[0,2)$ ← TIME

$\boxed{\text{CASE 2}}$ $t \in (2,3)$

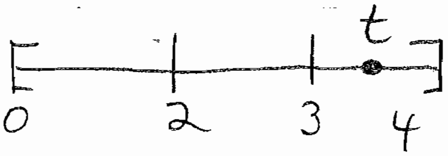


$$f'(t) = 6(\underset{R-L}{t-2})(\underset{L-R}{t-3}) < 0$$ VELOCITY NEGATIVE

$$\underbrace{pos \cdot pos \cdot neg}_{neg}$$

OBJECT MOVE LEFT ON $(2,3)$ ← TIME

$\boxed{\text{CASE 3}}$ $t \in (3,4]$



TIME LINE

$$f'(t) = \underbrace{\underset{pos}{6}\,\underset{pos}{\overset{R-L}{(t-2)}}\,\underset{pos}{\overset{R-L}{(t-3)}}}_{pos} > 0$$ VELOCITY POSITIVE

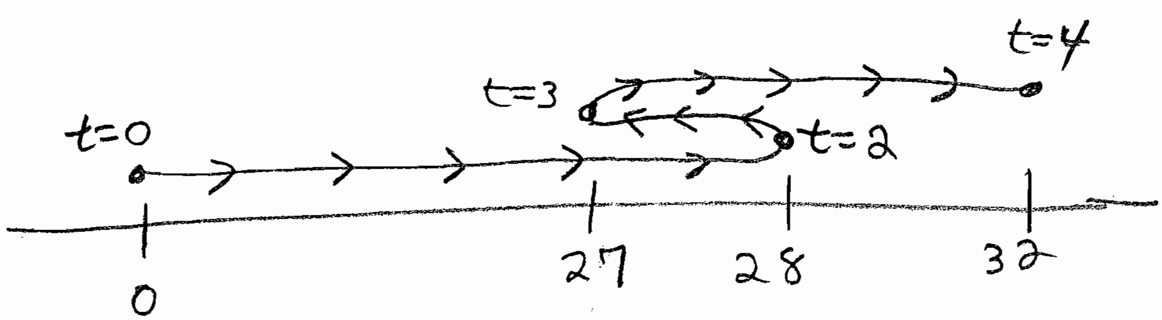
OBJECT MOVES RIGHT ON $(3,4]$.

$\uparrow$ TIME

5. SKETCH A SCHEMATIC DIAGRAM OF THE PARTICLE MOTION ON THE COORDINATE LINE (NOT ON THE TIME LINE)

$$f(0) = 2(0^3) - 15(0^2) + 36(0) = 0$$

$$f(4) = 2(4^3) - 15(4^2) + 36(4)$$
$$= 2(64) - 15(16) + 144 = 32$$



6. FIND THE TOTAL DISTANCE TRAVELLED FOR TIME $t \in [0,4]$

TOTAL DISTANCE $=$ [MOVING RIGHT $t \in [0,2)$]

$+$ [MOVING LEFT $t \in (2,3)$] $+$

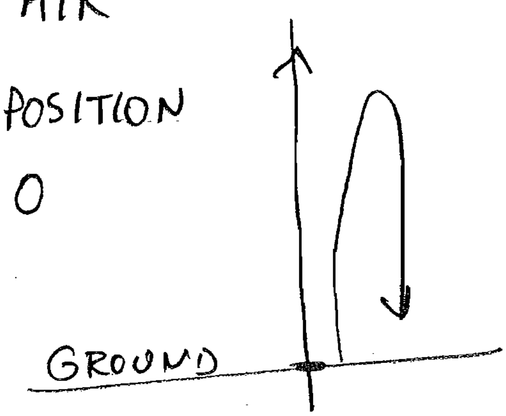
[MOVING RIGHT $t \in (3,4]$]

$= |f(2) - f(0)| + |f(3) - f(2)| + |f(4) - f(3)|$

$= |28 - 0| + |27 - 28| + |32 - 27|$

$= 28 + 1 + 5 = 34$ FEET

D. VERTICAL COORDINATE LINE (UP POSITIVE). AN OBJECT THROWN UP IN THE AIR



TRANSLATIONS: $f(t) =$ POSITION AT TIME t. GROUND $= 0$

1. WHAT TIME DOES THE OBJECT REACH MAXIMUM HEIGHT? FIND TIME WHEN VELOCITY IS 0 (OBJECT STOPPED).
(I.E. FIND t_0 WHEN $f'(t_0) = 0$)

2. HOW HIGH DOES THE OBJECT GO?

FIND TIME t_0 FOR MAXIMUM HEIGHT. EVALUATE $f(t_0)$.

3. HOW FAST IS THE OBJECT TRAVELLING WHEN IT HITS THE GROUND?

FIND TIME t_1 WHEN $f(t_1) = 0$.

EVALUATE $f'(t_1)$.

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SUPPOSE $s(t) = 98t - 4.9t^2$ IS THE POSITION OF AN OBJECT AT TIME t (sec) (POSITION IN METERS) WHEN THE OBJECT IS THROWN STRAIGHT UP. GROUND = 0 POSITION

A) HOW HIGH DOES IT GO?

B) HOW FAST IS IT GOING WHEN IT HITS THE GROUND.

A.) FIND THE TIME THE VELOCITY IS 0, THEN SUBSTITUTE THAT TIME IN THE POSITION FUNCTION.

$$v(t) = 98 - 9.8t = s'(t)$$

$$98 - 9.8t = 0$$

$$98 = 9.8t$$

$$10 = \frac{98}{9.8} = t$$

$$s(10) = 98(10) - 4.9(10^2)$$

$$= 980 - 4.9(100) = 980 - 490 = 490\text{ m}$$

MAXIMUM HEIGHT = 490 m

B. HOW FAST WHEN HITS GROUND:

1ST FIND TIME HITS GROUND

2nd PUT THAT TIME IN VELOCITY FUNCTION

$$s(t) = 98t - 4.9t^2 = 0$$

$$t(98 - 4.9t) = 0$$

$t = 0$ OR $98 - 4.9t = 0$

WE WANT $98 = 4.9t$

$$20 = \frac{98}{4.9} = t$$

$$V(t) = 98 - 9.8t$$

$$V(20) = 98 - 9.8(20) = 98 - 196$$

$$= -98 \text{ m/sec}$$

C. HIGHER DERIVATIVES APPLIED TO PHYSICS: FOR $f(t)$ BEING THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME t:

(CONTINUED ON NEXT PAGE)

$f'(t)$ IS THE INSTANTANEOUS <u>VELOCITY</u> OF THE OBJECT AT TIME t

$f''(t) = a(t)$ THE INSTANTANEOUS ACCELERATION OF THE OBJECT AT TIME t (THE INSTANTANEOUS RATE OF CHANGE OF VELOCITY WITH RESPECT TO TIME)

FOR t IN SECONDS AND $f(t)$ IN FEET

$v(t) = f'(t)$ UNITS feet/sec

$a(t) = v'(t) = f''(t)$ UNITS ft/sec^2

INTERPRETATION OF $f''(9) = a(9) = 32 \frac{\text{ft}}{\text{sec}^2}$:

<u>IF</u> THE RATE REMAINED CONSTANT AT $32 \frac{\text{ft}}{\text{sec}^2}$, FOR EACH ELAPSED SECOND, THE VELOCITY WOULD INCREASE AT $32\ \text{ft}/\text{sec}$ PER SECOND (i.e. $32 \frac{\text{ft}}{\text{sec}^2}$ CAN BE READ 32 FEET PER SECOND PER SECOND)

D VELOCITY, ACCELERATION

EXAMPLE: $f(t)$ IS THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME t. t: SEC

$f(t) = t^3 - 15t^2$ $f(t)$: FEET

FIND THE VELOCITY WHEN THE ACCELERATION IS 0.

<u>TRANSLATION:</u> FIND $f'(t)$ WHEN $f''(t) = 0$.

$f'(t) = 3t^2 - 30t$

$f''(t) = 6t - 30$

$6t - 30 = 0$

$6t = 30$ $t = 5$ TIME WHEN ACCELERATION IS 0

$V(5) = f'(5) = 3(5^2) - 30(5)$

$= 75 - 150 = -75 \frac{FT}{SEC}$

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HOMEWORK

PAGES 230, 231: 1, 8, 9, 10

EXPONENTIAL GROWTH & DECAY

A. <u>ASSUMPTION</u>: POPULATION GROWTH RATE IS PROPORTIONAL TO THE SIZE OF THE POPULATION. SAID AS A DIFFERENTIAL EQUATION: $P'(t) = kP(t)$ K CONSTANT (OR $\frac{dP}{dt} = kP$) WHERE $P(t)$ IS THE POPULATION AT TIME t.

B. NOTE: WHEN $P(t) = Ce^{kt}$, $P'(t) = kCe^{kt} = kP(t)$
SO $P(t) = Ce^{kt}$ SATISFIES $P'(t) = kP(t)$.
LET $t = 0$. $P(0) = Ce^{k0} = C$. SO $P(t) = P(0)e^{kt}$
SATISFIES $P'(t) = kP(t)$. IT CAN BE SHOWN THAT ALL SOLUTION TO $P'(t) = kP(t)$ ARE IN THIS FORM

C.

EXPONENTIAL GROWTH EQUATION
$P(t) = P(0)e^{kt}$
SOLUTION TO $P'(t) = kP(t)$

OTHER NOTATION YOU MAY SEE FOR $P(t) = P(0)e^{kt}$ IS

$$y(t) = y_0 e^{kt}$$

WHERE $y_0 = y(0)$

D. EXAMPLE: THE TABLE DESCRIBES A POPULATION AT VARIOUS TIMES

YEAR	POPULATION
1950	2000
1960	2300

USING 1950 AS TIME $t=0$ AND HENCE THE YEAR 1960 AS TIME $t=10$,

1) FIND THE EXPONENTIAL GROWTH EQUATION FOR THIS POPULATION

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$$P(t) = P(0)\,e^{kt}$$

$$P(t) = 2000\,e^{kt}$$ SINCE $P(0) = 2000$

$$P(10) = 2000\,e^{k10} = 2300$$

$$e^{k10} = \frac{2300}{2000} = \frac{23}{20}$$

$$e^{10k} = \frac{23}{20}$$

$$\ln e^{10k} = 10k = \ln \frac{23}{20}$$

$$k = \frac{1}{10} \ln \frac{23}{20}$$

EXPONENTIAL GROWTH EQUATION

IS $P(t) = 2000\,e^{\left(\frac{1}{10} \ln \frac{23}{20}\right)t}$

2) FIND THE POPULATION ESTIMATE ACCORDING TO $P(t)$ FOR 1974.

1974 CORRESPONDS TO $t = 24$

$$P(24) = 2000\,e^{\left(\frac{1}{10} \ln \frac{23}{20}\right)24}$$

E. TERMINOLOGY: "ASSUME THE GROWTH RATE IS PROPORTIONAL TO THE SIZE OF THE POPULATION" ≡ "THE POPULATION SATISFIES THE EXPONENTIAL GROWTH EQUATION."

F. RADIOACTIVE DECAY

THE RADIOACTIVE MASS DECAYS AT A RATE PROPORTIONAL TO THE AMOUNT OF MASS REMAINING.

$$M'(t) = k\,M(t)$$

NOTE: SAME FORM AS $P'(t) = k\,P(t)$

SO THE SOLUTION TO THE DIFFERENTIAL EQUATION IS

$$M(t) = M(0)\,e^{kt}$$

HOWEVER, FOR <u>DECAY</u> $k < 0$.

FOR <u>GROWTH</u> k WAS GREATER THAN 0

G. <u>HALF-LIFE</u>: THE TIME FOR $\frac{1}{2}$ OF THE MASS TO DECAY.

H. <u>EXAMPLE</u>. A RADIOACTIVE SUBSTANCE HAS A HALF LIFE OF 20 YEARS. AT TIME 0 THERE IS A MASS OF 100 g PRESENT.

1) FIND THE EQUATION $M(t) = M(0)e^{kt}$ FOR THIS PROBLEM

$$M(t) = 100e^{kt}$$

$$M(20) = 50 = 100e^{k20}$$

$$\frac{50}{100} = e^{k20}$$

$$\frac{1}{2} = e^{k20}$$

$$\ln\frac{1}{2} = \ln e^{k20} = k20$$

$$\frac{1}{20}\ln\frac{1}{2} = k$$

$$M(t) = 100e^{\left(\frac{1}{20}\ln\frac{1}{2}\right)t}$$

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2) REWRITE THIS EQUATION TO WHERE THE BASE IS $\frac{1}{2}$, NOT e

$$M(t) = 100\, e^{\frac{t}{20} \ln \frac{1}{2}} = 100\, e^{\ln\left(\frac{1}{2}^{\frac{t}{20}}\right)}$$

$$M(t) = 100 \left(\frac{1}{2}\right)^{\frac{t}{20}}$$

3) AT WHAT TIME WILL 30 g OF THE MATERIAL BE LEFT?

$$30 = 100 \left(\frac{1}{2}\right)^{\frac{t}{20}}$$

$$\frac{30}{100} = \frac{3}{10} = \left(\frac{1}{2}\right)^{\frac{t}{20}}$$

$$\ln \frac{3}{10} = \ln\left(\frac{1}{2}\right)^{\frac{t}{20}} = \frac{t}{20} \ln \frac{1}{2}$$

$$\frac{20 \ln \frac{3}{10}}{\ln \frac{1}{2}} = t$$

I. SINCE THERE IS THE SAME DIFFERENTIAL EQUATION $P' = kP$, $M' = kM$ FOR BOTH GROWTH AND DECAY,

$$y' = ky$$

IS CALLED THE LAW OF GROWTH AND DECAY

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HOMEWORK

PAGES 239, 240: 3, 5, 6b, 8abc, 10

EXPONENTIAL GROWTH & DECAY (CONT.)

FOR $P(t) = P(0)e^{kt}$, $P'(t)$ IS THE GROWTH RATE

A. **RELATIVE GROWTH RATE:** $\dfrac{P'(t)}{P(t)}$

(RECALL $P'(t)$ IS THE GROWTH RATE.)

RECALL THE DIFFERENTIAL EQUATION FOR EXPONENTIAL GROWTH AND DECAY IS

$$P'(t) = K\,P(t), \quad \text{SO}$$

$$\boxed{K = \frac{P'(t)}{P(t)}}.$$

$\dfrac{P'(t)}{P(t)}$ IS THE RELATIVE GROWTH RATE.

THE CONSTANT K IS THE RELATIVE GROWTH RATE WHEN $P'(t) = K\,P(t)$.

THE GROWTH RATE AT TIME t, AS OPPOSED TO THE RELATIVE GROWTH RATE, IS $P'(t)$ AND IS NOT NECESSARILY CONSTANT.

RECALL $P'(t) = K\,P(t)$ WHEN SOLVED YIELDS $P(t) = P(0)\,e^{kt}$

K, THE RELATIVE GROWTH RATE IS CONSTANT

SO IF A POPULATION IS ASSOCIATED WITH $P(t) = 200e^{.3t}$, THE RELATIVE GROWTH RATE IS .3, A CONSTANT

B. <u>TERMINOLOGY</u> THAT MEANS THE SAME THING.

1) THE RATE OF GROWTH (DECAY) IS PROPORTIONAL TO THE AMOUNT PRESENT
2) THE AMOUNT AT ANY TIME SATISFIES THE LAW OF GROWTH AND DECAY
3) THE RELATIVE GROWTH RATE OF THE AMOUNT IS CONSTANT.

ALL 3 MEAN THAT THE FOLLOW APPLIES:

$$\left.\begin{aligned} P'(t) &= k\,P(t) \\ P(t) &= P(0)e^{kt} \\ \frac{P'(t)}{P(t)} &= k \end{aligned}\right\}$$

THESE ALL MEAN THE SAME.

C. CONTINUOUSLY COMPOUNDED INTEREST

$$\boxed{\underline{\text{LEMMA}} \quad \lim_{\omega \to \infty} \left(1 + \frac{1}{\omega}\right)^{\omega} = e}$$

1.) 5% INTEREST ON \$100 COMPOUNED ANNUALLY

END OF 1 YEAR

$$100 + (.05)100 = 100(1+.05)$$

END OF 2 YEARS

$$[100(1+.05)] + (.05)[100(1+.05)]$$

$$= [100(1+.05)](1+.05) = 100(1+.05)^2$$

END OF 3 YEARS

$$[100(1+.05)^2] + (.05)[100(1+.05)^2] =$$

$$[100(1+.05)^2](1+.05) = 100(1+.05)^3$$

$$\vdots$$

AFTER n YEARS

$100(1+.05)^n \leftarrow$ 5% ON 100 DOLLARS COMPOUNDED ANNUALLY

2) 5% INTEREST ON \$100 COMPOUNDED SEMI-ANNUALLY (TWICE A YEAR) (2 A YEAR)

AFTER 6 MONTHS (1 PERIOD)

$$100 + \left(\frac{.05}{2}\right)100 = 100\left(1+\frac{.05}{2}\right)$$

AFTER 1 YEAR (2 PERIODS)

$$\left[100\left(1+\frac{.05}{2}\right)\right] + \left(\frac{.05}{2}\right)\left[100\left(1+\frac{.05}{2}\right)\right] =$$

$$\left[100\left(1+\frac{.05}{2}\right)\right]\left(1+\frac{.05}{2}\right) = 100\left(1+\frac{.05}{2}\right)^2$$

AFTER 3 PERIODS

$$100\left(1+\frac{.05}{2}\right)^3$$

$$\vdots$$

AFTER n PERIODS

$$100\left(1+\frac{.05}{2}\right)^n$$

3) 5% INTEREST ON \$100 COMPOUNDED m TIMES A YEAR, n PERIODS

$$100\left(1+\frac{.05}{m}\right)^n$$

4) INTEREST RATE r ON PRINCIPAL P_0
m TIMES A YEAR, n PERIODS

$$P_0\left(1+\frac{r}{m}\right)^n$$

$$= P_0\left(1+\frac{1}{\frac{m}{r}}\right)^n$$

$$= P_0\left(1+\frac{1}{\frac{m}{r}}\right)^{\frac{m}{r}(r)\left(\frac{n}{m}\right)}$$

(TO FIND THE AMOUNT AFTER t YEARS
$n = tm$)

$$= P_0\left(1+\frac{1}{\frac{m}{r}}\right)^{\frac{m}{r}(r)\left(\frac{tm}{m}\right)}$$

$$= P_0\left(1+\frac{1}{\frac{m}{r}}\right)^{\frac{m}{r}(rt)}$$

$$= P_0\left[\left(1+\frac{1}{\frac{m}{r}}\right)^{\frac{m}{r}}\right]^{rt}$$

5) INTEREST RATE r ON PRINCIPAL P_0 <u>COMPOUNDED CONTINUOUSLY</u> AFTER t YEARS

$$\lim_{m \to \infty} P_0 \left[\left(1 + \frac{1}{\frac{m}{r}} \right)^{\frac{m}{r}} \right]^{rt} =$$

$$P_0 \left[\lim_{m \to \infty} \left(1 + \frac{1}{\frac{m}{r}} \right)^{\frac{m}{r}} \right]^{rt} =$$

$$P_0 \left[\lim_{\omega \to \infty} \left(1 + \frac{1}{\omega} \right)^{\omega} \right]^{rt} \underline{\underline{\text{LEMMA}}}$$

$$P_0 e^{rt}$$

D. \$100 IS INVESTED FOR 5 YEARS AT AN ANNUAL INTEREST RATE OF 7% COMPOUNDED INSTANTOUSLY (COMPOUNDED CONTINUOUSLY). WHAT IS THE AMOUNT AFTER 5 YEARS

$$A = P_0 e^{rt} = 100 e^{(.07)5} = 100 e^{.35}$$

$$\approx \$141.91$$

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$\boxed{\text{HOMEWORK}}$

PAGES 239, 241: 4, 18

RELATED RATES

A. IMPLICIT DIFFERENTIATION INVOLVING MORE THAN ONE FUNCTION OF t.

$$(1+t^2)^{\frac{1}{2}} + (5+t^3)^{\frac{2}{3}} = 7$$

TAKE DERIVATIVE OF BOTH SIDES W.R.T t

$$\frac{1}{2}(1+t^2)^{-\frac{1}{2}}(2t) + \frac{2}{3}(5+t^3)^{-\frac{1}{3}} 3t^2 = 0$$

LET $x = 1+t^2$ AND $y = 5+t^3$

$$x^{\frac{1}{2}} + y^{\frac{2}{3}} = 7$$

TAKE $\frac{d}{dt}$ OF BOTH SIDES

$$\frac{1}{2}x^{-\frac{1}{2}}\frac{dx}{dt} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dt} = 0$$

IN $x^{\frac{1}{2}} + y^{\frac{2}{3}} = 7$ WE ARE TREATING BOTH x ANY y AS FUNCTIONS OF t.

$\frac{dx}{dt}$ AND $\frac{dy}{dt}$ ARE RATES

THIS EQUATION SHOWS HOW THE <u>RATES</u> ARE <u>RELATED</u> (HENCE, "<u>RELATED</u> <u>RATES</u>")

B. RELATED RATE PROBLEMS GENERALLY HAVE 2 PARTS: 1) AN "ALL TIME" MOTION PICTURE AND 2) A FROZEN MOMENT IN TIME

1) ALL TIME MOTION PICTURE:

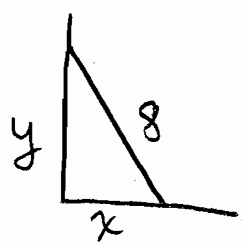
WILL HAVE ALL TIME EQUATIONS YOU CAN TAKE THE DERIVATIVE OF.

2) FROZEN MOMENT IN TIME

IT WILL HAVE SPECIFIC CONSTANT INFORMATION. YOU SUBSTITUTE THAT CONSTANT INFORMATION INTO THE ALL TIME EQUATIONS AND THEIR DERIVATIVES

C. <u>RELATED RATE PROBLEM</u>: AN 8-FOOT LADDER IS PLACED AGAINST A VERTICAL WALL. THE BOTTOM OF THE LADDER SLIPS AWAY FROM THE WALL AT A RATE OF $2\,\frac{FT}{SEC}$. HOW FAST IS THE TOP OF THE LADDER SLIPPING DOWN THE WALL WHEN THE BOTTOM OF THE LADDER IS 4 FEET FROM THE WALL?

MOTION PICTURE



$$\frac{dx}{dt} = 2\,\frac{FT}{SEC}$$

ALL TIME EQUATION

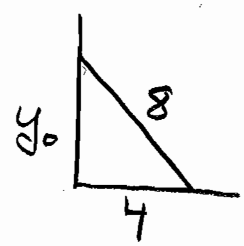
$$x^2 + y^2 = 8^2$$

$\frac{d}{dt}$ OF BOTH SIDES:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

FROZEN MOMENT IN TIME



$$4^2 + y_0^2 = 8^2$$

$$y_0^2 = 64 - 16 = 48$$

$$y_0 = \sqrt{48} = \sqrt{16 \cdot 3}$$

$$= 4\sqrt{3}$$

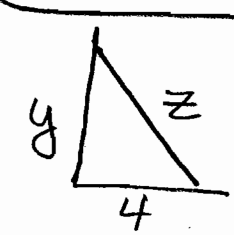
SUBSTITUTE F.M.I.T INTO RELATED RATE EQ.

$$4(2) + 4\sqrt{3}\,\frac{dy}{dt} = 0. \quad 4\sqrt{3}\,\frac{dy}{dt} = -8.$$

$$\frac{dy}{dt} = -8/(4\sqrt{3}) = \frac{-2}{\sqrt{3}}\,\frac{FT}{SEC}$$ ← ANSWER

D. <u>RELATED RATE PROBLEM</u>: AN 8-FOOT EXTENSION LADDER IS LEANING AGAINST A VERTICAL WALL AND STARTS COLLAPSING AT A RATE OF 3 FEET PER SECOND. THE BOTTOM OF THE LADDER IS FIXED AT 4 FEET FROM THE WALL. HOW FAST IS THE TOP OF THE LADDER SLIDING DOWN THE WALL WHEN THE LADDER IS 6 FEET LONG?

MOTION PICTURE



$$\frac{dz}{dt} = -3\,\frac{\text{ft}}{\text{sec.}}$$

ALL TIME EQUATION

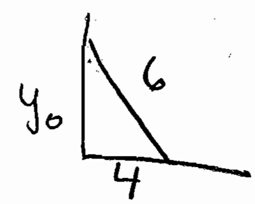
$$4^2 + y^2 = z^2$$

$\frac{d}{dt}$ OF BOTH SIDES

$$0 + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$y\frac{dy}{dt} = z\frac{dz}{dt}$$

FROZEN MOMENT IN TIME



$$4^2 + y_0^2 = 6^2$$

$$16 + y_0^2 = 36$$

$$y_0^2 = 20$$

$$y_0 = \sqrt{20} = 2\sqrt{5}$$

SUBSTITUTE F.M.I.T INTO RELATED RATE EQ.

$$(2\sqrt{5})\frac{dy}{dt} = (6)(-3)$$

$$\frac{dy}{dt} = \frac{-18}{2\sqrt{5}}\,\frac{\text{FT}}{\text{SEC}}$$

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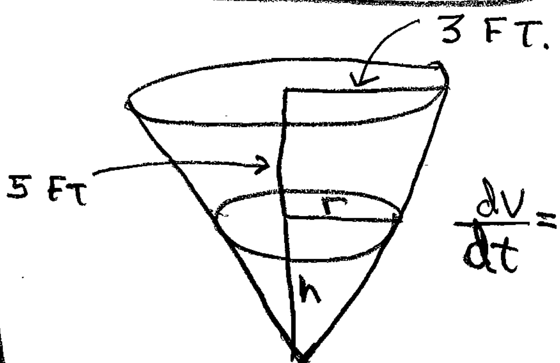
HOMEWORK

PAGE 245, 246: 7, 9, 11, 15, 20

MORE RELATED RATES

A. A CONE SHAPED WATER TANK HAS THE POINT AT THE BOTTOM. IT HAS A RADIUS OF 3 FEET AND A HEIGHT OF 5 FEET. WATER IS BEING PUMPED IN AT A RATE OF 4 CUBIC FEET PER MINUTE. AT WHAT RATE IS THE WATER LEVEL RISING WHEN THE WATER IS 2 FEET DEEP?

MOTION PICTURE	FROZEN MOMENT IN TIME



$$\frac{dV}{dt} = 4\frac{ft^3}{min}$$

ALL TIME EQ.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{h} = \frac{3}{5}$$

$$r = \frac{3}{5}h$$

$$V = \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 h$$

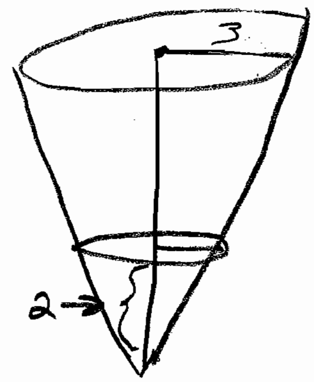
MOTION PICTURE (CONT.)

$$V = \frac{3}{25}\pi h^3$$

$\frac{d}{dt}$ OF BOTH SIDES

$$\frac{dV}{dt} = \frac{9}{25}\pi h^2 \frac{dh}{dt}$$

FROZEN MOMENT IN TIME (CONT.)



SUBSTITUTE F.M.I.T. INTO RELATED RATE EQ.

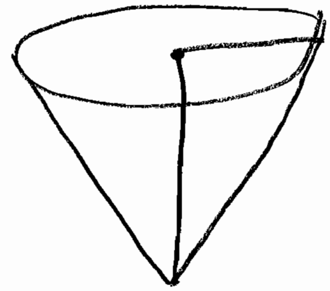
$$4 = \frac{9}{25}\pi(2)^2\frac{dh}{dt}$$

$$4 = \frac{36\pi}{25}\frac{dh}{dt}$$

$$\frac{4(25)}{36\pi} = \frac{dh}{dt} = \frac{25}{9\pi}\ \frac{\text{FT}}{\text{MIN}}$$ ← ANSWER

HOMEWORK

PAGE 246: PROBLEM 23
(CONVERT TO cm)



PAGE 246: PROBLEM 24
(CONVERT TO FEET)

DIFFERENTIALS

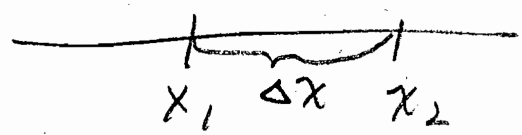
Δx AND Δy WERE SEEN ON PAGES 102-103

A. THE NOTATION CAN BE CONFUSING. SOME PEOPLE USE IT, SO YOU NEED TO SEE IT. CAN BE USED AS A MEMORY DEVICE FOR A LATER TOPIC: INTEGRATION

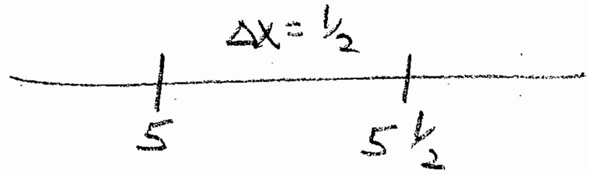
B. Δx READ "DELTA x"

WHEN ASKED "WHAT IS Δx?" IT IS ASSUMED THERE IS A START POSITION x_1 AND A FINISH POSITION x_2

$$\Delta x = x_2 - x_1$$



Δx IS ONE SYMBOL

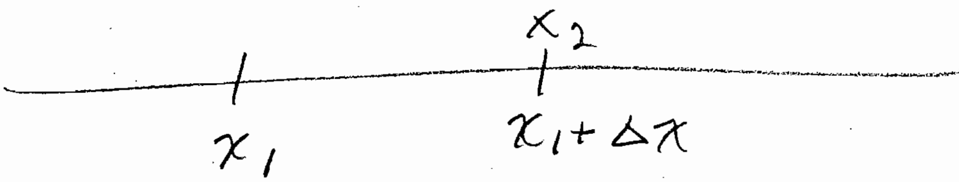


1. LET $x_1 = 5$
   LET $x_2 = 5\frac{1}{2}$
   $\Delta x = x_2 - x_1 = 5\frac{1}{2} - 5 = \frac{1}{2}$

2. LET $x_1 = 5$. LET $x_2 = 4\frac{1}{2}$
   $\Delta x = x_2 - x_1 = 4\frac{1}{2} - 5 = -\frac{1}{2}$

Δx CAN BE POSITIVE
Δx CAN BE NEGATIVE

3. NOTE $x_2 = x_1 + \Delta x$



C. Δy "READ Δy"

TO TALK ABOUT Δy YOU NEED

1. $y = f(x)$ A FUNCTION
2. Δx (i.e. $x_1 \leftarrow$ START $x_2 \leftarrow$ FINISH)

$$\Delta y = f(x_1 + \Delta x) - f(x_1) = f(x_2) - f(x_1)$$

D. FOR Δx SMALL $\boxed{\Delta y \approx f'(x_1)\Delta x}$

REASON: $f'(x_1) = \lim\limits_{h \to 0} \dfrac{f(x_1 + h) - f(x_1)}{h}$

$$= \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

FOR Δx CLOSE TO 0 (i.e. SMALL)

$$f'(x_1) \approx \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

$$f'(x_1)\Delta x \approx f(x_1 + \Delta x) - f(x_1) = \Delta y$$

$$\Delta y \approx f'(x_1)\Delta x$$

E. DIFFERENTIAL NOTATION

YOU TAKE DIFFERENTIALS OF FUNCTIONS OF 1 VARIABLE. THE DIFFERENTIAL OF A FUNCTION OF 1 VARIABLE IS A FUNCTION OF 2 VARIABLES.

LET $y = f(x)$ f IS A FUNCTION OF 1 VARIABLE.

df OR dy IS NOTATION FOR THE DIFFERENTIAL OF f.

DEFINITION: $df(x_1, \Delta x) = f'(x_1)\Delta x$

(i.e. $dy(x_1, \Delta x) = f'(x_1)\Delta x$)

SUPPOSE TEMPORARILY $y = f(x) = x$.

$dy(x_1, \Delta x) = f'(x_1)\Delta x = 1 \cdot \Delta x = \Delta x = dx(x_1, \Delta x)$

SINCE $y = x$

SINCE $f'(x) = 1$

SO $dx(x_1, \Delta x) = \Delta x$, WHICH

SOME BOOKS WRITE AS $dx = \Delta x$ WITH THE EVALUATION PAIR $(x_1, \Delta x)$ UNDERSTOOD. THE FORMULA

END OF TEMPORARY $y = f(x) = x$

$$dy\,(x_1, \Delta x) = f'(x_1)\,\Delta x$$

BECOMES

$$dy\,(x_1, \Delta x) = f'(x_1)\,dx\,(x_1, \Delta x)$$

SOMETIMES WRITTEN AS

$$\boxed{dy = f'(x)\,dx}$$

WITH UNDERSTOOD EVALUATION PAIR $(x_1, \Delta x)$ LEFT OFF

F. TAKING <u>DIFFERENTIALS</u> VS. TAKING <u>DERIVATIVES</u>

DIFFERENTIALS ARE DIFFERENT FROM DERIVATIVES. THE FOLLOWING CHART SHOULD HELP CLARIFY.

DIFFERENTIAL NOTATION CAN CONFUSE CLEAR MINDS, BUT MANY BOOKS USE IT, SO IT IS BEST TO GET FAMILIAR ENOUGH WITH IT SO THAT YOU CAN AT LEAST TAKE DIFFERENTIALS CLEARLY

CHART

DERIVATIVE TAKING VS. DIFFERENTIAL TAKING

FUNCTION	DERIVATIVE	DIFFERENTIAL
$y = x^3$	$y' = 3x^2$	$dy = 3x^2 dx$
$y = t^3$	$y' = 3t^2$	$dy = 3t^2 dt$
$w = \sin v^2$	$\frac{dw}{dv} = w' = (\cos v^2)2v$	$dw = (\cos v^2)2v\,dv$

G. SOME BOOKS DO NOT GIVE DERIVATIVE FORMULA BUT DIFFERENTIAL FORMULAS:

LET $y = \sin x^3$ $\quad dy = (\cos x^3)3x^2 dx$

$$\left.\begin{aligned} \text{LET } u &= x^3 \\ du &= 3x^2 dx \end{aligned}\right\} \text{SO } dy = \cos u\, du$$

SOME DIFFERENTIAL FORMULAS

$y = \cos u \qquad dy = -\sin u\, du$

$y = \tan u \qquad dy = \sec^2 u\, du$

$y = e^u \qquad dy = e^u du$

H. A USE FOR DIFFERENTIALS: TO APPROXIMATE f(GRUBBY) BY f(SIMPLE) WHERE SIMPLE IS CLOSE TO GRUBBY

1. RECALL $dy = f'(x)\Delta x \approx \Delta y$ SO,
2. $\Delta y = f(x+\Delta x) - f(x) \approx f'(x)\Delta x = dy$
3. $$\boxed{\underbrace{f(x+\Delta x)}_{\text{GRUBBY}} \approx f(\underset{\substack{\downarrow\\ \text{SIMPLE}}}{x}) + \underbrace{f'(x)\Delta x}_{\text{DIFFERENTIAL}}}$$

4. APPROXIMATE $\sqrt{16.2}$ USING DIFFERENTIALS (i.e. 3. ABOVE)

YOU NEED: FUNCTION f, SIMPLE x, AND GRUBBY $x+\Delta x$

$f(x) = \sqrt{x} = x^{1/2}$

SIMPLE $x = 16$

GRUBBY $x + \Delta x = 16.2$

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TO APPLY THE FORMULA YOU NEED $f'(x)$ AND Δx. RECALL $f(x) = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$16.2 = x + \Delta x = 16 + \Delta x.$$

$$.2 = 16.2 - 16 = \Delta x$$

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$$

$$\sqrt{16.2} = f(16.2) \approx f(16) + f'(16)(.2)$$

$$\sqrt{16.2} \approx \sqrt{16} + \frac{1}{2\sqrt{16}}(.2)$$

$$= 4 + \frac{1}{2(4)}(.2) = 4 + \frac{2}{2(4)(10)}$$

$$= 4 + \frac{1}{40} = \frac{160+1}{40} = \frac{161}{40}$$

APPROXIMATING $\sqrt{16.2}$ BY DIFFERENTIALS

$$\sqrt{16.2} \approx \frac{161}{40} = 4.025$$

APPROXIMATING $\sqrt{16.2}$ BY CALCULATOR

$$\sqrt{16.2} \approx 4.024922359$$

THEY DIFFER BY $\approx .000077641$ CLOSE!

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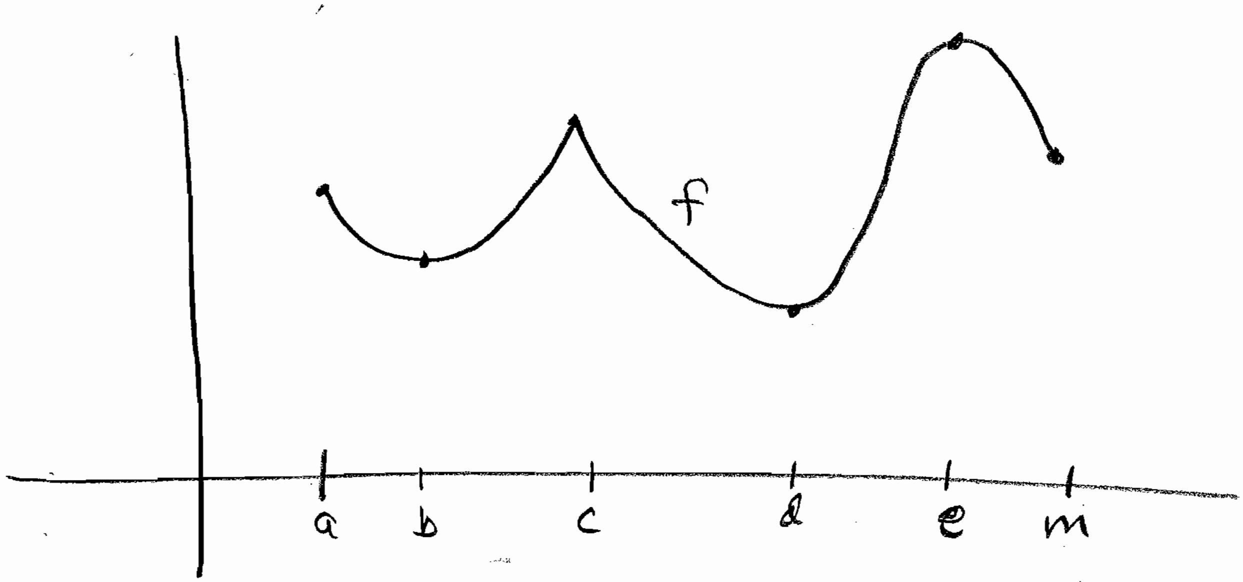
PAGE 252: 11, 13, 14, 23, 28

INTRODUCTION TO MAXIMUM AND MINIMUM VALUES



LET $I = [a, m]$

DEFINITIONS:

A. ABSOLUTE MAXIMUM VALUE OF f ON I: $p \in I$

$f(p)$ IS THE ABSOLUTE MAXIMUM VALUE OF f ON I IFF FOR EACH $x \in I$, $f(x) \leq f(p)$.

{AN ABSOLUTE MAXIMUM VALUE FOR f OCCURS AT p} $f(e)$ IS THE ABSOLUTE MAX VALUE OF f ON I ABOVE. AN ABSOLUTE MAX VALUE OCCURS AT e.

B. $f(p)$ IS THE ABSOLUTE MIN VALUE FOR f ON I IFF FOR EACH $x \in I$, $f(p) \leq f(x)$.

{AN ABSOLUTE MIN VALUE FOR f OCCURS AT p}. $f(d)$ IS THE ABSOLUTE MIN VALUE OF f ON I.

C. ABSOLUTE MAX VALUES AND ABSOLUTE MIN VALUES ARE CALLED ABSOLUTE EXTREME VALUES (I.E. ABSOLUTE EXTREMA ← PLURAL) SINGULAR: ABSOLUTE EXTREMUM

D. (RELATIVE) LOCAL MAX AND MIN VALUES DEFINITIONS:

$f(p)$ IS A LOCAL MAXIMUM VALUE FOR f IFF THERE IS AN OPEN INTERVAL (h,k) CONTAINING p SUCH THAT FOR ALL $x \in (h,k)$, $f(x) \leq f(p)$.

$f(p)$ IS A (RELATIVE) LOCAL MINIMUM VALUE FOR f IFF THERE IS AN OPEN INTERVAL (h,k) CONTAINING p SUCH THAT FOR ALL $x \in (h,k)$, $f(p) \leq f(x)$

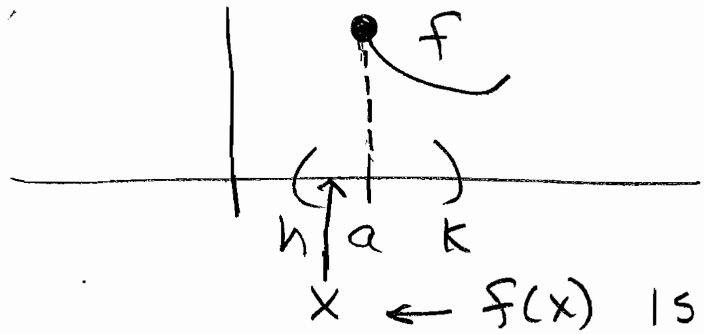
LOCAL MAX AND LOCAL MIN VALUES ARE CALLED LOCAL EXTREME VALUES.

$f(p)$ IS A LOCAL EXTREME VALUE MEANS A LOCAL EXTREME VALUE OCCURS AT p.

LOCAL MIN VALUES IN PREVIOUS PICTURE $f(b)$ AND $f(d)$

LOCAL MAX VALUES: $f(c)$ AND $f(e)$

E. END POINTS CANNOT BE LOCAL EXTREMA

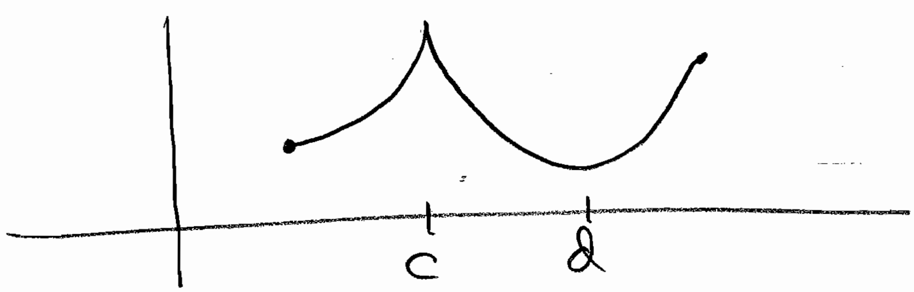


$\leftarrow f(x)$ is not defined, so

it is impossible for

for all $x \in (h,k)$ $f(x) \le f(a)$

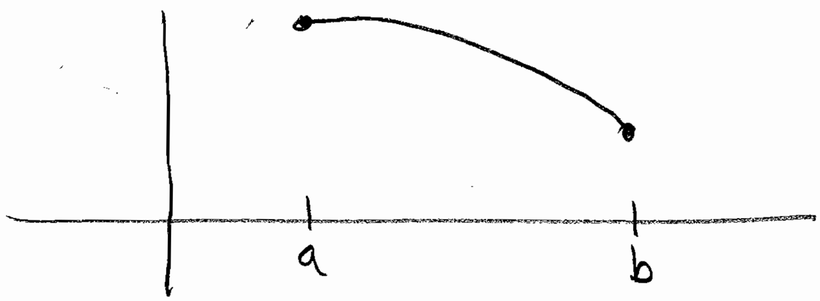
F. ABSOLUTE EXTREMA CAN BE LOCAL EXTREMA



$f(c)$ LOCAL AND ABSOLUTE MAX

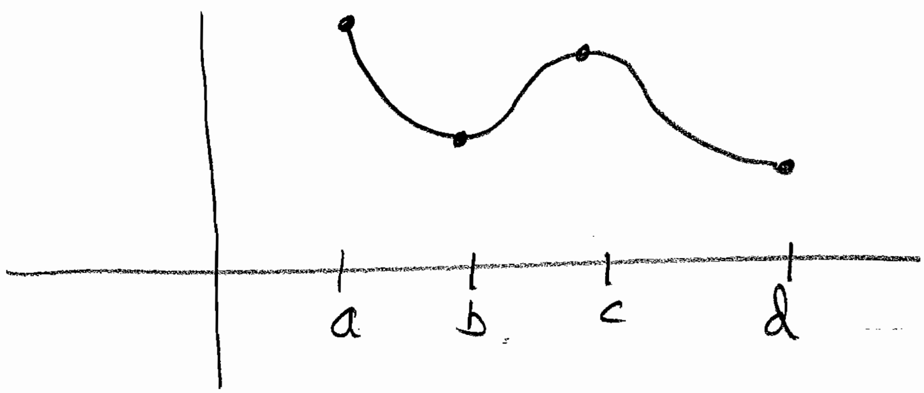
$f(d)$ LOCAL AND ABSOLUTE MIN

G. IT IS POSSIBLE THAT ABSOLUTE EXTREMA ARE NOT LOCAL EXTREMA



f(a) ABSOLUTE MAX, NOT LOCAL MAX
f(b) ABSOLUTE MIN, NOT LOCAL MIN

H. LOCAL EXTREMA ARE NOT NECESSARILY ABSOLUTE EXTREMA



f(b) AND f(c) ARE LOCAL EXTREMA BUT NOT ABSOLUTE EXTREMA
f(a) AND f(d) ARE THE ABSOLUTE EXTREMA.

I. <u>CRITICAL POINTS</u>: (LEARN WHAT THEY ARE NOW; THEY WILL RELATE TO LOCAL EXTREMA) ALSO CALLED <u>CRITICAL NUMBERS</u>.

C IS A CRITICAL NUMBER FOR f <u>IFF</u>

1) $c \in \text{dom}(f)$

AND

2) <u>EITHER</u> $f'(c) = 0$ <u>OR</u> $f'(c)$ DOES NOT EXIST

J. <u>EXAMPLE</u>: $f(x) = x^{1/3}$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

$0 \in \text{dom}(f)$

$f'(0)$ DOES NOT EXIST (DIVISION BY 0)

SO 0 IS A CRITICAL NUMBER

K. <u>EXAMPLE</u> $f(x) = x^{-1}$. $f'(x) = -x^{-2}$

$f'(x) = \frac{-1}{x^2}$. SO $f'(0)$ DOES NOT EXIST. BUT 0 IS NOT A CRITICAL NUMBER SINCE $0 \notin \text{dom}(f)$. $f(x) = x^{-1} = \frac{1}{x}$

L. <u>EXAMPLE</u> $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2}$

$f'(x) = \frac{1}{2}(x^2)^{-\frac{1}{2}} 2x = \dfrac{x}{(x^2)^{\frac{1}{2}}} = \dfrac{x}{|x|}$

$0 \in dom(f)$

$f'(0)$ DOES NOT EXIST (0 IN DENOMINATOR)

0 IS A CRITICAL NUMBER

M. <u>EXAMPLE</u>: FOR $f(x) = x^{2/3}(5-x)$

FIND ALL CRITICAL NUMBERS

$f'(x) = x^{\frac{2}{3}}(-1) + (5-x)\frac{2}{3}x^{-\frac{1}{3}} = \dfrac{-x^{\frac{2}{3}} + 2(5-x)}{3x^{1/3}}$

$f'(x) = \dfrac{-3x^{\frac{2}{3}+\frac{1}{3}} + 2(5-x)}{3x^{1/3}} = \dfrac{-3x + 10 - 2x}{3x^{1/3}}$

$f'(x) = \dfrac{-5x+10}{3x^{1/3}} = \dfrac{-5(x-2)}{3x^{1/3}}$

$0, 2 \in dom(f)$

$f'(2) = 0$. $f'(0)$ DOES NOT EXIST (0 IN DENOMINATOR)

0, 2 ARE CRITICAL NUMBERS

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HOMEWORK

PAGES 277, 278:

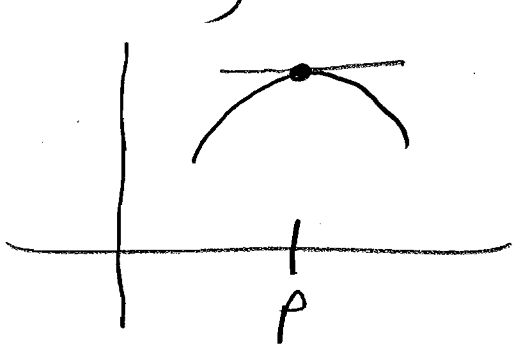
3, 4, 5, 29, 31, 34, 39, 40, 41, 43

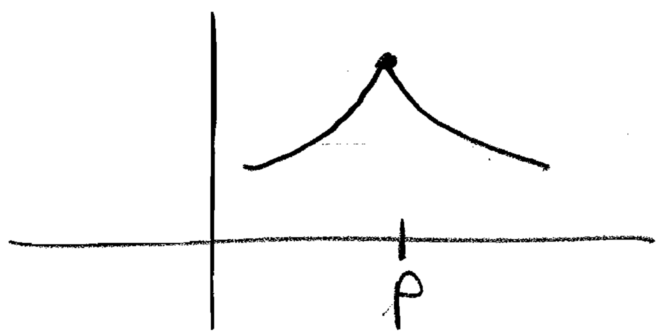
MAX\MIN VALUES ON A CLOSED INTERVAL

A. **THEOREM:** IF f HAS A LOCAL MAXIMUM OR A LOCAL MINIMUM AT p THEN p IS A CRITICAL NUMBER (I.E. $f'(p)=0$ OR $f'(p)$ DOES NOT EXIST)

INTUITION FOR LOCAL MAXIMUM:

AS IT HEADS TO $f(p)$ IT MUST TURN AROUND AND COME DOWN. IT EITHER DOES IT SMOOTHLY ($f'(p)=0$) OR WITH A SHARP POINT ($f'(p)$ DOES NOT EXIST).





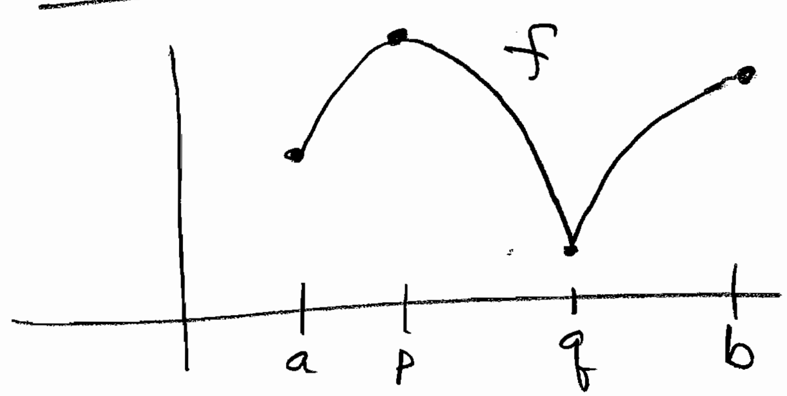
EXTREME VALUE THEOREM

B. THEOREM: IF f IS CONTINUOUS ON $[a,b]$, THEN THERE IS A $p \varepsilon [a,b]$ SUCH THAT $f(p)$ IS THE ABSOLUTE MAX VALUE OF f ON $[a,b]$ AND THERE IS A $q \varepsilon [a,b]$ SUCH THAT $f(q)$ IS THE ABSOLUTE MIN VALUE OF f ON $[a,b]$

NOTE: CONTINUOUS

NOTE: ON A CLOSED INTERVAL.

C. EXAMPLE.



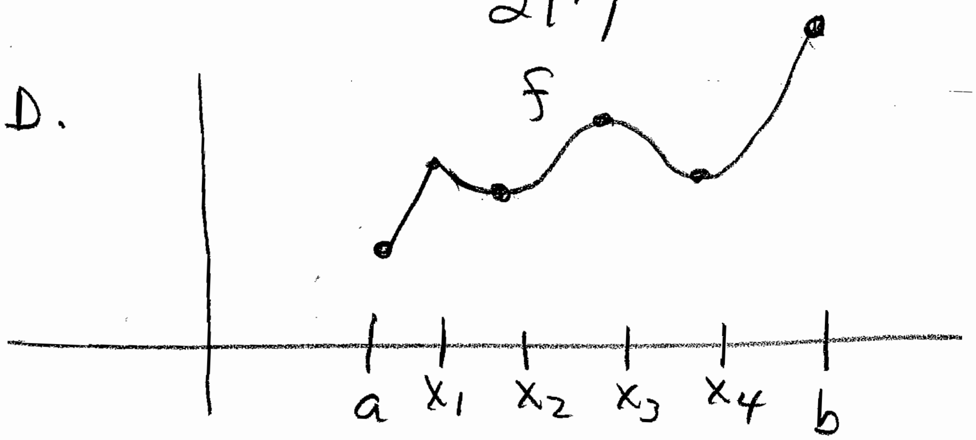
$f(p)$ ABSOLUTE MAX VALUE ON $[a,b]$

p IS A CRITICAL NUMBER

$f(q)$ ABSOLUTE MIN VALUE ON $[a,b]$

q IS A CRITICAL NUMBER.

D.



$f(b)$ ABS. MAX VALUE OF f ON $[a,b]$
b IS AN END POINT
$f(a)$ ABS. MIN VALUE OF f ON $[a,b]$
a IS AN END POINT
NOTE x_1, x_2, x_3, AND x_4 ARE CRITICAL NUMBERS IN (a,b) BUT THE ABSOLUTE EXTREMA DID NOT OCCUR THERE IN THIS EXAMPLE

BUT IN BOTH OF THE PREVIOUS EXAMPLES THE ABSOLUTE EXTREME VALUES (THAT WE KNOW HAVE TO EXIST) OCCURRED EITHER AT ONE OF THE END POINTS OF $[a,b]$ OR AT A CRITICAL NUMBER IN (a,b).

THIS LEADS TO . . . $\longrightarrow$

E. METHOD FOR FINDING ABSOLUTE EXTREMA OF CONTINUOUS f ON $[a,b]$

1. FIND ALL CRITICAL NUMBERS $c_1, c_2, \ldots, c_K$ OF f IN (a,b)

   NOTE: THEY MUST BE IN (a,b)

2. EVALUATE f AT THE END POINTS OF $[a,b]$ AND AT THE CRITICAL NUMBERS (i.e. FIND $f(a), f(b), f(c_1), f(c_2), \ldots, f(c_K)$)

3. THE LARGEST VALUE IN THE LIST IS THE ABSOLUTE MAX VALUE; THE SMALLEST VALUE IN THE LIST IS THE ABSOLUTE MIN VALUE.

THIS LEADS US TO THE JUGULAR #2 PROBLEM TYPE: FINDING ABSOLUTE EXTREMA FOR A FUNCTION f ON A CLOSED INTERVAL $[a,b]$.

F. FIND THE ABSOLUTE EXTREME <u>VALUES</u> FOR $f(x) = 3x^{\frac{1}{3}}(x+1)^2$

1) ON $[-8, 1]$

2) ON $[-\frac{5}{8}, 1]$

WORK PART 1)

FIND ALL CRITICAL NUMBERS IN $(-8, 1)$

$$f'(x) = 3x^{\frac{1}{3}}2(x+1) + (x+1)^2 x^{-\frac{2}{3}}$$

$$f'(x) = (x+1)\left[6x^{\frac{1}{3}} + \frac{(x+1)}{x^{\frac{2}{3}}}\right]$$

$$f'(x) = (x+1)\left[\frac{6x^{\frac{1}{3}+\frac{2}{3}} + (x+1)}{x^{\frac{2}{3}}}\right]$$

$$f'(x) = \frac{(x+1)(7x+1)}{x^{\frac{2}{3}}}$$

$f'(-1) = 0 \quad f'(-\frac{1}{7}) = 0 \quad f'(0)$ DOES NOT EXIST

CRITICAL NUMBERS IN $(-8, 1)$ ARE $-1, -\frac{1}{7}, 0$ SINCE THEY ARE IN dom(f)

FIND $f(-8)$, $f(1)$, $f(-1)$, $f(-\frac{1}{7})$, $f(0)$

$f(-8) = 3(-8)^{\frac{1}{3}}(-8+1)^2 = 3(-2)(-7)^2 = -294$

$f(1) = 3(1)^{\frac{1}{3}}(1+1)^2 = 3(2^2) = 12$

$f(-1) = 3(-1)^{\frac{1}{3}}(-1+1)^2 = 3(-1)(0^2) = 0$

$f(-\frac{1}{7}) = 3(-\frac{1}{7})^{\frac{1}{3}}(-\frac{1}{7}+1)^2 = 3(-\frac{1}{7})^{\frac{1}{3}}(\frac{6}{7})^2 \approx -1.15$

$f(0) = 3(0^{\frac{1}{3}})(0+1)^2 = 0$

$f(-8) = -294$ ABS. MIN VALUE

$f(1) = 12$ ABS. MAX VALUE

PART 2)

FIND ABS. EXTREMA ON $[-\frac{5}{8}, 1]$

1. ALL THE CRITICAL NUMBERS IN $(-\frac{5}{8}, 1)$ ARE $-\frac{1}{7}$, 0

NOTE: -1 IS A CRITICAL NUMBER, BUT NOT IN $(-\frac{5}{8}, 1)$

FIND $f(-\frac{5}{8})$, $f(1)$, $f(-\frac{1}{7})$, $f(0)$

$$f\left(-\frac{5}{8}\right) = 3\left(-\frac{5}{8}\right)^{\frac{1}{3}}\left(-\frac{5}{8}+1\right)^2 = 3\,\frac{(-5)^{\frac{1}{3}}}{2}\left(\frac{9}{64}\right) \approx -.36$$

$$f(1) = 3(1)^{\frac{1}{3}}(1+1)^2 = 3(2^2) = 12$$

$$f\left(-\frac{1}{7}\right) = 3\left(-\frac{1}{7}\right)^{\frac{1}{3}}\left(-\frac{1}{7}+1\right)^2 = \frac{-3}{\sqrt[3]{7}}\left(\frac{36}{49}\right) \cong -1.15$$

$$f(0) = 3(0^{\frac{1}{3}})(0+1)^2 = 0$$

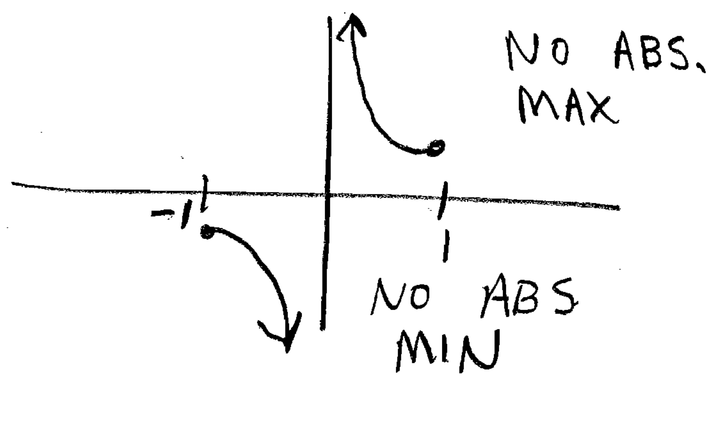
$f(-\frac{1}{7}) \cong -1.15$ ABS. MIN VALUE ON $[-\frac{5}{8}, 1]$

$f(1) = 12$ ABS. MAX VALUE ON $[-\frac{5}{8}, 1]$

G. NOTE: f MUST BE CONTINUOUS ON $[a,b]$ TO ASSURE THE EXISTENCE OF ABSOLUTE EXTREMA

$f(x) = \frac{1}{x}$

ON $[-1, 1] - \{0\}$



HOMEWORK

Page 278 : 55, 56

ALSO,

FOR $f(x) = x^{\frac{1}{3}}(x-3)^2$ FIND THE ABSOLUTE EXTREMA FOR f ON $[-1, 8]$

ROLLE'S THEOREM AND MEAN VALUE THEOREM

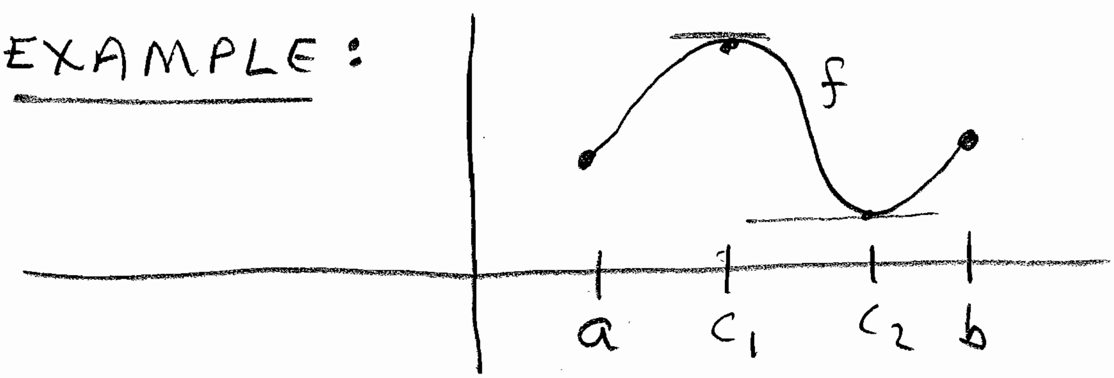
ROLLE'S THEOREM

A. IF ① f IS CONTINUOUS ON $[a,b]$, ② f IS DIFFERENTIABLE ON (a,b), AND ③ $f(a) = f(b)$, THEN THERE IS A $c \in (a,b)$ SUCH THAT

$$f'(c) = 0.$$

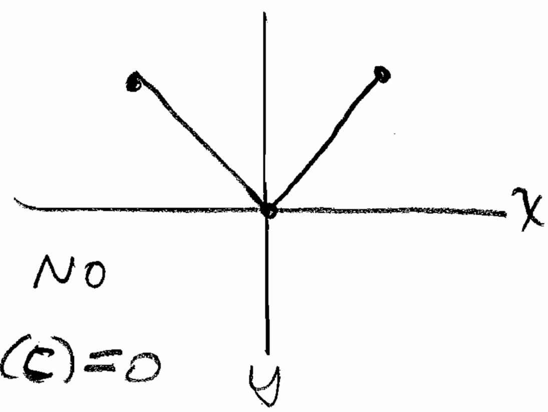
(PROOF NOT GIVEN)

B. EXAMPLE:



"THERE IS" IN MATH MEANS "THERE IS AT LEAST ONE". IN THIS PICTURE THERE ARE 2 c's SUCH THAT $f'(c) = 0$ NAMELY c_1 AND c_2 (i.e. $f'(c_1) = 0$, $f'(c_2) = 0$)

C. EXAMPLE: $f(x) = |x|$ on $[-1, 1]$
HAS ① f IS CONTINUOUS ON $[-1, 1]$
AND ② $f(-1) = f(1)$, BUT f IS
NOT DIFFERENTIABLE
ON $(-1, 1)$ SINCE IT IS
NOT DIFFERENTIABLE
AT 0. NOTE, THERE IS NO
$c \in (-1, 1)$ SUCH THAT $f'(c) = 0$

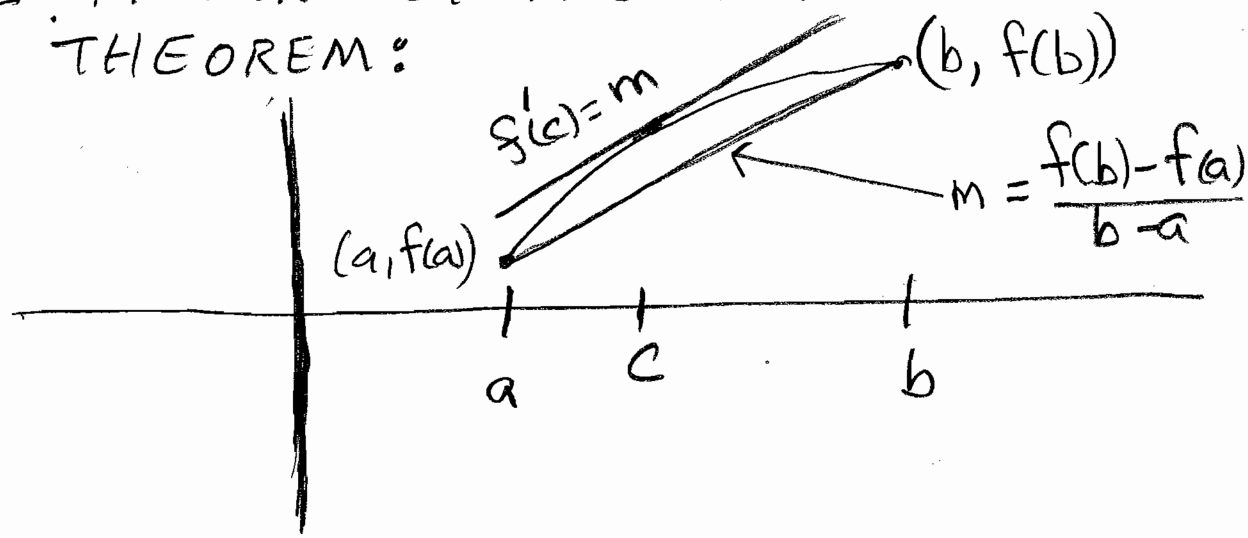


NO SURPRISE... THE HYPOTHESIS OF
ROLLE'S THEOREM IS NOT SATISFIED.

D. **MEAN VALUE THEOREM FOR DERIVATIVES**: IF ① f IS CONTINUOUS ON $[a, b]$ AND ② f IS DIFFERENTIABLE ON (a, b), THEN THERE IS A $c \in (a, b)$ SUCH THAT

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

E. PICTURE OF THE MEAN VALUE THEOREM:



THE SLOPE OF THE STRAIGHT LINE BETWEEN $(a, f(a))$ AND $(b, f(b))$ IS

$$m = \frac{f(b) - f(a)}{b-a}$$

WHEN THE MEAN VALUE THEOREM HYPOTHESIS IS SATISFIED WE KNOW THERE IS A $c \in (a,b)$ SUCH THAT

THE TANGENT LINE TO THE GRAPH OF f AT $(c, f(c))$ IS PARALLEL

TO THE LINE BETWEEN

$(a, f(a))$ AND $(b, f(b))$

F. LET $f(x) = x^3 - 2x$ ON $[1, 4]$

f IS CONTINUOUS ON $[1, 4]$ SINCE IT IS A POLYNOMIAL

$f'(x) = 3x^2 - 2$ EXISTS FOR EACH $x \in (1, 4)$ SO f IS DIFFERENTIABLE ON $(1, 4)$. SO THE MEAN VALUE THEOREM HYPOTHESIS IS SATISFIED. HENCE THERE IS A $c \in (1, 4)$ SUCH THAT $f'(c) = \frac{f(4) - f(1)}{4 - 1}$. FIND ONE SUCH c.

$f(4) = 4^3 - 2(4) = 64 - 8 = 56$

$f(1) = 1^3 - 2(1) = 1 - 2 = -1$

$$\frac{f(4) - f(1)}{4 - 1} = \frac{56 - (-1)}{3} = \frac{57}{3} = 19$$

$f'(c) = 3c^2 - 2 = 19$

$3c^2 = 21$. $c^2 = 7$

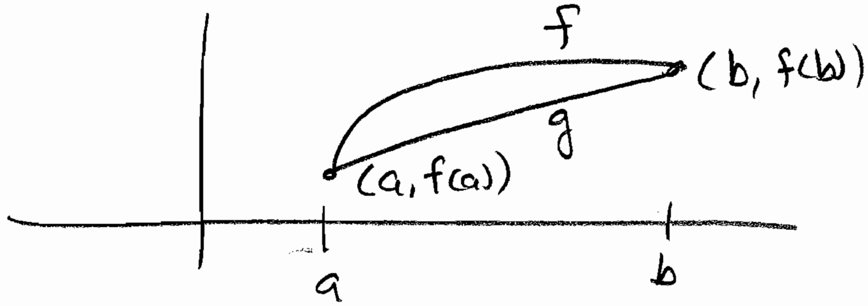
$\sqrt{c^2} = \sqrt{7}$. $|c| = \sqrt{7}$. $c = \pm\sqrt{7}$

$-\sqrt{7} \notin (1, 4) \leftarrow -\sqrt{7}$ NOT ACCEPTABLE

ANSWER $c = +\sqrt{7}$

G. PROOF IDEA FOR MEAN VALUE THEOREM FOR DERIVATIVES:

1. SUPPOSE f IS CONTINUOUS ON $[a,b]$ AND f IS DIFFERENTIABLE ON (a,b). (SHOW THERE IS A $c \in (a,b)$ SUCH THAT $f'(c) = \frac{f(b)-f(a)}{b-a}$)



2. LET g BE THE STRAIGHT LINE BETWEEN $(a, f(a))$ AND $(b, f(b))$)
3. LET $h(x) = f(x) - g(x)$.
4. $h(a) = f(a) - g(a) = 0 = f(b) - g(b) = h(b)$
5. h IS CONT. ON $[a,b]$, AND DIFF. ON (a,b)
6. BY ROLLE'S THEOREM, THERE IS A $c \in (a,b)$ SUCH THAT $h'(c) = 0$
7. $0 = h'(c) = f'(c) - g'(c) = 0$
8. $f'(c) = g'(c) = \frac{f(b)-f(a)}{b-a}$ = SLOPE OF STRAIGHT LINE g

H. THEOREM: IF FOR ALL $x \in (a,b)$, $f'(x) = 0$, THEN f IS CONSTANT ON (a,b) (i.e. FOR EVERY $x_1, x_2 \in (a,b)$, IF $x_1 < x_2$ THEN $f(x_1) = f(x_2)$)

PROOF: 1. ASSUME FOR ALL $x \in (a,b)$ $f'(x) = 0$ AND $x_1, x_2 \in (a,b)$ AND $x_1 < x_2$. (SHOW $f(x_1) = f(x_2)$)

2. SO f IS CONTINUOUS ON $[x_1, x_2]$ SINCE IT IS DIFFERENTIABLE ON $[x_1, x_2]$. ALSO, f IS DIFF. ON (x_1, x_2). BY THE MEAN VALUE THEOREM APPLIED TO $[x_1, x_2]$, THERE IS A $c \in (x_1, x_2)$ SUCH THAT

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \stackrel{1}{=} f'(c)$$

3. $f(x_2) - f(x_1) = 0$

4. $f(x_2) = f(x_1)$

I. **THEOREM**: **IF** FOR ALL $x \in (a,b)$, $f'(x) = g'(x)$, **THEN** f AND g DIFFER BY A CONSTANT ON (a,b).

PROOF: 1. ASSUME FOR ALL $x \in (a,b)$, $f'(x) = g'(x)$. (SHOW f AND g DIFFER BY A CONSTANT ON (a,b).)

2. LET $h(x) = f(x) - g(x)$ FOR ALL $x \in (a,b)$.

3. FOR ALL $x \in (a,b)$, $h'(x) = f'(x) - g'(x) \overset{1}{=} 0$

4. BY THE PREVIOUS THEOREM, h IS CONSTANT ON (a,b), SO THERE IS A CONSTANT C SUCH THAT FOR ALL $x \in (a,b)$, $h(x) = C$

5. FOR ALL $x \in (a,b)$, $h(x) \overset{2}{=} f(x) - g(x) = C$ (REASONS: LINES 2 AND 4)

6. FOR ALL $x \in (a,b)$, $f(x) \overset{5}{=} g(x) + C$

7. f AND g DIFFER BY A CONSTANT ON (a,b).

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HOMEWORK

PAGE 285, 286: 1, 2, 5, 11, 12, 13, 15

FIRST DERIVATIVE: INCREASING, DECREASING, TEST FOR LOCAL EXTREMA

A. **THEOREM**: IF FOR ALL $x \in (a,b)$, $f'(x) > 0$, THEN f IS INCREASING ON (a,b)

PROOF: 1. ASSUME FOR ALL $x \in (a,b)$, $f'(x) > 0$.
(SHOW f IS INCREASING ON (a,b)).
(SHOW FOR ALL $x_1, x_2 \in (a,b)$, IF $x_1 < x_2$, THEN $f(x_1) < f(x_2)$.)

2. ASSUME $a < x_1 < x_2 < b$
(SHOW $f(x_1) < f(x_2)$)

3. SINCE f IS CONTINUOUS ON $[x_1, x_2]$ AND DIFFERENTIABLE ON (x_1, x_2) THEN BY THE MEAN VALUE THEOREM, THERE IS A $c \in (x_1, x_2)$ SUCH THAT

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \overset{1}{>} 0$$

4. $f(x_2) - f(x_1) > 0$ 3

5. $f(x_2) > f(x_1)$ [i.e. $f(x_1) < f(x_2)$]

B **THEOREM**: IF FOR ALL $x \in (a,b)$, $f'(x) < 0$, THEN f IS DECREASING ON (a,b)

PROOF SIMILAR TO PREVIOUS THEOREM.

C. FOR $f(x) = -x^3 + 12x$, FIND THE OPEN INTERVALS WHERE f INCREASES AND WHERE f DECREASES.

$$f'(x) = -3x^2 + 12 = -3(x^2 - 4)$$

$$f'(x) = -3(x-2)(x+2)$$

-3	$-$	$-$	$-$
$x-(-2)$	L–R $-$	R–L $+$	R–L $+$
$x-2$	L–R $-$	L–R $-$	R–L $+$
	$f' < 0$	$f' > 0$	$f' < 0$
	f DEC.	f INC	f DEC

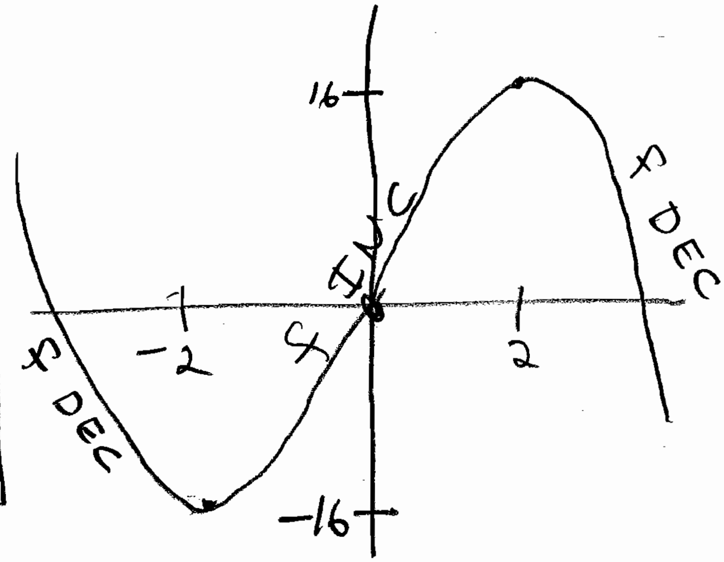
Division points: -2, 2

f DECREASES ON $(-\infty, -2), (2, \infty)$

f INCREASES ON $(-2, 2)$

LET'S SKETCH

x	$f(x)$
0	0
-2	$-(-2)^3+12(-2)$ $8-24=-16$
2	$-2^3+12(2)=$ $-8+24=16$



D. FIRST DERIVATIVE TEST FOR LOCAL EXTREMA: SUPPOSE c IS A CRITICAL NUMBER OF CONTINUOUS f.

IF $\underset{h}{(}\overset{f'>0}{\underset{f \text{ INC}}{\rule{2em}{0.4pt}}}\underset{c}{|}\overset{f'<0}{\underset{f \text{ DEC}}{\rule{2em}{0.4pt}}}\underset{k}{)}$, THEN $f(c)$ IS A LOCAL MAX. VALUE

IF $\underset{h}{(}\overset{f'<0}{\underset{f \text{ DEC}}{\rule{2em}{0.4pt}}}\underset{c}{|}\overset{f'>0}{\underset{f \text{ INC}}{\rule{2em}{0.4pt}}}\underset{k}{)}$, THEN $f(c)$ IS A LOCAL MIN. VALUE

IF $\underset{h}{(}\overset{f'>0}{\underset{f \text{ INC}}{\rule{2em}{0.4pt}}}\underset{c}{|}\overset{f'>0}{\underset{f \text{ INC}}{\rule{2em}{0.4pt}}}\underset{k}{)}$ OR $\underset{h}{(}\overset{f'<0}{\underset{f \text{ DEC}}{\rule{2em}{0.4pt}}}\underset{c}{|}\overset{f'<0}{\underset{f \text{ DEC}}{\rule{2em}{0.4pt}}}\underset{k}{)}$,

THEN $f(c)$ IS NEITHER A LOCAL MAX NOR A LOCAL MIN VALUE

E. IN THE PREVIOUS EXAMPLE,

$$f(x) = -x^3 + 12x$$

$$f'(x) = -3(x-2)(x+2)$$

0, 2 CRITICAL NUMBERS

$f' < 0$		$f' > 0$		$f' < 0$
f DEC	-2	f INC	2	f DEC

$f(-2) = -16$ LOCAL MIN VALUE

$f(2) = 16$ LOCAL MAX VALUE

F. FOR $f(x) = \frac{3}{2} x^{\frac{2}{3}} (x-1)^2$,

FIND ALL LOCAL EXTREMA BY THE FIRST DERIVATIVE TEST.

$$f'(x) = \frac{3}{2} x^{\frac{2}{3}} 2(x-1) + (x-1)^2 x^{-\frac{1}{3}}$$

$$f'(x) = 3 x^{\frac{2}{3}} (x-1) + \frac{(x-1)^2}{x^{\frac{1}{3}}}$$

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$$f'(x) = \frac{3x(x-1) + (x-1)^2}{x^{1/3}}$$

$$f'(x) = \frac{(x-1)(3x+(x-1))}{x^{\frac{1}{3}}} = \frac{(x-1)(4x-1)}{x^{1/3}}$$

$$f'(x) = \frac{(x-1)4(x-\frac{1}{4})}{x^{1/3}}$$

CRITICAL NUMBERS: $0, \frac{1}{4}, 1$

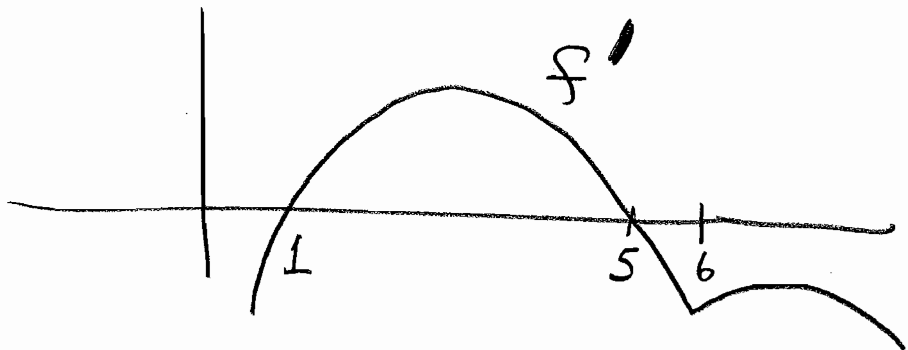
4	$+$	$+$	$+$	$+$
$x^{\frac{1}{3}}$	$-$	$+$	$+$	$+$
$x-1$	L-R $-$	L-R $-$	L-R $-$	R-L $+$
$x-\frac{1}{4}$	L-R $-$	L-R $-$	R-L $+$	R-L $+$
	$f'<0$	$f'>0$	$f'<0$	$f'>0$
	f DEC	f INC	f DEC	f INC

Boundaries: 0, $\frac{1}{4}$, 1

LOCAL MIN OCCURS AT $0, 1$

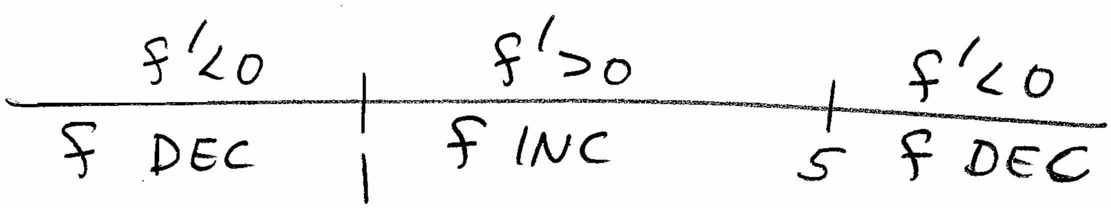
LOCAL MAX OCCURS AT $\frac{1}{4}$

6. BELOW IS THE GRAPH OF f'. FIND ALL LOCAL EXTREMA FOR f.



$f'(c)=0$ AT $1, 5$ $\qquad$ $c \in \mathrm{dom}(f)$ FOR $c = 1, 5$

CRITICAL NUMBERS $1, 5$



$f(1)$ LOCAL MIN VALUE

$f(5)$ RELATIVE MAX VALUE

$\boxed{\text{HOMEWORK}}$

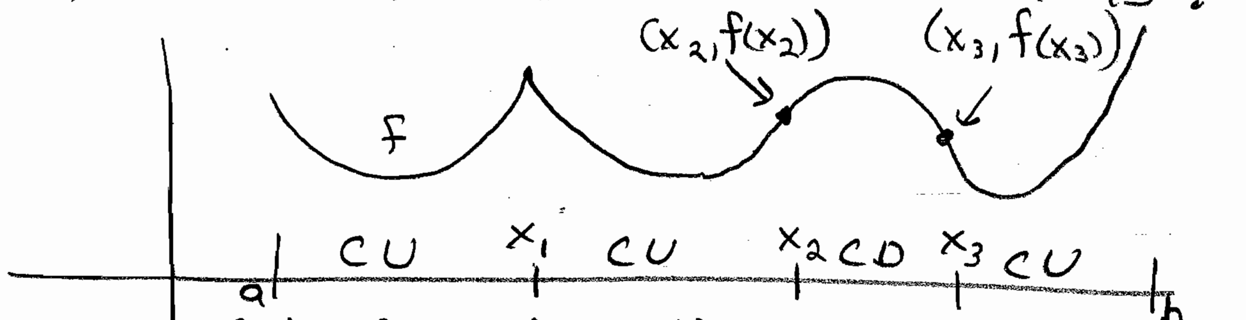
PAGES 295, 296: 5, 6, 9ab, 11ab, 22ac, 41ab

SECOND DERIVATIVE: CONCAVE UP, CONCAVE DOWN, 2^{nd} DERIVATIVE TEST FOR LOCAL EXTREMA

A. PORTIONS OF GRAPHS THAT ARE CONCAVE UPWARD:

B. PORTIONS OF GRAPHS THAT ARE CONCAVE DOWNWARD:

C. CONCAVITY AND INFLECTION POINTS:



CU = CONCAVE UP   CD = CONCAVE DOWN

<u>INFLECTION POINT DEFINITION</u> FOR POINT P ON THE GRAPH OF f: THE GRAPH CHANGES FROM CONCAVE UP TO CONCAVE DOWN AT P OR FROM CONCAVE DOWN TO CONCAVE UP AT P. $(x_2, f(x_2))$ AND $(x_3, f(x_3))$ ARE INFLECTION POINTS.

D. SECOND DERIVATIVE AND CONCAVITY:

SUPPOSE I IS AN INTERVAL.

[FOR ALL $x \in I$, $f''(x) > 0$] IMPLIES [f IS CONCAVE UP ON I].

[FOR ALL $x \in I$ $f''(x) < 0$] IMPLIES [f IS CONCAVE DOWN ON I].

E. CONCAVITY EXAMPLE: $f(x) = x^4 - 2x^3$

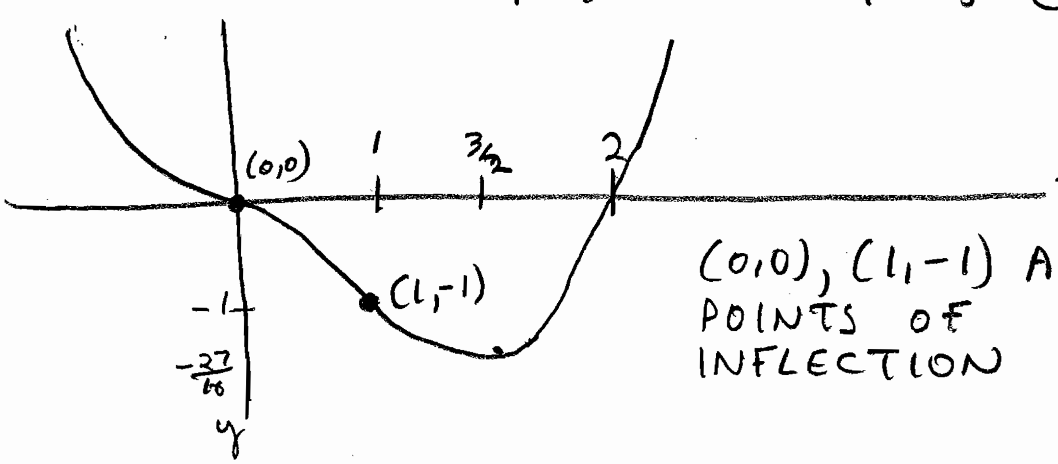
$f'(x) = 4x^3 - 6x^2$. $f''(x) = 12x^2 - 12x$

$f''(x) = 12x(x-1)$

12	+	+	+
x	−	+	+
$x-1$	L−R −	L−R −	R−L +

$f'' > 0$ | 0 | $f'' < 0$ | 1 | $f'' > 0$

f CONCAVE UP | f CD | f CU



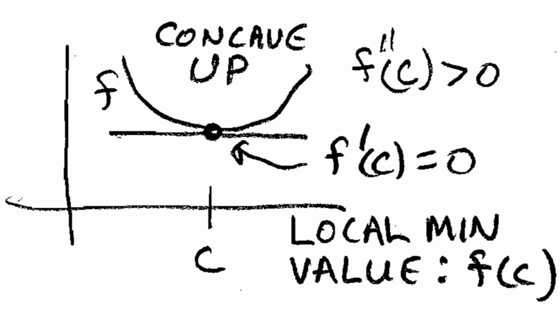
(0,0), (1,−1) ARE POINTS OF INFLECTION

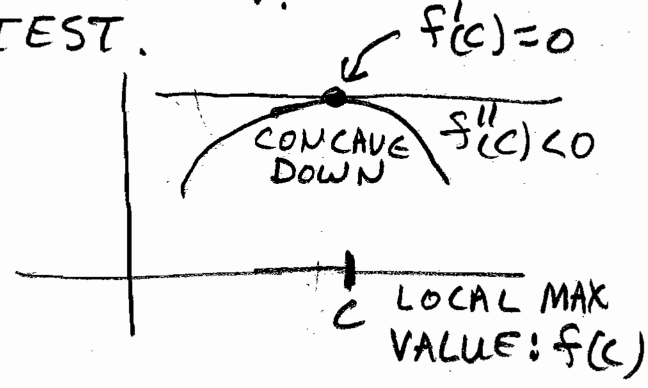
F. SECOND DERIVATIVE TEST FOR LOCAL EXTREMA: SUPPOSE $c \in (a,b)$ AND f'' IS CONTINUOUS ON (a,b).
FOR $f'(c) = 0$,
IF $f''(c) > 0$, THEN $f(c)$ IS A LOCAL MIN VALUE
IF $f''(c) < 0$, THEN $f(c)$ IS A LOCAL MAX VALUE

NOTE: THE 2nd DERIVATIVE TEST APPLIES ONLY TO THOSE TYPES OF CRITICAL NUMBERS c WHERE $f'(c) = 0$; IF $f'(c)$ DOES NOT EXIST AND c IS A CRITICAL NUMBER THEN TRY THE FIRST DERIVATIVE TEST TO CHECK FOR LOCAL EXTREMA.

NOTE: THE 2nd DERIVATIVE TEST IS ONLY FOR FINDING LOCAL NOT ABSOLUTE EXTREMA.

NOTE: FOR THE ASSUMPTIONS OF THE 2nd DERIVATIVE TEST, IF $f''(c) = 0$ YOU KNOW NOTHING ABOUT LOCAL EXTREMA. TRY THE FIRST DERIVATIVE TEST.





G. <u>EXAMPLE</u>: FOR $f(x) = 2x^3 - 3x^2 - 12x$, FIND ALL LOCAL EXTREMA BY THE SECOND DERIVATIVE TEST.

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

$f'(x) = 6(x-2)(x+1)$

$f'(2) = 0$ AND $f'(-1) = 0$

$f''(x) = 12x - 6$

$f''(2) = 12(2) - 6 = 24 - 6 = 18 > 0$

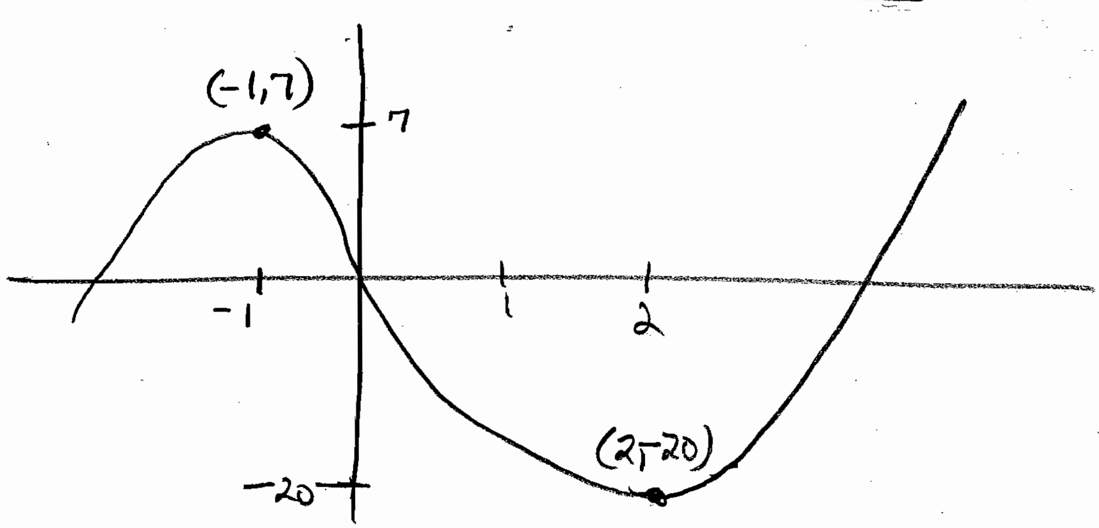
$f(2) = 2(2^3) - 3(2^2) - 12(2) = 16 - 12 - 24$

$= -20$ IS A LOCAL <u>MIN</u> VALUE

$f''(-1) = 12(-1) - 6 = -18 < 0$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) = -2 - 3 + 12$

$= 7$ IS A LOCAL <u>MAX</u> VALUE



H. **EXAMPLE**: SKETCH **A** GRAPH THAT SATISFIES:

1. $f'' < 0$ on $(-\infty, -1)$ AND $(-1, 3)$
2. $f'(-1)$ DOES NOT EXIST
3. $f' < 0$ ON $(-\infty, -1), (1, 4)$
4. $f' > 0$ ON $(-1, 1), (4, \infty)$
5. $f'' > 0$ ON $(3, \infty)$
6. $f(0) = 3$
7. $f'(1) = 0 = f'(4)$

f DECREASES ON $(-\infty, -1)$ AND IS CONCAVE DOWN ON $(-\infty, -1)$ 1, 3

f INCREASES ON $(-1, 1)$ AND IS CONCAVE DOWN ON $(-1, 1)$ 1, 4

HORIZONTAL TANGENT AT $(1, f(1))$ 7

f DECREASES ON $(1, 3)$ AND IS CONCAVE DOWN ON $(1, 3)$ 1, 3

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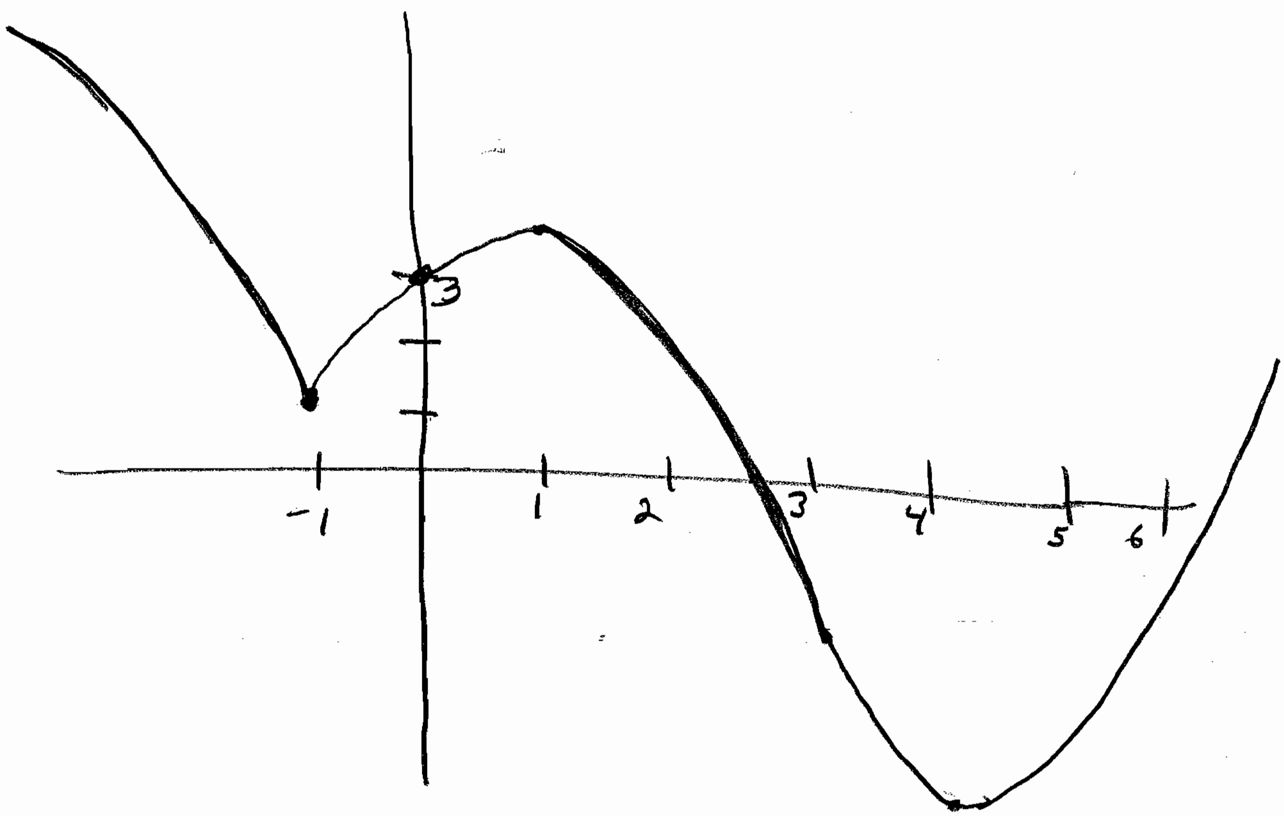
f DECREASES ON (3,4) AND IS CONCAVE UP ON (3,4) 3, 5

HORIZONTAL TANGENT AT (4, f(4)) 7

f INCREASES AND IS CONCAVE UP ON (4, ∞) 4, 5



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HOMEWORK

9c, 11c, 22b, 41c

pages 295–296

In addition, do this:
We are to have worked the other parts of these problems involving the first derivative test. For each of these problems find the inflection points (if any), find the intervals where they are concave up and/or concave down, then sketch (using the previous first derivative information.)

INDETERMINANT FORMS AND L'HOSPITAL'S RULE

A. OUR PAST: WE RELIED ON ALGEBRA

$$\lim_{x \to 1} \frac{x-1}{x^2-1} \leftarrow \text{FORM } \frac{0}{0} \text{ INDETERMINANT}$$

<u>ALG. HELP</u> $$\lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2}$$

B. WHAT ABOUT $\lim_{x \to 0} \frac{\sin x}{x} \leftarrow \frac{0}{0}$ INDET. FORM

THIS IS A JOB FOR L'HOSPITAL'S RULE!

C. L'HOSPITAL'S RULE:

IF $\lim_{x \to a} T(x) = 0$ AND $\lim_{x \to a} B(x) = 0$

AND $\lim_{x \to a} \frac{T'(x)}{B'(x)} = L$ (A NUMBER OR $\pm\infty$),

THEN $$\lim_{x \to a} \frac{T(x)}{B(x)} = \lim_{x \to a} \frac{T'(x)}{B'(x)} = L$$

D. $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} \overset{\frac{0}{0}}{\underset{L.R.}{=}} \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$

E. PROOF IDEA FOR C (PRECEDING) WHEN $T(a) = B(a) = 0$

ASSUME BETWEEN "IF" & "THEN" OF C.

$$\lim_{x \to a} \frac{T'(x)}{B'(x)} = \frac{\lim_{x \to a} T'(x)}{\lim_{x \to a} B'(x)} \overset{\text{CONT. ASSUME}}{=} \frac{T'(a)}{B'(a)} =$$

$$\frac{\lim_{x \to a} \frac{T(x)-T(a)}{x-a}}{\lim_{x \to a} \frac{B(x)-B(a)}{x-a}} = \lim_{x \to a} \frac{\frac{T(x)-T(a)}{x-a}}{\frac{B(x)-B(a)}{x-a}} =$$

$$\lim_{x \to a} \frac{T(x)-\overbrace{T(a)}^{0}}{B(x)-\underbrace{B(a)}_{0}} = \lim_{x \to a} \frac{T(x)}{B(x)}$$

F. OTHER INDETERMINANT FORMS BESIDES $\frac{0}{0}$: $\frac{\pm\infty}{\pm\infty}$

L'HOSPITAL'S RULE STILL APPLIES

ALSO FOR $\lim_{x \to a^{\pm}}$ OR $\lim_{x \to \pm\infty}$

G. CAN APPLY L'HOSPITAL'S RULE MORE ONCE

$$\lim_{x\to\infty} \frac{3x^2}{e^{2x}} \overset{\frac{\infty}{\infty}}{\underset{L.R.}{=}} \lim_{x\to\infty} \frac{6x}{2e^{2x}} \overset{\frac{\infty}{\infty}}{\underset{L.R.}{=}} \lim_{x\to\infty} \frac{6}{4e^{2x}} = 0$$

H. DO NOT TRY TO APPLY L'HOSPITAL'S RULE WHEN IT DOES NOT APPLY.

$$\lim_{x\to \frac{\pi}{2}^-} \frac{\sin x}{x} \overset{\text{FALSE}}{\underset{L.R.}{=}} \lim_{x\to \frac{\pi}{2}^-} \frac{\cos x}{1} = 0 \quad \underline{\text{WRONG!!}}$$

THIS IS NOT AN INDETERMINANT FORM

$$\lim_{x\to \frac{\pi}{2}^-} \frac{\sin x}{x} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

I INDETERMINANT FORM $0 \cdot \infty$

EACH OF THESE HAS A DIFFERENT ANSWER.

$$\lim_{x\to 0^+} \overset{0\cdot\infty}{x \cdot \frac{7}{x}} = 7 \qquad \lim_{x\to 0^+} \overset{0\cdot\infty}{x^2 \cdot \frac{1}{x}} = 0 \qquad \lim_{x\to 0^+} \overset{0\cdot\infty}{x \cdot \frac{1}{x^2}} = +\infty$$

SO $0 \cdot \infty$ IS AN INDETERMINANT FORM

J. A TECHNIQUE FOR SOLVING $0 \cdot \infty$

$$L(x) \cdot R(x) = \frac{R(x)}{\frac{1}{L(x)}}$$

$$\lim_{x \to 0^+} x^2 \ln x \overset{0 \cdot (-\infty)}{=\!=} \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x^2}} \overset{\frac{-\infty}{\infty}\ L.R.}{=\!=}$$

$$\lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x \to 0^+} \frac{1}{x} \cdot \left(\frac{-x^3}{2}\right) =$$

$$\lim_{x \to 0^+} \left(-\frac{x^2}{2}\right) = 0$$

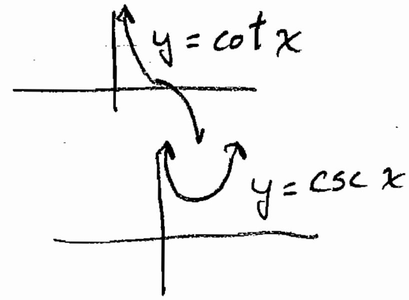
K. $\boxed{\text{INDETERMINANT FORM } \infty - \infty}$

EACH OF THESE HAVE DIFFERENT ANS.

$$\lim_{x \to \infty} \overset{\infty}{(7+x)} - \overset{\infty}{x} = 7 \qquad \lim_{x \to \infty} \overset{\infty}{x} - \overset{\infty}{(10+x)} = -10$$

SO INDETERMINANT

L. $\lim_{x\to 0^+} \overset{\infty}{\cot x} - \overset{\infty}{\csc x} =$



$$\lim_{x\to 0^+} \frac{\cos x}{\sin x} - \frac{1}{\sin x} =$$

$$\lim_{x\to 0^+} \frac{(\cos x)-1}{\sin x} \overset{\frac{0}{0}}{\underset{L.R.}{=}} \lim_{x\to 0^+} \frac{-\sin x}{\cos x} = \frac{0}{1} = 0$$

M. OTHER INDETERMINANT FORMS:

0^0	∞^0	1^∞
↑	↑	↑
DOES IT GO TO 0 OR 1 OR ___	DOES IT GO TO ∞ OR 1 OR ___	DOES IT GO TO 1 OR ∞ OR ___

<u>SOLUTION:</u> RECALL $\square \overset{*}{=} e^{\ln \square}$

TO FIND $\lim y$ ← ORIGINAL PROBLEM

FIND $\lim \ln y = L$, THEN

$$\lim y \overset{*}{=} \lim e^{\ln y} = e^{\lim \ln y} = e^{L}$$

N. FIND $\lim_{x\to 0^+} (1+\sin 2x)^{\csc x}$ FORM 1^∞

LET $y = (1+\sin 2x)^{\csc x}$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \ln (1+\sin 2x)^{\csc x} =$$

$$\lim_{x\to 0^+} \overset{\infty}{(\csc x)} \cdot \overset{0}{\ln (1+\sin 2x)} =$$

$$\lim_{x\to 0^+} \frac{\ln(1+\sin 2x)}{\sin x} \overset{\frac{0}{0}}{\underset{L.R.}{=}} \lim_{x\to 0^+} \frac{\frac{2\cos 2x}{1+\sin 2x}}{\cos x}$$

$$= \frac{\frac{2\cos 0}{1+\sin 0}}{\cos 0} = \frac{\frac{2}{1+0}}{1} = 2. \text{ SO...}$$

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y}$$

$$= e^{\lim_{x\to 0^+} \ln (1+\sin 2x)^{\csc x}}$$

$$= e^2 \leftarrow \text{ANSWER}$$

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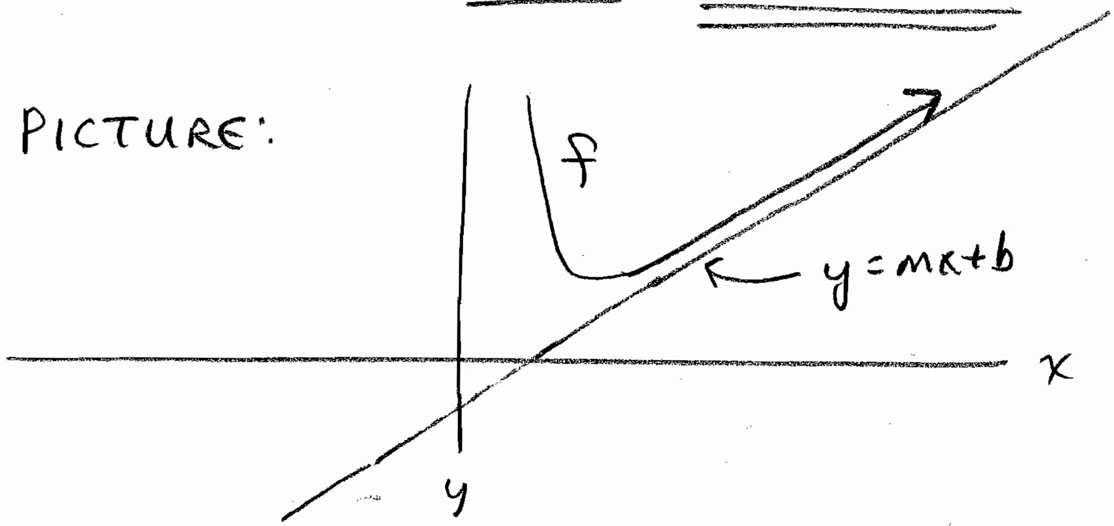
PAGES 304-305 : 9, 11, 19, 21, 27, 31,
41, 44, 45, 46, 53, 57, 60

CURVE SKETCHING USING MANY WEAPONS

A. NEW WEAPON: SLANT ASYMPTOTES

PICTURE:



INTUITIVELY: THE GRAPH GETS AND STAYS CLOSE TO THE LINE $y = mx + b$ AS x APPROACHES $\pm\infty$

DEFINITION: $y = mx + b$ IS A SLANT ASYMPTOTE FOR $y = f(x)$ IF

$$\lim_{x \to \pm\infty} \left[f(x) - (mx + b) \right] = 0$$

B. EXAMPLE OF A SLANT ASYMPTOTE:

FIND A SLANT ASYMPTOTE FOR

$$f(x) = \frac{x^2}{x-1}$$

$$\begin{array}{r|l} & \quad x+1 \\ \hline x-1 & x^2+0x+0 \\ & \ominus\, x^2 \overset{\oplus}{-} x \\ \hline & \quad x+0 \\ & \ominus\, x \overset{\oplus}{-} 1 \\ \hline & \qquad 1 \end{array}$$

$$f(x) = \frac{x^2}{x-1} = (x+1) + \frac{1}{x-1}$$

$y = x+1$ IS A SLANT ASYMPTOTE FOR $f(x) = \frac{x^2}{x-1}$ SINCE

$$\lim_{x\to\infty} \left[\frac{x^2}{x-1} - (x+1)\right] =$$

$$\lim_{x\to\infty} \left[(x+1) + \frac{1}{x-1} - (x+1)\right] = \lim_{x\to\infty} \frac{1}{x-1}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x}(x-1)} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{1}{x}} = \frac{0}{1-0} = 0$$

C. USING MANY WEAPONS TO ANALYZE AND SKETCH A GRAPH
JUGULAR # 3 TYPE OF PROBLEM

FOR $y = f(x) = \dfrac{x^2}{x-1}$

1. FIND $\mathrm{dom}(f)$
2. FIND x-INTERCEPTS AND y-INTERCEPTS
3. FIND ASYMPTOTES OF THE TYPE
   a. HORIZONTAL
   b. VERTICAL
   c. SLANT
4. FIND INTERVALS WHERE f INCREASES AND WHERE f DECREASES
5. FIND ANY LOCAL EXTREMA
6. FIND INTERVALS WHERE f IS CONCAVE UP AND WHERE f IS CONCAVE DOWN
7. FIND ANY POINTS OF INFLECTION
8. PLOT OTHER POINTS IF NECESSARY
9. FIND ANY ABSOLUTE EXTREMA
10. SKETCH.

1. $\boxed{\text{FIND dom(f)}}$ $f(x) = \dfrac{x^2}{x-1} = y$

$\text{dom}(f) = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$

2. $\boxed{\text{FIND } x \text{ AND } y \text{ INTERCEPTS}}$

x- INTERCEPT : SET $y = 0$

$y = \dfrac{x^2}{x-1} = 0$. $x^2 = 0$. $\boxed{x = 0}$

y- INTERCEPT : SET $x = 0$

$y = \dfrac{x^2}{x-1} = \dfrac{0^2}{0-1} = 0$ $\boxed{y = 0}$

3. $\boxed{\text{FIND ASYMPTOTES}}$

(i) HORIZONTAL $\lim\limits_{x \to \infty} \dfrac{x^2}{x-1} =$

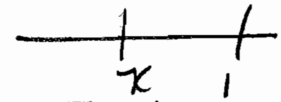
$\lim\limits_{x \to \infty} \dfrac{\frac{1}{x}(x^2)}{\frac{1}{x}(x-1)} = \lim\limits_{x \to \infty} \dfrac{x}{1 - \frac{1}{x}} = \infty$

NOT A NUMBER ANSWER.

$\lim\limits_{x \to -\infty} \dfrac{x^2}{x-1} = \lim\limits_{x \to -\infty} \dfrac{x}{1 - \frac{1}{x}} = -\infty$

NOT A NUMBER.

NO HORIZONTAL ASYMPTOTES

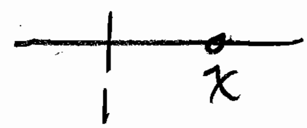


(ii) VERTICAL: $\displaystyle\lim_{x\to 1^-} \frac{\overset{\text{APP POS}}{x^2}}{\underset{\text{L-R NEG APP. } \to 0}{x-1}} = -\infty$

$x=1$ IS A VERTICAL ASYMPTOTE

FOR CURVE SKETCHING, WE FIND

$\displaystyle\lim_{x\to 1^+} \frac{\overset{\text{APP } \to 1 \text{ POS}}{x^2}}{\underset{\text{R-L POS APP } \to 0}{x-1}} = +\infty$



(iii) SLANT: AS SEEN EARLIER,

$\displaystyle\lim_{x\to\infty}\left[\frac{x^2}{x-1}-(x+1)\right]=0$ SO

$y=x+1$ IS A SLANT ASYMPTOTE AS $x\to+\infty$

FOR CURVE SKETCHING, WE FIND

$\displaystyle\lim_{x\to-\infty}\left[\frac{x^2}{x-1}-(x+1)\right]=$ AS SEEN EARLIER

$\displaystyle\lim_{x\to-\infty}\left[(x+1)+\frac{1}{x-1}-(x+1)\right]=\lim_{x\to-\infty}\frac{1}{x-1}=$

$\displaystyle\lim_{x\to-\infty}\frac{\frac{1}{x}}{\frac{1}{x}(x-1)}=\lim_{x\to-\infty}\frac{\frac{1}{x}}{1-\frac{1}{x}}=\frac{0}{1-0}=0$

$y=x+1$ IS A SLANT ASYMPTOTE AS $x\to-\infty$

4. $\boxed{\text{INTERVALS OF INCREASE\DECREASE}}$

$$f(x) = \frac{x^2}{x-1}$$

$$f'(x) = \frac{(x-1)2x - x^2(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2}$$

$$f'(x) = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

x	$-$	$+$	$+$	$+$
$x-2$	L−R $-$	L−R $-$	L−R $-$	R−L $+$
$(x-1)^2$	$+$	$+$	$+$	$+$

$f'>0$ 0 $f'<0$ 1 $f'<0$ 2 $f'>0$

f INC f DEC f DEC f INC

f INCREASES ON $(-\infty, 0)$, $(2, \infty)$

f DECREASES ON $(0, 1)$, $(1, 2)$

5. $\boxed{\text{FIND ANY LOCAL EXTREMA}}$

BY THE FIRST DERIVATIVE TEST, USING THE CHART ON THE PREVIOUS PAGE:

ONLY CRITICAL NUMBERS ARE: 0, 2
(NOTE $1 \notin \operatorname{dom}(f)$)

$f(0) = \frac{0^2}{0-1} = 0$ LOCAL MAX VALUE

$f(2) = \frac{2^2}{2-1} = 4$ LOCAL MIN VALUE

6. $\boxed{\text{INTERVALS OF CONCAVE UP\textbackslash DOWN}}$

$f(x) = \frac{x^2}{x-1}$. $f'(x) = \frac{x^2-2x}{(x-1)^2}$

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x)2(x-1)}{(x-1)^4}$$

$$f''(x) = \frac{(x-1)^2 2(x-1) - (x^2-2x)2(x-1)}{(x-1)^4}$$

$$f''(x) = \frac{2(x-1)\left[(x-1)^2 - (x^2-2x)\right]}{(x-1)^4}$$

$$f''(x) = \frac{2\left[x^2-2x+1-x^2+2x\right]}{(x-1)^3}$$

$$f''(x) = \frac{2}{(x-1)^3}$$

2		$+$			$+$
$(x-1)^3$	$(L-R)^3$	$-$		$(R-L)^3$	$+$
	$f'' < 0$		1	$f'' > 0$	
	f CONCAVE DOWN			f CONCAVE UP	

f CONCAVE UP ON $(1, \infty)$

f CONCAVE DOWN ON $(-\infty, 1)$

7. $\boxed{\text{FIND ANY POINTS OF INFLECTION}}$

NONE. $1 \notin \mathrm{dom}(f)$

8. $\boxed{\text{PLOT OTHER POINTS IF NECESSARY}}$

x	y
0	0
2	4
$\frac{3}{2}$	$\frac{9}{2}$
$\frac{1}{2}$	$-\frac{1}{2}$

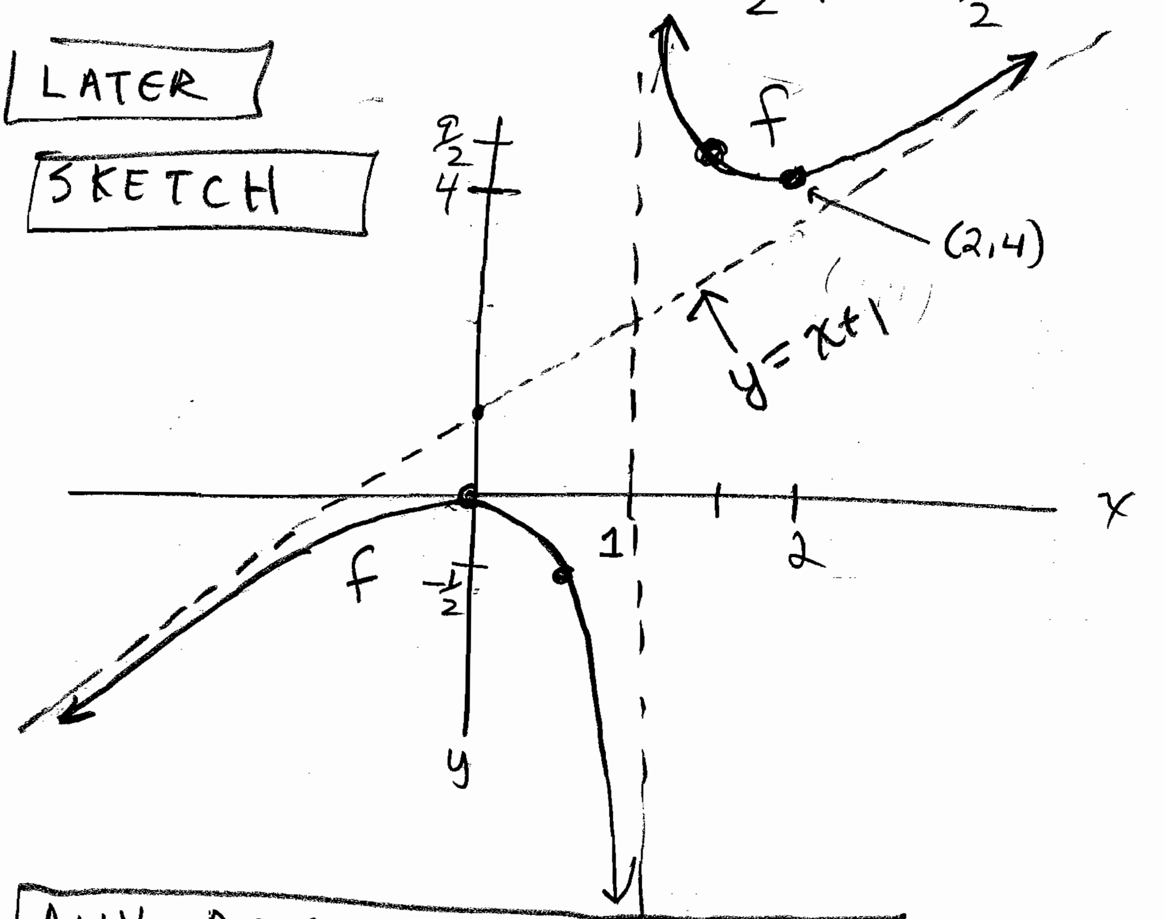
$$f\left(\tfrac{3}{2}\right) = \frac{\left(\frac{3}{2}\right)^2}{\frac{3}{2}-1} = \frac{\frac{9}{4}}{\frac{1}{2}} =$$

$$\frac{9}{4} \cdot \frac{2}{1} = \frac{18}{4} = \frac{9}{2}$$

$$f\left(\tfrac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^2}{\frac{1}{2}-1} = \frac{\frac{1}{4}}{-\frac{1}{2}} = -\frac{1}{2}$$

9. $\boxed{\text{LATER}}$

10. $\boxed{\text{SKETCH}}$



9. $\boxed{\text{ANY ABSOLUTE EXTREMA}}$

NONE

HOMEWORK

A. FIND ALL SLANT ASYMPTOTES FOR $f(x) = \dfrac{x^2+1}{x-2}$

B. DO ALL THE ANALYSIS SKETCHING STEPS LISTED ON PAGE 253* FOR

1. $f(x) = \dfrac{x^2}{x^2-1}$

2. $f(x) = 3x^4 - 4x^3 - 12x^2$

* OF THESE NOTES

WORD PROBLEMS INVOLVING MAXIMIZATION AND MINIMIZATION

A. WISDOM STEPS FOR SOLVING THIS TYPE OF WORD PROBLEM

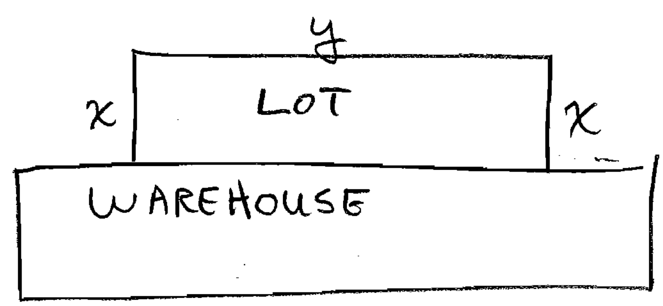
1. <u>READ</u> THE PROBLEM
2. DRAW A <u>PICTURE</u>
3. NAME THE <u>VARIABLES</u> (ESPECIALLY LOOK FOR THE ENTITY, M, TO BE MAXIMIZED OR MINIMIZED)
4. GET <u>EQUATIONS</u> RELATING THE VARIABLES
5. GET M AS A FUNCTION OF ONE VARIABLE
6. FIND THE DOMAIN OF M.
7. USE THE CALCULUS\ALGEBRA TECHNIQUES STUDIED TO <u>FIND</u> <u>AND</u> <u>PROVE</u> THE DESIRED ABS. MAX\MIN INFORMATION
8. BE SURE TO ANSWER THE QUESTIONS.

HAVE DISCIPLINE TO DO THESE STEPS

B. A MAN WANTS TO BUILD A RECTANGULAR LOT TO GIVE MAXIMUM AREA ALONG THE WALL OF HIS WAREHOUSE. ONE SIDE OF THE LOT WILL BE THE WALL OF HIS WAREHOUSE. HE WILL USE 400 FEET OF FENCING. WHAT ARE THE DIMENSIONS THAT MAXIMIZE AREA? (NO FLAT RECTANGLES WITH NO WIDTH OR NO LENGTH ALLOWED.)

1. READ PROBLEM

2. PICTURE



3. VARIABLES

x: WIDTH

y: LENGTH

A : AREA ← TO BE MAXIMIZED

4. EQUATIONS

$$A = xy$$

$$x + y + x = 400 \quad (\text{i.e. } 2x + y = 400)$$

5. GET A TO 1 VARIABLE

$$y = 400 - 2x$$

$$A = xy = x(400 - 2x) = 400x - 2x^2$$

6. <u>DOMAIN</u> OF A:

$A > 0$

$400x - 2x^2 > 0$

$400x > 2x^2$

$200x > x^2$ $\qquad x > 0$

$200 > x$

$\mathrm{dom}(A) = (0, 200) = \{x \mid 0 < x < 200\}$

7. <u>FIND</u> AND <u>PROVE</u> <u>MAX</u> INFO.

$A(x) = 400x - 2x^2$

$A'(x) = 400 - 4x$

$A'(x) = -4(x - 100)$

-4	$-$	$-$
$x - 100$	L-R $-$	R-L $+$
	$A' > 0$ A INC	$A' < 0$ A DEC

$0 \qquad 100 \qquad 200$

A(100) IS THE <u>ABS.</u> MAX VALUE ON (0, 200) SINCE A INCREASES ON (0, 100) AND DECREASES ON (100, 200)

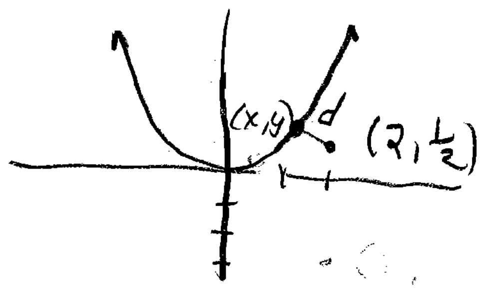
8. <u>ANSWER</u>: DIMENSIONS $y = 400 - 2x = 400 - 2(100) = 200$

100 FT WIDE BY 200 FEET LONG

C. FOR THE PARABOLA $y = x^2$, FIND THE POINT ON THE PARABOLA CLOSEST TO THE POINT $(2, \frac{1}{2})$

1. READ

2. PICTURE



3. VARIABLES

x, y : COORDINATES OF POINTS ON THE PARABOLA.

d : DISTANCE FROM $(2, \frac{1}{2})$ TO (x,y) ON THE PARABOLA

4. EQUATIONS

$$d = \sqrt{(x-2)^2 + (y - \tfrac{1}{2})^2}$$

$$d^2 = (x-2)^2 + (y - \tfrac{1}{2})^2$$

$$y = x^2$$

MINIMIZE d^2

5. GET d^2 TO A FUNCTION OF 1 VARIABLE

$$d^2 = (x-2)^2 + (x^2 - \tfrac{1}{2})^2 = f(x)$$

MINIMIZE $f(x)$

6. FIND DOMAIN OF $f(x)$.

$\mathrm{dom}(f) =$ REALS

7. FIND\PROVE ABS. MIN

$$f'(x) = 2(x-2) + 2\left(x^2 - \tfrac{1}{2}\right)2x$$
$$= 2x - 4 + 4x^3 - 2x$$
$$= 4x^3 - 4 = 4(x^3 - 1)$$
$$= 4(x-1)(x^2+x+1)$$
$$= 4(x-1)\left(\left(x^2+x+\tfrac{1}{4}\right)+\tfrac{3}{4}\right)$$
$$= 4(x-1)\left(\left(x+\tfrac{1}{2}\right)^2+\tfrac{3}{4}\right)$$

4		$+$		$+$
$x-1$	L-R	$-$	R-L	$+$
$\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$		$+$		$+$
		$f'<0$	1	$f'>0$
		f DEC		f INC

$f(1)$ IS THE ABSOLUTE MIN VALUE FOR f SINCE f DECREASE TO 1 AND THEN INCREASES

$x = 1, \quad y = x^2 = 1^2 = 1$

8: ANSWER

$(1,1)$ IS THE POINT CLOSEST TO $(2, \frac{1}{2})$. $(1,1)$ IS ON THE PARABOLA $y = x^2$

$\boxed{\text{HOMEWORK}}$

PAGES 328, 329: 3, 5, 6, 14, 33

PAGES 268-276 ARE OMITTED.

ANTIDERIVATIVES

A. **DEF.** F IS AN ANTIDERIVATIVE OF f ON INTERVAL I IFF FOR EVERY $x \in I$, $F'(x) = f(x)$

B. EXAMPLES:

1. $F(x) = \sin x^2$ IS AN ANTIDERIVATIVE OF $f(x) = (\cos x^2)2x$ since $F' = f$

2. LET $f(x) = 3x^2$. $F(x) = x^3$ IS AN ANTIDERIVATIVE OF f SINCE $F' = f$

3. LET $f(x) = \dfrac{1}{\sqrt{1-x^2}}$. $F(x) = \arcsin x$ IS AN ANTIDERIVATIVE OF f, SINCE $F' = f$

C. ANTIDERIVATIVES JUST DIFFER BY A CONSTANT FROM EACH OTHER:

SUPPOSE F AND G ARE BOTH ANTIDERIVATIVES OF f.

HENCE $F' = G' = f$.

SINCE, F AND G HAVE THE SAME DERIVATIVE, THEY DIFFER BY A CONSTANT I.E. FOR ALL $x \in I$, AN INTERVAL, $G(x) = F(x) + C$.

D. <u>THE</u> <u>MOST</u> <u>GENERAL</u> <u>ANTIDERIVATIVE</u> <u>OF</u> <u>f</u> IS REALLY A FAMILY OF FUNCTIONS

$$\{ F(x) + C \mid C \text{ IS A CONSTANT AND } F' = f \}$$

BUT IS TRADITIONALLY DENOTED BY

$F(x) + C$ WITH C CONSIDERED AN ARBITRARY CONSTANT.

E. EXAMPLES OF MOST GENERAL ANTIDERIVATIVES

GIVEN FUNCTION f	MOST GENERAL ANTIDERIVATIVE $F(x) + C$
$f(x) = (\cos x^2) 2x$	$\sin x^2 + C$
$f(x) = 3x^2$	$x^3 + C$

F. ANTIDERIVATIVE FORMULAS TO KNOW

GIVEN FUNCTION f	MOST GENERAL ANTIDERIVATIVE $F(x)+c$
$x^n \quad n \neq -1$	$\dfrac{x^{n+1}}{n+1} + c$
$\dfrac{1}{x} = x^{-1}$	$\ln\lvert x\rvert + c$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\sec x \tan x$	$\sec x + c$
$\csc^2 x$	$-\cot x + c$
$\csc x \cot x$	$-\csc x + c$
$\dfrac{1}{1+x^2}$	$\arctan x + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$

G. <u>THEOREM</u> IF F IS AN ANTIDERIVATIVE OF f AND G IS AN ANTIDERIVATIVE OF g, THEN

$cF(x)$ IS AN ANTIDERIVATIVE OF $cf(x)$

AND

$F(x)+G(x)$ IS AN ANTIDERIVATIVE OF $f(x)+g(x)$

H. <u>EXAMPLES</u>: FIND THE MOST GENERAL ANTIDERIVATIVE OF

1. $h(x) = 5x^2 - 3\cos x + \dfrac{4}{\sqrt{1-x^2}}$

   ANSWER: $H(x) = 5\left(\dfrac{x^3}{3}\right) - 3\sin x + 4\sin^{-1}(x) + C$

2. $h(x) = -7\sec x \tan x - 5x^8 + \dfrac{2}{1+x^2}$

   $h(x) = -7[\sec x \tan x] - 5(x^8) + 2\left[\dfrac{1}{1+x^2}\right]$

   ANSWER:

   $H(x) = -7\sec x - 5\left(\dfrac{x^9}{9}\right) + 2\arctan x + C$

3. FOR $f'(x) = 15x^2 - \frac{7}{2\sqrt{x}}$, FIND $f(x)$ WHERE $f(4) = 310$.

NOTE: IN RELATING TO OUR PREVIOUS NOTATION f' CORRESPONDS TO f
f CORRESPONDS TO F

$$f'(x) = 15(x^2) - \frac{7}{2}x^{-\frac{1}{2}}$$

$$f(x) = 15\left(\frac{x^3}{3}\right) - \frac{7}{2}\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$$

$$f(x) = 5x^3 - \frac{7}{2}\cdot\frac{2}{1}\sqrt{x} + c$$

$$f(x) = 5x^3 - 7\sqrt{x} + c$$

$$f(4) = 310 = 5(4^3) - 7\sqrt{4} + c$$

$$310 = 5(64) - 7(2) + c$$

$$310 = 320 - 14 + c$$

$$310 = 306 + c$$

$$4 = c$$

ANSWER $f(x) = 5x^3 - 7\sqrt{x} + 4$

I. ANTIDERIVATIVES APPLIED TO VELOCITY AND ACCELERATION

RECALL: IF $s(t)$ IS THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME t, THEN

$s'(t) = v(t)$ VELOCITY AT TIME t

$s''(t) = v'(t) = a(t)$ ACCELERATION AT TIME t.

ACCELERATION DUE TO GRAVITY

$$a(t) = -32\ \frac{ft}{sec^2} \qquad a(t) = -9.8\ \frac{m}{sec^2}$$

J. VELOCITY, ACCELERATION EXAMPLE

AN OBJECT IS THROWN STRAIGHT UP WITH AN INITIAL VELOCITY OF $64\ \frac{ft}{sec}$. HOW HIGH DOES THE OBJECT GO? AT WHAT TIME DOES IT HIT THE GROUND?

$$s''(t) = v'(t) = a(t) = -32$$

$$s'(t) = v(t) = -32t + C_0$$

$$V(0) = -32(0) + C_0 = 64$$

$$C_0 = 64$$

$$V(t) = f'(t) = -32t + 64$$

AT MAX HEIGHT $V(t) = 0$

$$-32t + 64 = 0$$

$$64 = 32t$$

$$2 = t \leftarrow \text{TIME REACHES MAX HEIGHT}$$

$$f(t) = -16t^2 + 64t + C_1$$

AT START TIME ($t=0$), POSITION $= 0$

$$f(0) = -16(0^2) + 64(0) + C_1 = 0$$

$$C_1 = 0$$

$$f(t) = -16t^2 + 64t$$

$$f(2) = -16(2^2) + 64(2)$$

$$= -16(4) + 128 = \underline{\underline{64}} \leftarrow \text{POSITION AT TIME REACHES MAX HEIGHT}$$

TIME OBJECT HITS GROUND, I.E. POSITION $= 0$

$$f(t) = -16t^2 + 64t = 0$$

$$-16t(t-4) = 0$$

$t = 0$ ← START TIME

$t = 4$ ← GROUND HITTING TIME

HOMEWORK

PAGE 345: 1, 3, 5, 7, 9, 11, 12, 13, 20, 27, 29, 33, 37

BUILD-UP TO DEFINITE INTEGRAL

A. $\sum$ SIGMA NOTATION

1. $$\boxed{\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k}$$

   i, j, k ARE DUMMY VARIABLES

2. EXAMPLE $\sum_{i=1}^{3} (2i-7) =$

   $(2(1)-7) + (2(2)-7) + 2(3)-7$

3. $$\boxed{\sum_{i=1}^{n} c = nc \qquad c \text{ CONSTANT}}$$

4. EXAMPLE $\sum_{i=1}^{3} 7 = 3(7) = 21$

   <u>REASON</u>: $\sum_{i=1}^{3} 7 = \sum_{i=1}^{3} a_i$ WHERE $a_i = 7$

   $= a_1 + a_2 + a_3 = 7 + 7 + 7 = 3(7)$

5. $$\boxed{\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i \qquad c \text{ CONSTANT}}$$

6. EXAMPLE $\sum_{i=1}^{3} 5a_i = 5\sum_{i=1}^{n} a_i$

7. EXAMPLE $\sum_{i=1}^{n} \frac{5}{n}(i-3) = \frac{5}{n}\sum_{i=1}^{n}(i-3)$

i IS THE DUMMY VARIABLE FOR Σ

n IS A CONSTANT FOR THIS Σ NOTATION.

8. EXAMPLE $\sum_{j=1}^{k} \frac{5ikj^2}{n} = \frac{5ik}{n}\sum_{j=1}^{k} j^2$

9. $$\boxed{\sum_{i=1}^{n}(a_i \pm b_i) = \left(\sum_{i=1}^{n} a_i\right) \pm \left(\sum_{i=1}^{n} b_i\right)}$$

10. EXAMPLE $\sum_{i=1}^{n}\left(5i^2 - \frac{2}{n}i\right) =$

$$\sum_{i=1}^{n} 5i^2 - \sum_{i=1}^{n} \frac{2}{n}i = 5\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i$$

11. FALSE $\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$

$$\sum_{i=1}^{2} a_i b_i = a_1 b_1 + a_2 b_2 \neq a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$$

$$(a_1 + a_2)(b_1 + b_2) = \left(\sum_{i=1}^{2} a_i\right)\left(\sum_{i=1}^{2} b_i\right)$$

B. $\sum$ FORMULAS

1. $1 + 2 + 3 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

C. $\sum_{i=1}^{n} \left(\frac{3}{n} i^2 + 5i\right) = \frac{3}{n}\left(\sum_{i=1}^{n} i^2\right) + 5 \sum_{i=1}^{n} i$

$$= \frac{3}{n} \frac{n(n+1)(2n+1)}{6} + 5 \frac{n(n+1)}{2}$$

D. <u>NOTE</u>: $1^2+2^2+3^2=1+4+9=14$

$$1^2+2^2+3^2=\sum_{i=1}^{3} i^2 = \frac{3(3+1)(2(3)+1)}{6} = \frac{3(4)(7)}{6} = 14$$

E. <u>PARTITION</u> OF $[a,b]$

$\{x_0, x_1, x_2, \ldots, x_n\}$ IS A PARTITION OF $[a,b]$ IF AND ONLY IF (IFF)

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$$

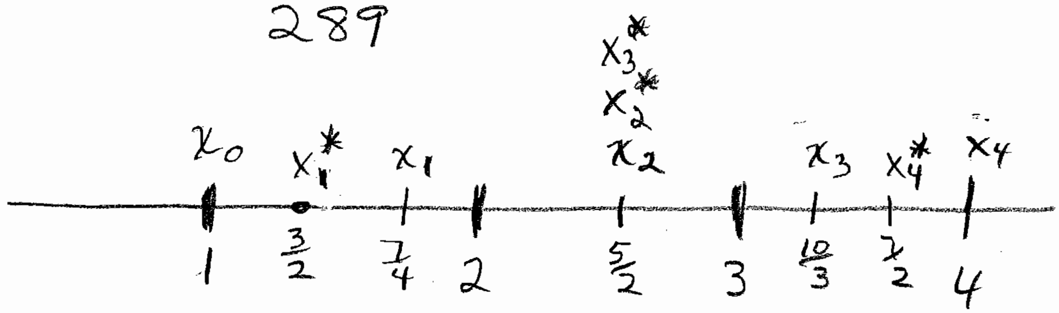
F. <u>SAMPLE</u> <u>POINTS</u> OF A PARTITION

$x_1^*, x_2^*, x_3^*, \ldots, x_n^*$ ARE SAMPLE POINTS FOR PARTITION $\{x_0, x_1, \ldots, x_n\}$ OF $[a,b]$ IFF $x_0 \le x_1^* \le x_1$,

$x_1 \le x_2^* \le x_2, \ldots$

IN GENERAL: FOR EVERY $i \in \{1,2,\ldots,n\}$ $x_i^* \in [x_{i-1}, x_i]$

G.



$x_0 = 1$; $x_1 = \frac{7}{4}$; $x_2 = \frac{5}{2}$; $x_3 = \frac{10}{3}$; $x_4 = 4$

PARTITIONS $[1, 4]$ WITH SAMPLE POINTS $x_1^* = \frac{3}{2}$; $x_2^* = \frac{5}{2}$; $x_3^* = \frac{5}{2}$; $x_4^* = \frac{7}{2}$

H. REGULAR PARTITION : ALL THE SUBINTERVALS $[x_{i-1}, x_i]$ ARE THE SAME LENGTH

$\Delta x_1 = x_1 - x_0$; $\Delta x_2 = x_2 - x_1$;

$\Delta x_i = x_i - x_{i-1}$.

FOR REGULAR PARTITION

$\Delta x_1 = \Delta x_2 = \Delta x_3 = \cdots = \Delta x_n$

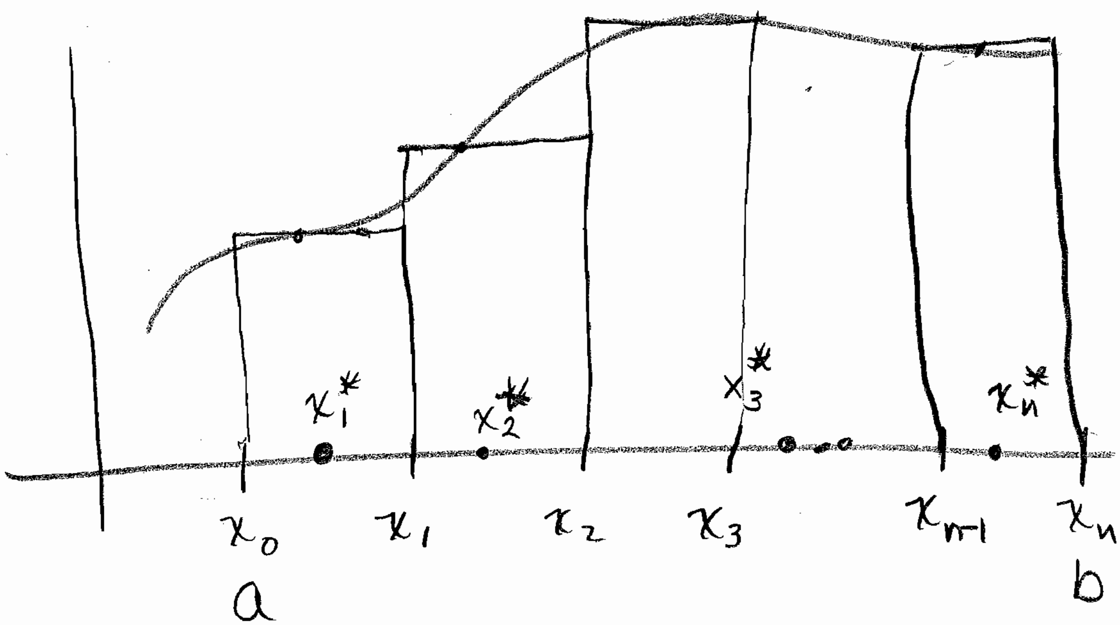
$= \Delta x \leftarrow$ NOTATION FOR SUBINTERVAL LENGTH OF REGULAR PARTITION.

I. RIEMANN SUM FOR CONTINUOUS FUNCTION f ON $[a,b]$ FOR PARTITION $\{x_0, x_1, x_2, \ldots, x_n\}$ WITH SAMPLE POINTS $x_1^*, x_2^*, \ldots, x_n^*$.

$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

$$= \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

PICTURE



J. FOR NOW WE HAVE ALL PARTITIONS <u>REGULAR</u>. ALL THAT IS NEEDED IS a, b, n TO DETERMINE $x_0, x_1, x_2, \ldots, x_n$ AND Δx

FOR $[a,b] = [2,5]$ AND $n = 4$, FIND $x_0, x_1, x_2, x_3, x_4,$ AND Δx

GENERALLY

$$\Delta x = \frac{b-a}{n} = \frac{5-2}{4} = \frac{3}{4}$$

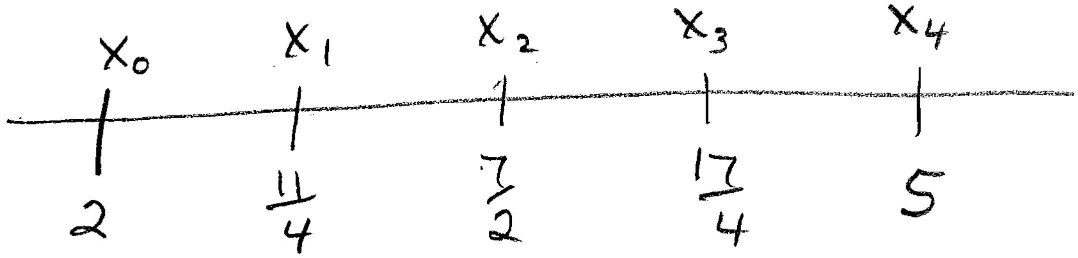
$$x_0 = 2$$

$$x_1 = 2 + \Delta x = 2 + \frac{3}{4} = \frac{11}{4}$$

$$x_2 = 2 + 2\Delta x = 2 + 2\left(\frac{3}{4}\right) = 2 + \frac{6}{4} = \frac{14}{4} = \frac{7}{2}$$

$$x_3 = 2 + 3\Delta x = 2 + 3\left(\frac{3}{4}\right) = 2 + \frac{9}{4} = \frac{17}{4}$$

$$x_4 = 2 + 4\Delta x = 2 + 4\left(\frac{3}{4}\right) = 5$$



K **HOMEWORK**

1. EVALUATE (a) $\sum_{i=1}^{3}(2i-1)$

(b) $\sum_{i=1}^{n}(3i^2+4)$ ← USE FORMULAS

(c) $\sum_{i=1}^{n}\frac{5}{n}(2i^3-7i)$ ← USE FORMULAS

2. FOR $[a,b]=[3,17]$ and $n=5$, FIND $x_0, x_1, x_2, x_3, x_4, x_5$ THAT MAKE UP THE REGULAR PARTITION. ALSO MAKE UP SOME ACCEPTABLE SAMPLE POINTS $x_1^*, x_2^*, \ldots, x_5^*$. $\Delta x =$

3. DRAW A CLEAR PICTURE OF THE RIEMANN SUM FOR $f(x)=2x+1$ FOR THE PARTITION IN PROBLEM 2 WITH THE SAMPLE POINTS YOU CHOSE. (LIKE THE PICTURE ON PAGE 30 OF THE NOTES)

THE DEFINITE INTEGRAL

A DEF. OF THE DEFINITE INTEGRAL FOR CONTINUOUS FUNCTION f ON $[a,b]$, DENOTED $\int_a^b f(x)\,dx$,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

RECALL $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ IS A REGULAR PARTITION, $\Delta x = \dfrac{b-a}{n}$, AND $x_i^* \in [x_{i-1}, x_i]$ FOR $i \in \{1, 2, \ldots, n\}$

FOR CONTINUOUS FUNCTIONS THIS LIMIT WILL ALWAYS EXIST.

NOTE: THE DEFINITE INTEGRAL IS A NUMBER.

RECOGNIZING A LIMIT OF RIEMANN SUMS AS A DEFINITE INTEGRAL.

A. RECALL SAMPLE POINTS:

$x_i^* \in [x_{i-1}, x_i]$ SO x_i^* COULD BE x_{i-1} OR x_i^* COULD BE x_i OR ANY POINT IN (x_{i-1}, x_i).

B. LET $f(x) = \dfrac{\ln(x^2+1)}{2x}$

$$\int_1^3 f(x)\,dx = \int_1^3 \frac{\ln(x^2+1)}{2x}\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \quad [\text{LET } x_i^* = x_i]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\ln(x_i^2+1)}{2x_i}\Delta x$$

C. $\lim_{n\to\infty} \sum_{i=1}^{n} 3x_i^5 \tan^2 x_i \,\Delta x =$ WHAT DEFINITE INTEGRAL ON $[0, \frac{\pi}{4}]$?

ANSWER: $\int_0^{\frac{\pi}{4}} 3x^5 \tan^2 x \, dx$

NOTE: $x_i^* = x_i$

D. $\lim_{n\to\infty} \sum_{i=1}^{n} (3x_{i-1}^4 + 2x_{i-1})\Delta x =$ WHAT DEFINITE INTEGRAL ON $[-1, 2]$?

ANSWER $\int_{-1}^{2} 3x^4 + 2x \, dx$

NOTE: $x_{i-1} = x_i^*$

B. UNDERSTANDING THE NOTATION:

$$\int_3^5 2x^3+3x\,dx$$

x IS A DUMMY VARIABLE
INTERVAL OF INTEGRATION IS $[3,5]$

FUNCTION TO INTEGRATE: $f(x)=2x^3+3x \quad x\varepsilon[3,5]$

$$\int_{-1}^7 2t-5\,dt$$

t IS A DUMMY VARIABLE
INTERVAL OF INTEGRATION IS $[-1,7]$

FUNCTION TO INTEGRATE: $f(t)=2t-5 \quad t\varepsilon[-1,7]$

C. EVALUATE $\int_2^5 4x^2-7x\,dx$ BY DEFINITION

$$\Delta x=\frac{b-a}{n}=\frac{5-2}{n}=\frac{3}{n}$$

$x_0=2$

$x_1=2+\Delta x=2+\frac{3}{n}$

$x_2=2+2\Delta x=2+2\left(\frac{3}{n}\right)$

$x_3=2+3\Delta x=2+3\left(\frac{3}{n}\right)$

$\vdots$

$x_i=2+i\Delta x=2+i\left(\frac{3}{n}\right)$

PICK x_i^* IN $[x_{i-1},x_i]$ AS x_i

$f(x) = 4x^2 - 7x$

$$\int_2^5 4x^2 - 7x\,dx \overset{DEF.}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2 + i\left(\frac{3}{n}\right)\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[4\left(2 + i\left(\frac{3}{n}\right)\right)^2 - 7\left(2 + i\left(\frac{3}{n}\right)\right)\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[4\left(4 + \frac{12}{n}i + \frac{9}{n^2}i^2\right) - 14 - \frac{21}{n}i\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[16 + \frac{48}{n}i + \frac{36}{n^2}i^2 - 14 - \frac{21}{n}i\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[2 + \frac{27}{n}i + \frac{36}{n^2}i^2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\sum_{i=1}^{n} 2 + \frac{27}{n}\sum_{i=1}^{n} i + \frac{36}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[2n + \frac{27}{n}\frac{n(n+1)}{2} + \frac{36}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} 6 + \frac{81}{2}\left(\frac{n+1}{n}\right) + 18\left(\frac{2n^2+3n+1}{n^2}\right)$$

$$= \lim_{n\to\infty} 6 + \frac{81}{2}\left(\frac{1+\frac{1}{n}}{1}\right) + 18\left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{1}\right)$$

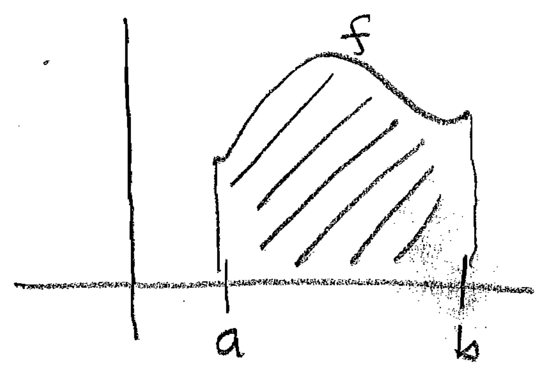
$$= 6 + \frac{81}{2}\left(\frac{1+0}{1}\right) + 18\left(\frac{2+0+0}{1}\right) = 42 + \frac{81}{2} = \frac{165}{2}$$

HOMEWORK: (PAGE 377) 17, 18, 19, 21, 23, 24

D. DEFINITE INTEGRALS AND AREA.

1. FOR $f \geq 0$ ON $[a,b]$, $\int_a^b f(x)\,dx$

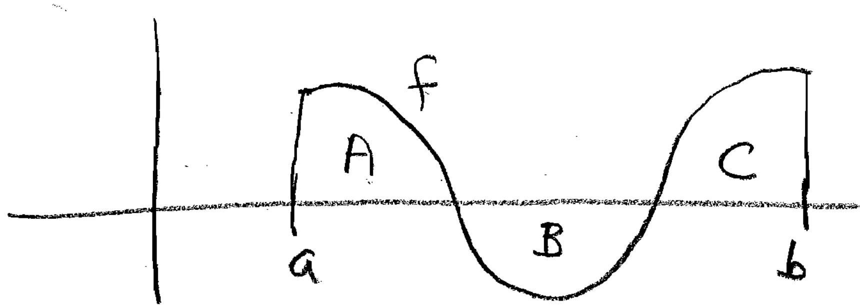
IS THE AREA OF THE REGION BOUNDED BY $x=a$, $x=b$, x-axis, and graph of f on $[a,b]$



THE PICTURE IN SECTION I OF THE PREVIOUS SECTION SHOULD GIVE AN INDICATION OF WHY THIS IS TRUE.

2. IN GENERAL $\int_a^b f(x)\,dx =$

(AREA ABOVE x-AXIS) − (AREA BELOW x-AXIS)

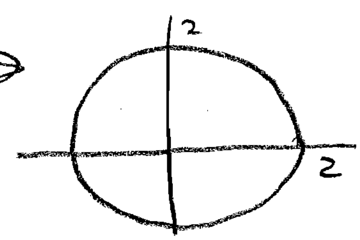


IF $\left\{\begin{array}{l} \text{AREA OF } A = 7 \\ \text{AREA OF } \underline{B} = 8 \\ \text{AREA OF } C = 9 \end{array}\right\}$, THEN $\int_a^b f(x)\,dx$

$= 7 - 8 + 9 = 8$

E. USING AREA PROPERTIES, FIND

$\int_0^2 \sqrt{4-x^2}\,dx$.

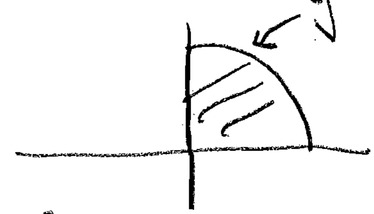


NOTE $x^2+y^2=4$

$y^2 = 4-x^2$

$\sqrt{y^2} = |y| = \sqrt{4-x^2}$

For $y \geq 0$, $y = \sqrt{4-x^2}$



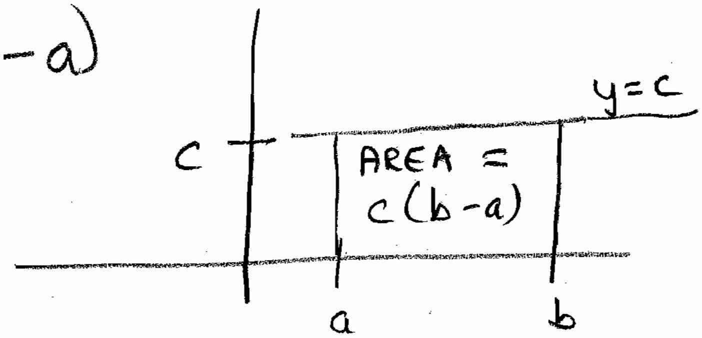
$\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}\left(\begin{array}{c}\text{AREA OF}\\ \text{CIRCLE OF}\\ \text{RADIUS 2}\end{array}\right) = \frac{1}{4}\pi 2^2 = \pi$

F. INTEGRAL PROPERTIES

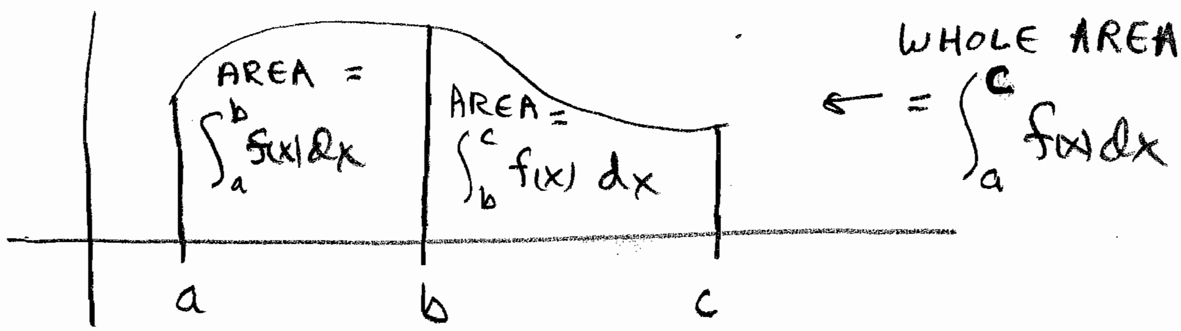
1. $$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

2. $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$ C CONSTANT

3. $$\int_a^b c\,dx = c(b-a)$$



4. $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



5. FOR $f \geq 0$ ON $[a,b]$ $\quad \int_a^b f(x)\,dx \geq 0$

6. FOR $f \leq g$ ON $[a,b]$ $\quad \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

G. DEFINITIONS

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

H. WRITE AS ONE INTEGRAL.

$$\int_1^7 3x-2\,dx - \int_5^7 3x-2\,dx =$$

$$\int_1^5 3x-2\,dx + \int_5^7 3x-2\,dx - \int_5^7 3x-2\,dx = \int_1^5 3x-2\,dx$$

I. WRITE AS ONE INTEGRAL.

$$\int_7^1 2-3x\,dx + \int_7^5 3x-2\,dx = -\int_1^7 2-3x\,dx - \int_5^7 3x-2\,dx$$

$$= \int_1^7 -(2-3x)\,dx - \int_5^7 3x-2\,dx = \int_1^7 (3x-2)\,dx - \int_5^7 3x-2\,dx$$

$$\overset{\text{H ABOVE}}{=} \int_1^5 3x-2\,dx$$

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$\boxed{\text{HOMEWORK}}$

PAGE 378: 35, 36, 37, 47

A. FUNDAMENTAL THEOREM OF CALCULUS (PART I)

If f IS CONTINUOUS ON $[a,b]$ AND

$$F(x) = \int_a^x f(t)\,dt,$$

THEN $F'(x) = f(x)$

B. <u>ILLUSTRATION</u>: Let $f(t) = 5$

$$F(x) = \int_2^x 5\,dt = 5(x-2) = 5x-10$$

$$F'(x) = 5 = f(x)$$

C. <u>EXAMPLE</u> LET $F(x) = \int_2^x \sqrt{1+t^2}\,dt$

$$F'(x) = \sqrt{1+x^2}$$

D. COMPOSITION INVOLVING FUNDAMENTAL THEOREM OF CALCULUS (PART I)

Let $h(x) = \int_5^{x^5} \sqrt{1+t^2}\,dt$

$$h'(x) = \left(\sqrt{1+(x^5)^2}\right) 5x^4$$

AN EXPLANATION FOLLOWS:

RECALL CHAIN RULE: $(F \circ g)'(x) = F'(g(x))g'(x)$

LET $F(x) = \int_5^x \sqrt{1+t^2}\,dt$

SO $F'(x) = \sqrt{1+x^2}$

LET $g(x) = x^5$. $(F \circ g)'(x) = F'(g(x))g'(x)$

$= F'(x^5)\,5x^4 = \left(\sqrt{1+(x^5)^2}\right)5x^4$

NOTE: $(F \circ g)(x) = F(g(x)) = F(x^5) = \int_5^{x^5} \sqrt{1+t^2}\,dt$

$= h(x)$ FROM PREVIOUS PAGE, SO $h'(x) =$

E. **FUND. THM. OF CALCULUS, PART II**

IF f IS CONTINUOUS ON $[a,b]$ AND F IS ANY ANTIDERIVATIVE OF f, (i.e. $F' = f$), THEN $\int_a^b f(x)\,dx = F(b) - F(a) = \Big[F(x)\Big]_a^b$

NOTATION DEFINED

F. $f(x) = 4x^2 - 7x \qquad F(x) = \frac{4}{3}x^3 - \frac{7}{2}x^2 \qquad F'(x) = f(x)$

$$\int_2^5 4x^2 - 7x \, dx = \left[\frac{4}{3}x^3 - \frac{7}{2}x^2\right]_2^5 =$$

$$\left[\frac{4}{3}(5^3) - \frac{7}{2}(5^2)\right] - \left[\frac{4}{3}(2^3) - \frac{7}{2}(2^2)\right] = \frac{165}{2}$$

$$F(5) \qquad - \qquad F(2)$$

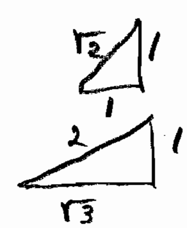
EARLIER WE CALCULATED THIS INTEGRAL BY DEFINITION

G. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \, dx = \left[-\cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$

$$\left(-\cos\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{4}\right) = -0 - \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

H. $$\int_{\frac{1}{\sqrt{3}}}^{1} \frac{8}{1+x^2} \, dx = 8 \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1+x^2} \, dx = 8\left[\tan^{-1} x\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= 8\left[\tan^{-1} 1 - \tan^{-1}\frac{1}{\sqrt{3}}\right] = 8\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$$



$$= 8\left[\frac{3\pi}{12} - \frac{2\pi}{12}\right] = 8\left[\frac{\pi}{12}\right] = \frac{2\pi}{3}$$

I. THE FUNDAMENTAL THEOREM OF CALCULUS RELATES THE TWO FIELDS OF DIFFERENTIAL AND INTEGRAL CALCULUS TOGETHER.

J. $$D_x \int_{x^2}^{7} \frac{1}{1-2t}\,dt = D_x - \int_{7}^{x^2} \frac{1}{1-2t}\,dt$$

$$= -D_x \int_{7}^{x^2} \frac{1}{1-2t}\,dt = -\left[\frac{1}{1-2(x^2)} \cdot 2x\right]$$

HOMEWORK

PAGE 388: 7, 8, 11, 12, 13, 15, 17, 19, 20, 23, 25, 26, 28, 29, 31, 33, 35, 36, 37, 38

INDEFINITE INTEGRAL

A. FUNDAMENTAL THEOREM OF CALCULUS SHOWS ANTIDERIVATIVES CAN BE USED TO CALCULATE DEFINITE INTEGRALS

IF $F'(x) = f(x)$, $\int_a^b f(x)\,dx = F(b) - F(a) = \left[F(x)\right]_a^b$

ANTIDERIVATIVE NOTATION WOULD BE HELPFUL... NOTATION AND A NEW NAME IS COMING!

B. NOTATION: $\int f(x)\,dx$ IS "THE INDEFINITE INTEGRAL OF $f(x)$ WITH RESPECT TO x.

C. DEFINITION:

$$\int f(x)\,dx = F(x) + C$$

IF AND ONLY IF $F'(x) = f(x)$

D. NOTE $\int 3x^2\,dx = x^3 + C$ SINCE $f(x) = 3x^2$, $F(x) = x^3$, AND $F'(x) = 3x^2 = f(x)$

E. TO CHECK TO SEE IF AN INDEFINITE INTEGRAL ANSWER IS CORRECT, JUST TAKE THE DERIVATIVE OF THE ANSWER AND SEE IF THAT EQUALS WHAT IS BETWEEN THE $\int$ AND THE dx

F. IS $\int \overbrace{3x^2+x^5}\,dx = x^3+\frac{1}{6}x^6+C$?

$$D_x\left(x^3+\frac{1}{6}x^6+C\right)=3x^2+6\cdot\frac{1}{6}x^5+0$$

$$=\underbrace{3x^2+x^5}.$$

F. DISTINCTION BETWEEN DEFINITE AND INDEFINITE INTEGRALS:

DEFINITE INTEGRAL $\int_a^b f(x)\,dx$

A NUMBER

INDEFINITE INTEGRAL $\int f(x)\,dx$

THE MOST GENERAL ANTIDERIVATIVE

A FAMILY OF FUNCTIONS OF THE FORM $F(x)+C$ WHERE $F'(x)=f(x)$

G. SOMETIMES YOU MAY SEE NOTATION LIKE THIS:

$$D_x \int f(x)\,dx = f(x)$$

ILLUSTRATED

$$D_x \int 3x^2\,dx = D_x\left(x^3 + C\right) = 3x^2$$

H. OTHER NOTATION YOU MAY SEE

$$\int F'(x)\,dx = F(x) + C$$

OR

$$\int D_x F(x)\,dx = F(x) + C$$

ILLUSTRATED

$$\int D_x x^3\,dx = \int 3x^2\,dx = x^3 + C$$

I ANTIDIFFERENTATION FORMULAS WRITTEN IN INDEFINITE INTEGRAL NOTATION

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1 \qquad \int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C \qquad \int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C \qquad \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

J. $$\int \frac{3x^2-5}{\sqrt{x}} dx = \int \frac{3x^2-5}{x^{\frac{1}{2}}} dx$$

$$= \int \frac{3x^2}{x^{\frac{1}{2}}} - \frac{5}{x^{\frac{1}{2}}} dx = \int 3x^{\frac{3}{2}} - 5x^{-\frac{1}{2}} dx$$

$$= \int 3x^{\frac{3}{2}} dx - \int 5x^{-\frac{1}{2}} dx =$$

$$3\int x^{\frac{3}{2}}dx - 5\int x^{-\frac{1}{2}}dx =$$

$$3\,\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 5\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{6}{5}x^{\frac{5}{2}} - 10x^{\frac{1}{2}} + C$$

NOTE THE PROPERTIES OF ANTI-DERIVATIVES USED

$$\int c\,f(x)\,dx = c\int f(x)\,dx$$

$$\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

AS YOU GET MORE FAMILIAR WITH INTEGRALS YOU CAN LEAVE OUT STEPS AND DO

$$\int 3x^{\frac{3}{2}} - 5x^{-\frac{1}{2}}dx = 3\,\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 5\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \ldots$$

K. $\displaystyle\int \frac{\sin 2\theta}{\sin\theta}\,d\theta = \int \frac{2\sin\theta\cos\theta}{\sin\theta}\,d\theta$

$= 2\int \cos\theta\,d\theta = 2\sin\theta + C$

NOTE: $2\int \cos\theta\,d\theta = 2\left[\sin\theta + C_1\right]$

$= 2\sin\theta + \underbrace{2C_1}_{\text{A CONSTANT}} = 2\sin\theta + C$

LET $2C_1 = C$

L. HELPFUL SOMETIMES TO EVALUATE A <u>DEFINITE</u> INTEGRAL BY FIRST FINDING THE <u>INDEFINITE</u> INTEGRAL

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\tan x}{\sec x}\,dx$

TIME OUT $\displaystyle\int \frac{\tan x}{\sec x}\,dx = \int \frac{1}{\sec x}\tan x\,dx$

$\displaystyle = \int \cos x\,\frac{\sin x}{\cos x}\,dx = \int \sin x\,dx = -\cos x + C$

TIME IN

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\tan x}{\sec x}\,dx = \Big[-\cos x\Big]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} - (-\cos 0)$

$= -0 - (-1) = 1$

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HOMEWORK

PAGE 397: 1, 3, 5, 7, 9, 11, 12, 15, 16, 17, 18; 43

HINT: $|x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x < 0 \end{cases}$

$$\int_a^b f(x)\,dx = \int_a^m f(x)\,dx + \int_m^b f(x)\,dx$$

BASIC

KNOWLEDGE

QUESTIONS

MAT 121
BASIC KNOWLEDGE QUESTIONS

1. GIVE THE DEFINITION OF A <u>FUNCTION</u>

2. LET f BE THE FUNCTION $\{(1,2),(5,3),(4,2)\}$

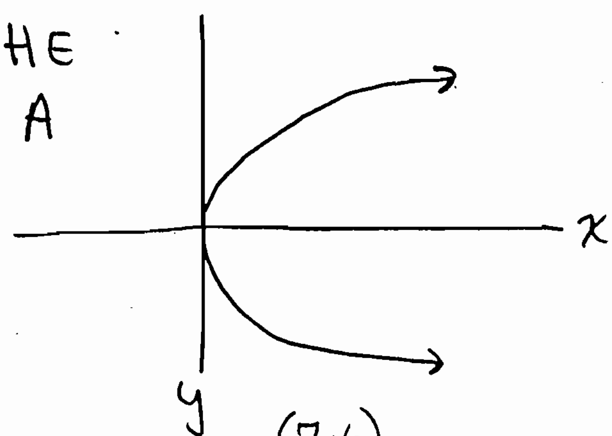
   $f(4) =$ ____

3. FOR A FUNCTION DEFINED BY AN EQUATION, THE <u>INDEPENDENT</u> VARIABLE IS ASSOCIATED WITH THE ____ TERMS OF THE ORDERED PAIRS OF THE FUNCTION

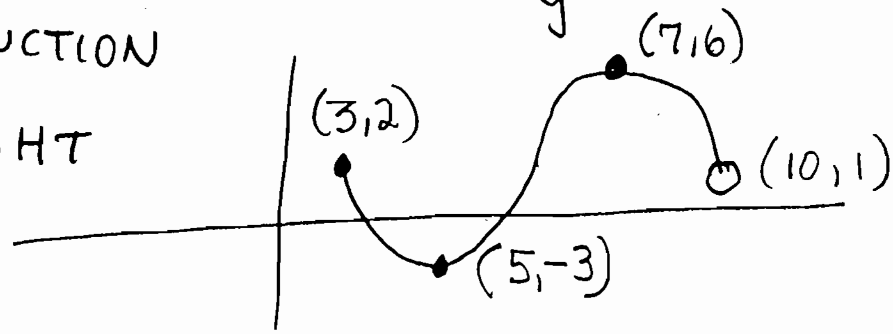
4. $f(x) = 3x^2 + 4x + 5$ $\quad f(x+h) =$ ____________

5. FOR $f(x) = \dfrac{1}{\sqrt{x-2}}$ $\quad \mathrm{dom}(f) =$ ____________

6. IS THE GRAPH AT THE RIGHT THE GRAPH OF A FUNCTION? WHY OR WHY NOT?



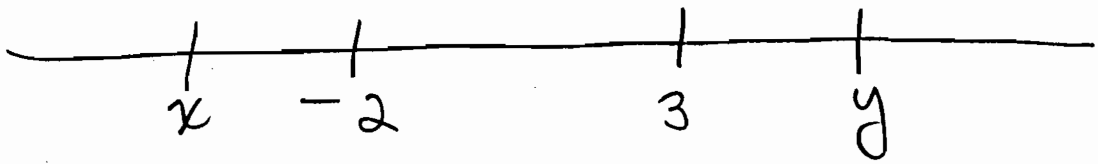
7. WHAT IS THE RANGE FOR THE FUNCTION AT THE RIGHT



MAT 121 BASIC KNOWLEDGE QUESTIONS

8. REMOVE ABSOLUTE VALUE SIGNS ACCORDING TO DEFINITION, TELL WHY.

$$x < -2 \qquad y > 3$$



$|x| =$

$|xy| =$

$|x - y| =$

9. $\sin \frac{\pi}{3} =$

10. $\sec \frac{\pi}{4} =$

11. $\cos 180^\circ =$

12. $\sec 0^\circ =$

13. $1 + \tan^2\theta =$ ______

14. $\cos^2\theta =$ ____________

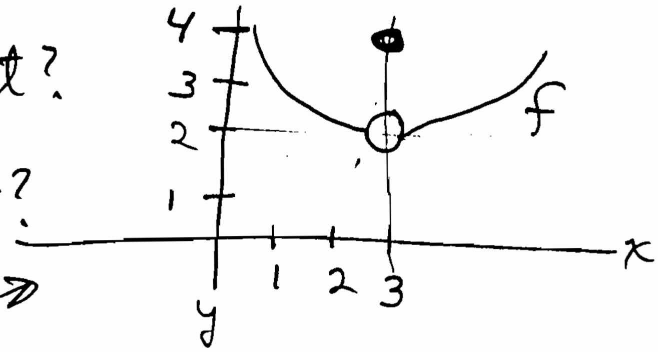
15. $\cos^2\theta + \sin^2\theta =$ ______

QUESTIONS 16-24 ARE OMITTED.

MAT 121 BASIC KNOWLEDGE QUESTIONS

25. DOES $\lim_{x \to 3} f(x)$ exist?

If yes, what is it?



26. $f(3) =$

27. TO EVALUATE $\lim_{x \to 0} \frac{\sqrt{x^2+25} - 5}{x^2}$,

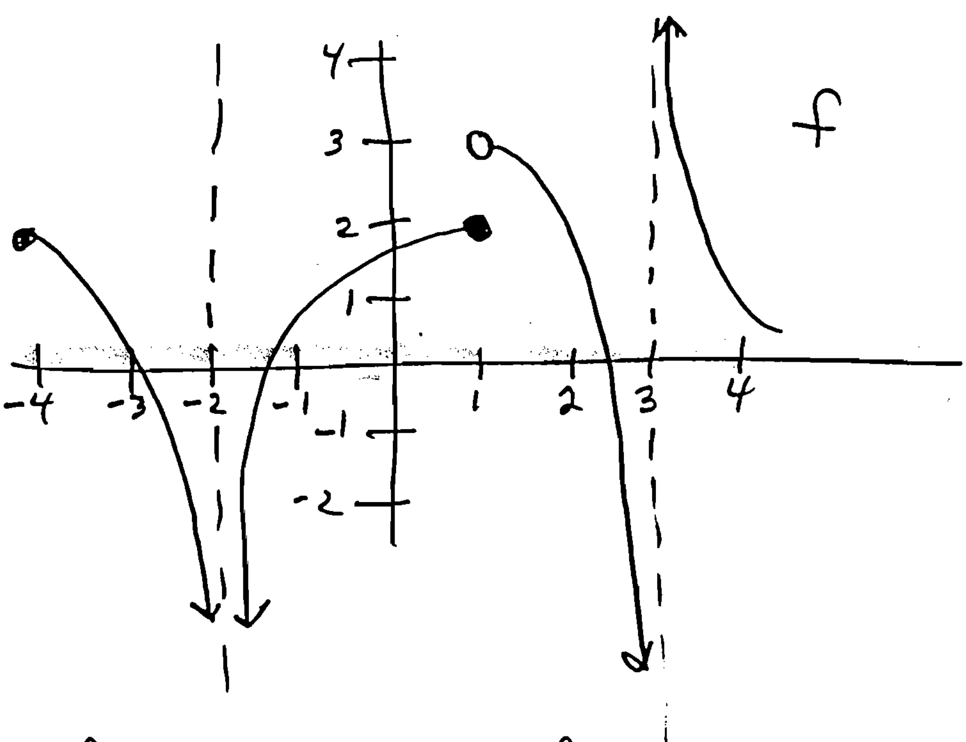
MULTIPLY TOP AND BOTTOM OF THE FRACTION BY WHAT?

28. WHEN x IS NEGATIVE, WHAT IS THE VALUE OF $\frac{|x|}{x}$? WHY?

MAT 121 BASIC KNOWLEDGE QUESTIONS

29. HOW IS $\lim_{x \to -5^-} f(x) = 7$ READ?

30. CONSIDER THE GRAPH BELOW



$\lim_{x \to 1^-} f(x) =$ $\qquad$ $\lim_{x \to 1^+} f(x) =$

$\lim_{x \to 1} f(x) =$ $\qquad$ $\lim_{x \to 3^-} f(x) =$

$\lim_{x \to -2^+} f(x) =$ $\qquad$ $\lim_{x \to -2} f(x) =$

MAT 121 BASIC KNOWLEDGE QUESTIONS

31. GIVE THE RIGOROUS DEFINITION FOR
$$\lim_{x \to 5} 3x - 7 = 8$$

32. WHAT ARE THE FIRST 3 STEPS OF A RIGOROUS PROOF OF $\lim_{x \to 5} 3x - 7 = 8$? (OMIT SPECIFYING δ)

33. WHAT ARE THE FIRST 3 STEPS OF A RIGOROUS PROOF OF $\lim_{x \to -4} 3x + 1 = -11$? (OMIT SPECIFYING δ)

34. NAME $f(x)$, $g(x)$, and "a" WHERE
$$\lim_{x \to a} [f(x) + g(x)] \neq \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

35. WHEN IS IT TRUE THAT
$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

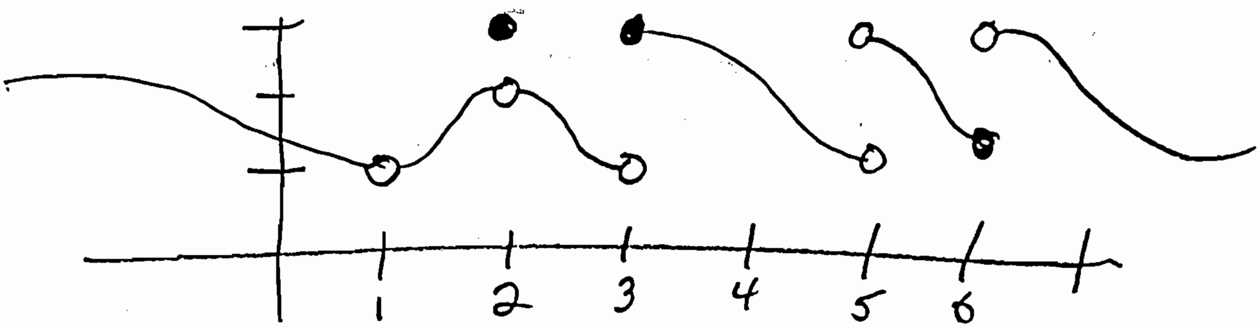
36. NAME $f(x)$, $g(x)$, and "a" WHERE
$$\lim_{x \to a} f(x) \cdot g(x) \neq \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$$

MAT 121 BASIC KNOWLEDGE QUESTIONS

37. GIVE THE DEFINITION OF f IS CONTINUOUS AT 5

38. GIVE THE DEFINITION THAT f IS CONTINUOUS ON $[3,6]$.

39. NAME THE PLACES WHERE f HAS A REMOVABLE DISCONTINUITY



40. $\lim\limits_{x \to -\infty} \frac{5}{x^2} =$

41. TO EVALUATE $\lim\limits_{x \to \infty} \frac{2x^3+7}{3x-5x^3+2}$

MULTIPLY TOP AND BOTTOM

BY ________________

42. STATE THE INTERMEDIATE VALUE THEOREM

MAT 121 BASIC KNOWLEDGE QUESTIONS

43. GIVE THE DEFINITION OF THE DERIVATIVE OF f AT a ($f'(a) =$ ____)

44. WHAT IS THE GRAPHICAL INTERPRETATION OF $f'(a)$?

45. IF $f(t)$ IS THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME t, THEN $f'(t)$ IS ________

46. GIVE THE EQUIVALENT FORMULATION OF $f'(a)$

47. $$\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = f'(a)$$

WHERE $f(x) =$ ____ AND $a =$ ____

48. IF $W = f(h)$

h: height (FEET)

W: WEIGHT (POUNDS)

THEN

$f'(h)$ IS THE INSTANTANEOUS

____ ____ ____ OF _____

WITH RESPECT TO ______

UNITS: _______

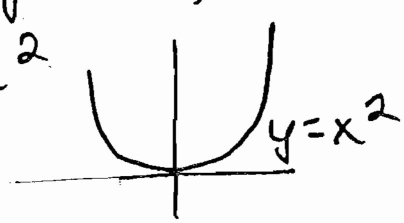
MAT 121 BASIC KNOWLEDGE QUESTIONS

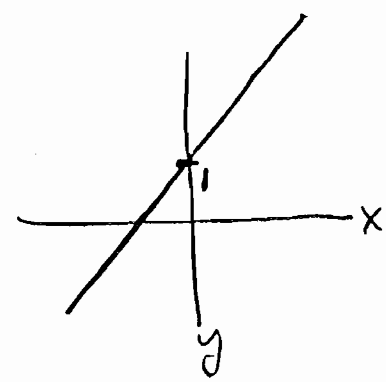
49. GIVE THE LEIBNIZ NOTATION FOR $f'(x)$ WHERE $y=f(x)$.

50. IF f IS DIFFERENTIABLE AT a, IS f NECESSARILY CONTINUOUS AT a? IF NO, GIVE AN EXAMPLE OF f AND a SO THAT f IS DIFFERENTIABLE AT a BUT NOT CONTINUOUS AT a.

51. IF f IS CONTINUOUS AT a, IS f NECESSARILY DIFFERENTIABLE AT a? IF NO, GIVE AN EXAMPLE OF f AND a SO THAT f IS CONTINOUUS AT a BUT NOT DIFFERENTIABLE AT a.

52. EXPLAIN WHY THE GRAPH AT THE RIGHT IS NOT THE GRAPH OF $y=f'(x)$ WHERE $y=f(x)=x^2$





53. For $y=f(x)$, GIVE THE PRIME NOTATION FOR $\dfrac{d^2y}{dx^2}$.

53A. $\dfrac{d\sqrt{x}}{dx} =$

MAT 121 BASIC KNOWLEDGE QUESTIONS

54. FOR $f(x) = \dfrac{2-3x}{5x+4}$, FIND $f'(x)$.

STATE ANSWER IN FORMULA FORM.

55. FOR $f(x) = (3-2x)(5x+20)$, FIND $f'(x)$.
DO NOT MULTIPLY OUT, BUT STATE ANSWER IN FORMULA FORM

56. LET $f(x) = \dfrac{5}{x}$. $f'(x) =$ ______

57–60 OMITTED

61. $\displaystyle\lim_{t \to 0} \frac{\sin t}{t} =$ ______

62. $\dfrac{d}{dx}(\csc x) =$ ______

63 $\dfrac{d}{dx}(\tan x) =$ ______

64. THE CHAIN RULE GIVES THE DERIVATIVE OF THE ______

65. GIVE THE FORMULA FOR THE CHAIN RULE

66. $\dfrac{d}{dx} e^{x^2+7} =$

67 $\dfrac{d}{dx}(1+\sin 2x)^8 =$

68 FOR $f(x) = \sec x^2$, $f'(x) =$

BASIC KNOWLEDGE QUESTIONS

69. ASSUME y IS A FUNCTION OF x.

a. $\frac{d}{dx}(y^4) =$

b. $\frac{d}{dx}(x^8 y^4) =$

70 $\frac{d}{dx} \sin^{-1}(e^{3x})$

71. $\frac{d}{dx} \tan^{-1}(e^{3x})$

72. $\frac{d}{dx} \sec^{-1}(e^{3x})$

73. FOR $y = f(x)$, GIVE THE PRIME NOTATION FOR $\frac{d^2y}{dx^2}$.

74. $\frac{d}{dx}\left(\ln(x^2+1)\right) =$

75. $\frac{d}{dx}\left(\log_7(x^2+1)\right) =$

76. IF $f(t)$ IS THE POSITION OF AN OBJECT ON A COORDINATE LINE AT TIME t, THEN $f''(t)$ IS THE ______

BASIC KNOWLEDGE QUESTIONS

76A SUPPOSE $f(t)$ IS THE POSITION OF AN OBJECT ON A HORIZONTAL COORDINATE LINE AT TIME t (POSITIVE TO THE RIGHT)

a. WHEN DOES THE OBJECT MOVE TO THE LEFT?

b. WHEN IS THE OBJECT STOPPED

76B SUPPOSE $f(t)$ IS THE POSITION OF AN OBJECT THROWN STRAIGHT UP ALONG A VERTICAL COORDINATE LINE (UP POSITIVE)

a) HOW DO YOU FIND MAXIMUM HEIGHT?

b) HOW DO YOU FIND HOW FAST THE OBJECT IS TRAVELLING WHEN IT HITS THE GROUND?

BASIC KNOWLEDGE QUESTIONS

77. GIVE THE EXPONENTIAL GROWTH EQUATION THAT IS THE SOLUTION TO $P'(t) = K\,P(t)$.

78. FOR A POPULATION COUNT $P(t)$, $P'(t)$ IS THE <u>GROWTH RATE</u>. THE RELATIVE GROWTH RATE IS THE FRACTION ______.

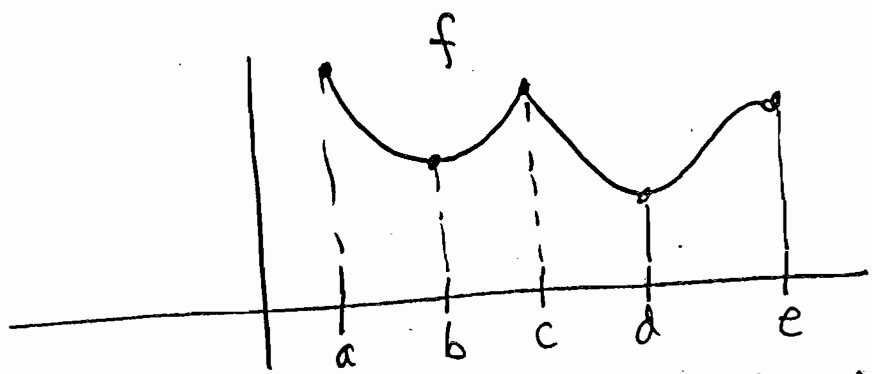
79. TAKE $\frac{d}{dt}$ OF BOTH SIDES

$$x^{\frac{1}{2}} + y^{\frac{2}{3}} = 7$$

80. FOR $y = f(x)$, GIVE THE DEFINITION OF THE DIFFERENTIAL OF y: $dy =$ ______

81. FOR $y = \sin x^2$, $dy =$ ______

82.



1) NAME WHERE LOCAL EXTREMA <u>OCCUR</u> THAT ARE NOT ABSOLUTE EXTREMA
2) WHAT IS THE ABSOLUTE MAX <u>VALUE</u>? IS IT A LOCAL MAX VALUE?
3) WHAT IS THE ABSOLUTE MIN <u>VALUE</u>? IS IT A LOCAL MIN VALUE?

83. DEFINITION OF C IS A CRITICAL NUMBER FOR FUNCTION f.

BASIC KNOWLEDGE QUESTIONS MAT 121

84. STATE THE METHOD FOR FINDING ABSOLUTE EXTREMA OF CONTINUOUS f ON $[a,b]$

85. $f(x) = x^{-1}$. $f'(x) = -x^{-2} = -\frac{1}{x^2}$

$f'(0)$ DOES NOT EXIST. WHY IS 0 NOT A CRITICAL NUMBER FOR f?

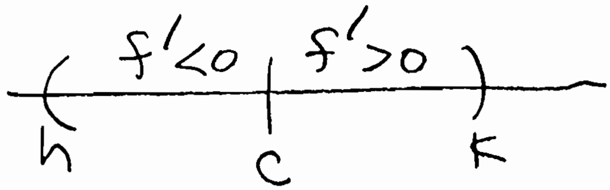
86. STATE ROLLE'S THEOREM

87. STATE THE MEAN VALUE THEOREM FOR DERIVATIVES.

88. COMPLETE ACCORDING TO A THEOREM WE STUDIED: IF FOR ALL $x \in (a,b)$, $f'(x) = g'(x)$, THEN ________________

89. IF FOR ALL $x \in (a,b)$, $f'(x) < 0$, THEN f ____________ ON (a,b)

90.



C IS A CRITICAL NUMBER FOR CONT. f

$f(c)$ IS A ________________

BASIC KNOWLEDGE QUESTIONS MAT 121

91. IF $f'' < 0$ ON (a,b), THEN f IS __________ ON (a,b)

92. IF f'' IS CONTINUOUS ON (a,b) AND $c \in (a,b)$. AND $f'(c) = 0$, THEN IF $f''(c) > 0$, THEN $f(c)$ IS ________

93. SUPPOSE THE GRAPH OF FUNCTION f CHANGES FROM CONCAVE UP TO CONCAVE DOWN AT THE POINT P ON THE GRAPH OF f, THEN P IS ____________

94. FOR $\lim\limits_{x \to 0} \frac{\sin x}{x}$, DOES L'HOSPITAL'S RULE APPLY? IF YES, WHAT FORM IS IT IN, THEN APPLY L'HOSPITAL'S RULE TO GET THE ANSWER.

95. TO APPLY L'HOSPITAL'S RULE TO THE FORM 1^∞, LET y BE WHAT YOU WANT TO TAKE THE LIMIT OF (I.E. YOU WANT $\lim y$)
FIND $\lim \ln y = L$
THEN $\lim y =$ _____ $=$ _____ $=$ _____

96. WHAT IS THE DEFINITION FOR $y = x+1$ BEING A SLANT ASYMPTOTE FOR $f(x) = \dfrac{x^2}{x-1}$?

BASIC KNOWLEDGE QUESTIONS MAT 121

97. WHAT ARE THE 8 STEPS LISTED IN THE COURSE NOTES FOR SOLVING MAX\MIN WORD PROBLEMS?

NO QUESTIONS 98, 99

100. WHAT IS THE MOST GENERAL ANTIDERIVATIVE FOR

$$f(x) = 7x^3$$

$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{5}{\sqrt{1-x^2}}$$

$$f(x) = \sin x$$

101 WHAT IS THE VALUE FOR THE ACCELERATION DUE TO GRAVITY?

THERE ARE NO QUESTIONS NUMBERED 102 – 117

118. $\sum_{i=1}^{3} 7 =$

119 $\sum_{i=1}^{n} i =$

120 NAME A PARTITION OF $[1,4]$.

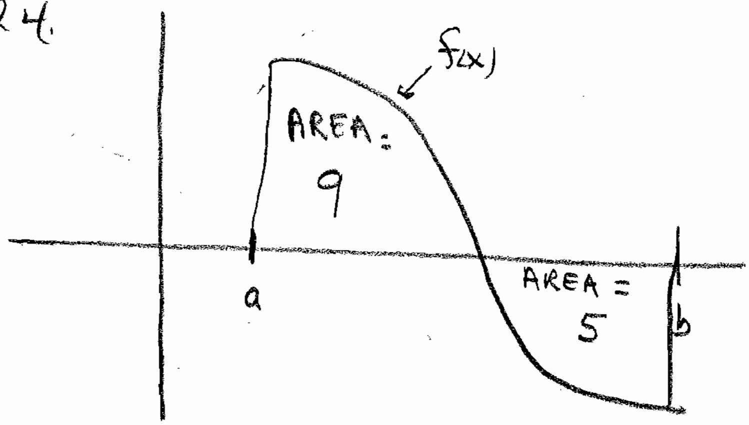
121. GIVE THE GENERAL FORM FOR A RIEMANN SUM IN $\sum$ NOTATION

122. FOR A REGULAR PARTITION OF $[2,5]$ WITH $n=4$, $\Delta x =$ ______

DEFINITE INTEGRAL QUESTIONS

123 WHAT EQUATION IS IN THE DEFINITION OF A DEFINITE INTEGRAL $\int_a^b f(x)\,dx =$

124.



$$\int_a^b f(x)\,dx =$$

125 $\int_2^7 3\,dx =$

126. $\int_3^8 7x\,dx = \_\_ \int_8^3 \_\_\_\_ dx$

FUNDAMENTAL THM OF CALCULUS

127 STATE THE FUNDAMENTAL THEOREM OF CALCULUS

a. PART I

b. PART II

128. $D_x \int_7^x \sqrt{1+t^2}\,dt =$

129. $D_x \int_7^{x^2} \sqrt{1+t^2}\,dt =$

130. $\int_1^2 3x^2\,dx =$ ← Tell the

131 $\int_0^{\frac{\pi}{2}} \sin x\,dx =$ ← steps

INDEFINITE INTEGRAL QUESTIONS

132. $\int 5x^2\,dx$

133. $\int \frac{4}{\sqrt{1-x^2}}\,dx$

134. $\int D_x F(x)\,dx =$ ________

135. $D_x \int f(x)\,dx =$ ________

136. $\int h(x)\,dx = x^5 + C$ iff $h(x) =$ ______

TRUTH GEMS

THE ANSWER
IS IN
THE BACK OF
THE BOOK

TRUTH GEM

BE IN THE WILL OF GOD FOR WHAT YOU DO

A. ROM 15:32 ....that I may come to you with joy by the WILL OF GOD, and may be refreshed together with you.

B. I come to you in the WILL OF GOD with joy and we will have refreshing math.

C. Being in the WILL OF GOD taking this course and faithfully, wisely studying you will flourish.

D. HEB. 10:36 For you have need of endurance, so that after you have done the WILL OF GOD you may receive the promise.

TG-2

TRUTH GEM

WISDOM

A. PR 1:7 WISDOM IS THE PRINCIPAL THING; THEREFORE GET WISDOM. AND IN ALL YOUR GETTING, GET UNDERSTANDING.

B. <u>DEFINITIONS</u>:

1. <u>KNOWLEDGE</u>: FACTS, GAINED INFORMATION
2. <u>UNDERSTANDING</u>: WHY A FACT IS A FACT.
3. <u>WISDOM</u>: BEING LED BY THE SPIRIT. KNOWING WHAT TO DO AT ANY MOMENT.

C. PRAY FOR WISDOM <u>IN</u> <u>FAITH</u>: JAMES 1:5
IF ANY OF YOU LACKS WISDOM, LET HIM ASK OF GOD, WHO GIVES TO ALL LIBERALLY AND WITHOUT REPROACH, AND IT WILL BE GIVEN HIM.

TRUTH GEM

BEGIN

A. ACTS 1:1 "... of all that Jesus BEGAN both to do and teach."

B. MK 4:1 "And again He BEGAN to teach by the sea."

C. For a task to be accomplished, you must BEGIN.

D. BEGINNINGS:

1. BEGIN to see yourself as a faithful, good math student
2. See changes you need to make, and BEGIN on those changes.
3. See and learn BEGINNINGS of different problem types.

TRUTH GEM

PRESS ON

A. PLP 3:12 NOT THAT I HAVE ALREADY ATTAINED... BUT I <u>PRESS</u> <u>ON</u>, THAT I MAY LAY HOLD OF THAT FOR WHICH CHRIST JESUS HAS ALSO LAID HOLD OF ME.

B. LIKE A DISTANCE RUNNER OR CRUISE CONTROL

1. CONSTANCY, STEADFASTNESS
2. PATIENCE POWER

GOOD WORDS

C. <u>PRESS</u> <u>ON</u> CYCLE DONE WITH A GOOD ATTITUDE (NOT AT LAST MINUTE)

HEAR THE NEW
REVIEW
STUDY
DO HOMEWORK

D. HEB 6:12 ... THAT YOU DO NOT BECOME SLUGGISH (LAZY), BUT IMITATE THOSE WHO THROUGH FAITH AND PATIENCE INHERIT THE PROMISES.

TRUTH GEM

COMPLETE IT

A. PLP 1:6 "... being confident of this very thing, that He who has BEGUN a good work in you will COMPLETE IT until the day of Jesus Christ.

B. THINGS GOD BEGAN, IN HIS WILL, GOD GIVES ABILITY AND PROVISION TO COMPLETE.

1. SEE THESE THINGS THROUGH TO THE END.
2. VICTORY IS SWEET

C. THERE CAN BE BARRIERS TO BREAK THROUGH AT THE END.

1. LIKE A TAPE AT THE END OF A RACE.
2. LIKE THE SOUND BARRIER.

TRUTH GEM

BE ESTABLISHED

A. Ps 90:17 And let the beauty of the Lord our God be upon us, and ESTABLISH the work of our hands for us;

B. Be established in the BEGIN - PRESS ON - COMPLETE IT cycle for working problems

C. Be established in knowing how to work certain problem types

D. Hebrew: established = koon : things brought into incontroovertible existence like:

1. Your nature to faithfully study
2. Your ability to work certain problem types

E. To learn better how to be ESTABLISHED

1. Is 54:14a In righteousness you shall be ESTABLISHED
2. We will learn of righteousness

TG-7

TRUTH GEM

RIGHTEOUSNESS

A. HEB 9:28a For He will finish the work and cut it short in RIGHTEOUSNESS;

B. RIGHTEOUSNESS = RIGHT STANDING WITH GOD BY FAITH.

PLP 3:9 and be found in Him, not having my own righteousness, which is from the law, but that which is through faith in Christ, the righteousness which is from God by faith.

C. BEING IN THE WILL OF GOD FOR WHAT YOU DO, IN RIGHT STANDING WITH GOD, THERE IS GREAT LIBERTY AND SPEEDUP IN WHAT YOU DO. (RIGHTEOUSNESS ENHANCED ACCELERATED LEARNING)

TG-8

TRUTH GEM

RIGHTEOUSNESS - SHALOM

A. IS 32:17 THE WORK OF RIGHTEOUSNESS (RIGHT STANDING) WILL BE PEACE (SHALOM: NOTHING MISSING, NOTHING BROKEN, COMPLETENESS, HEALTH, PROSPERITY, SAFETY, PEACE)

B. THE SHALOM LEARNING ENVIRONMENT: TEACHER & STUDENT AT PEACE AS THEY TEACH, STUDY, TAKE TESTS, & GRADE PAPERS.

C. "LET RIGHTEOUSNESS WORK FOR YOU" (ED TAYLOR)

D. SPEEDUP BY LEARNING IN A STATE OF SHALOM

TG-9

TRUTH GEM

RIGHTEOUSNESS - BOLD

A. PR. 28:1 THE WICKED FLEE WHEN NO ONE PURSUES, BUT THE RIGHTEOUS ARE AS BOLD AS A LION

B. PR. 30:30 A LION, WHICH IS MIGHTY AMONG BEASTS AND DOES NOT TURN AWAY FROM ANY;

C. LEARNING IS SLOWED DOWN BY WIMPILY, TIMIDLY TURNING AWAY FROM SOME MATH PROBLEMS.

D. LEARNING SPEEDUP BY BOLDLY (IN ACCORDANCE WITH YOUR RIGHTEOUS NATURE) TAKING ON THINGS ON YOUR PATH AND OVERCOMING.

E. NEXT, BOLDNESS AND HUMILITY, NOT CONTRADICTARY.

TRUTH GEM

RIGHTEOUSNESS: BOLD & HUMBLE

A. PLP 2:8a He (Jesus) HUMBLED Himself and became obedient..

B. James 4:6-7a "... God resists the proud, but gives grace to the HUMBLE." Therefore submit to God.

C. HUMILITY: Go God's way not your own way.

D. Some problems it takes boldness to solve. A false humility doctrine can cause someone to wimp out and be defeated.

E. There is a difference between boldness and aggression

TRUTH GEM

FAITH – NO FEAR

A. MK 5:36b "DO NOT BE AFRAID, ONLY BELIEVE." —— HAVE FAITH

B. FEAR IS HAVING MORE FAITH IN THE POWER OF THE DEVIL TO DO HARM THAN THE POWER OF GOD TO DO GOOD. Philip Derber

C. FAITH IS LIKE PUTTING IT IN DRIVE,
DOUBT IS LIKE PUTTING IT IN NEUTRAL,
FEAR IS LIKE PUTTING IT IN REVERSE
UNBELIEF IS LIKE PUTTING IT IN PARK.
John Paul

D. ROM 12:21 "DO NOT BE OVERCOME BY EVIL, BUT OVERCOME EVIL WITH GOOD."
GOOD OVERCOMES EVIL SO DO NOT FEAR.

E. FEAR ATTRACTS THE FEARED THING. FAITH IS THE SUPERNATURAL CONNECTION THAT RECEIVES THE OVERCOMING GOOD – THE THING BELIEVED FOR ... MATH UNDERSTANDING

TG-12

TRUTH GEM

GRACE

A. I Pet 4:10-11 part As each one has received a GIFT, minister it to one another as good stewards of the manifold GRACE of God... If anyone ministers let him do it as with the ABILITY which God supplies...

B. DEFINITION: GRACE - God's ability gift to live and function in the gifts and callings.

C. Grace is part of God's supernatural provision to do excellently what God has called us to do

D. 2 Cor 9:8 And God is able to make all GRACE abound toward you, that you, always having all sufficiency in all things, may have an abundance for every good work.

TRUTH GEM

GRACE - WORKS EFFECTIVELY

A. Parts of Gal 2:7-9 ...when they SAW that the gospel for the uncircumcised had been committed to me.. for He who... WORKED EFFECTIVELY in me toward the Gentiles, ... when James, Cephas, and John,... perceived the GRACE that had been given to me.

B. A GRACEFUL PERSON WORKS EFFECTIVELY

C. GRACE: GOD'S ABILITY GIFT TO LIVE AND FUNCTION IN THE GIFTS AND CALLINGS

D. BEING IN GOD'S WILL FOR TAKING THIS COURSE, THERE IS GRACE (SUPERNATURAL ABILITY TO WORK EFFECTIVELY) FOR YOU TO DO SO WELL IT CAN BE SEEN.

TG-14

TRUTH GEM

IDOLATRY

A. COL. 3:5b ... PUT TO DEATH YOUR MEMBERS WHICH ARE ON THE EARTH: FORNICATION, UNCLEANNESS, PASSION, EVIL DESIRE, AND COVETEOUSNESS, WHICH IS IDOLATRY.

B. IDOLS: PEOPLE WORSHIPPED IMAGES OF THINGS THAT ARE NOTHING (I COR 8:4b ... WE KNOW THAT AN IDOL IS NOTHING...)

C. MANY PEOPLE COVET THE IMAGE OF BEING EDUCATED, BUT THERE IS NO REALITY, NOTHING TO BACK UP THE IMAGE.

D. SOME WANT TO WORK IN GROUPS AND GET A GROUP GRADE WHEN IN TRUTH THEY DO NOT KNOW IT, OR WRITE A PAPER ON FEELINGS ABOUT MATH RATHER THAN DO MATH. THEY ARE CONTENT WITH THE GRADE, THE IMAGE (THE IDOL) EVEN THOUGH THEY DO NOT KNOW.

E. FROM I JN 5:21 ... KEEP YOURSELF FROM IDOLS. [DESIRE TO KNOW THE MATH, DESIRE TO BE ABLE TO OBJECTIVELY DEMONSTRATE THAT YOU KNOW.]

TRUTH GEM

DISCIPLINE

A. 2 Tim 1:7 For God has not given us a spirit of timidity, but of power and love and DISCIPLINE.

B. The undisciplined have problems with math.

C. Regular steady study (in wisdom) prevails, not just a big flurry the night before the test.

D. You will have to keep retaking life's test on discipline until you pass it.

E. For the Christian, DISCIPLINE IS a fruit of what we are given.
Gifts must be received.
Gifts must be cultivated.

TG-16

TRUTH GEM

MEDITATE

A. PS 1:2 BUT HIS DELIGHT IS IN THE LAW OF THE LORD AND IN HIS LAW HE MEDITATES* DAY AND NIGHT.

*PONDERS BY TALKING TO HIMSELF

B. WHAT YOU ARE IN THE WILL OF GOD TO LEARN CAN BE MEDITATED.

C. PICK A DEFINITION, THEOREM, OR PROBLEM DERIVATION.

1. CHEW ON IT WORD FOR WORD SEEKING UNDERSTADING
2. SPEAK IT OUT SO YOU CAN HEAR IT
3. WRITE IT DOWN OVER AND OVER
4. REVIEW IT

TG-17

TRUTH GEM

MAGNIFY THE SOLUTION AND NOT THE PROBLEM

A. PS 34:3 OH, MAGNIFY THE LORD WITH ME AND LET US EXALT HIS NAME TOGETHER

B. FOCUS ON WHAT YOU SEE IS TRUE AND GOOD AND CAN BE DONE. DO THAT. THE PROBLEM SHRINKS. SEE SOMETHING ELSE THAT IS TRUE AND GOOD AND CAN BE DONE. DO THAT. THE PROBLEM SHRINKS. PRESS ON DOING THIS UNTIL THE PROBLEM IS GONE!

C. PR 3:27 DO NOT WITHHOLD GOOD FROM THOSE TO WHOM IT IS DUE, WHEN IT IS IN THE POWER OF YOUR HAND TO DO SO.

D. JAMES 4:17 THEREFORE TO HIM WHO KNOWS TO DO GOOD AND DOES NOT DO IT, TO HIM IT IS SIN.

TG-18

TRUTH GEM

LITTLE BY LITTLE

A. DEUT. 7:22 AND THE LORD YOUR GOD WILL DRIVE OUT THOSE NATIONS BEFORE YOU <u>LITTLE BY LITTLE</u>; YOU WILL BE UNABLE TO DESTROY THEM AT ONCE, (EX 23:29 LEST THE LAND BECOME DESOLATE AND) LEST THE BEASTS OF THE FIELD BECOME TOO NUMEROUS FOR YOU.

B. LEARNING MATHEMATICS IS LIKE POSSESSING NEW TERRITORY. YOU MUST DO IT <u>LITTLE BY LITTLE</u> AND GET ESTABLISHED IN, CULTIVATE, PATROL\GUARD THAT GAINED WISDOM\KNOWLEDGE\UNDERSTANDING

C. TO TRY TO LEARN ALL AT ONCE, YOUR MIND IS STRETCHED TOO THIN (LAND BECOMES DESOLATE) YOU ARE UNABLE TO PATROL\GUARD AGAINST THE BEASTS OF THE FIELD (CONFUSION ATTEMPTS, LACK OF REMEMBERANCE, FEAR OF A HERD OF QUESTIONS COMING AT YOU ON A TEST)

D. DO NOT WAIT UNTIL RIGHT BEFORE THE TEST TO TRY TO LEARN DAYS OF WORK, WISDOM: LITTLE BY LITTLE.

TRUTH GEM

BE STEADFAST IN HOMEWORK

A. I COR 15:58 THEREFORE, MY BELOVED BRETHREN, BE STEADFAST, IMMOVABLE, ALWAYS ABOUNDING IN THE WORK OF THE LORD, KNOWING THAT YOUR LABOR IS NOT IN VAIN IN THE LORD.

B. BEING IN THE WILL OF GOD BY TAKING THIS COURSE MEANS YOU ARE DOING THE WORK OF THE LORD, ... SO... BE STEADFAST

C. DISTRACTIONS, LOWER PRIORITY THINGS WILL TRY TO TAKE YOU AWAY FROM "ALWAYS ABOUNDING" IN STUDY ... BE SET. DO NOT BE MOVED. HOLD FAST AND STUDY.

D. PS 16:8 I HAVE SET THE LORD ALWAYS BEFORE ME; BECAUSE HE IS AT MY RIGHT HAND I SHALL NOT BE MOVED.

TG-20

TRUTH GEM

PRIORITIZE

A. NEH 6:3 I AM DOING A GREAT WORK, SO THAT I CANNOT COME <u>DOWN</u>. WHY SHOULD THE WORK CEASE WHILE I LEAVE IT AND GO <u>DOWN</u> TO YOU?

B. HAVE GOD'S PRIORITIES FOR YOUR LIFE CLEAR, DECIDED, SET, ESTABLISHED... A DIVINE ORDER. WHEN SOMETHING COMES UP FOR A DECISION, DISCERN WHAT CATEGORY IT IS IN AND THE DECISION HAS ALREADY BEEN MADE.

C. BEING IN GOD'S WILL FOR TAKING THIS COURSE MEANS THIS COURSE IS A HIGH PRIORITY, SO REGULAR, NONDISTRACTED STUDY TIME IS A <u>HIGH</u> PRIORITY, SO... DO NOT LEAVE IT AND GO <u>DOWN</u> TO DO A LESSER PRIORITY.

TG-21

TRUTH GEM

OVERCOME

A. I JN 5:4 FOR WHATEVER IS BORN OF GOD OVERCOMES THE WORLD. AND THIS IS THE VICTORY THAT HAS OVERCOME THE WORLD— OUR FAITH.

B. BEING IN THE WILL OF GOD FOR TAKING THIS COURSE, AND HENCE DOING A GREAT WORK, YOU WILL BE COME AGAINST TO TRY TO STOP, HINDER, OR HARASS THE WORK OF REGULAR STUDY — OVERCOME IT WITH SUPERNATURAL HELP

C. WITH A BOLD, NONTIMID, STRONG AND COURAGEOUS INNER MAN — SAY NO AND RESIST AND OVERCOME THE OPPOSITION

D. EPH 3:16 ... THAT HE WOULD GRANT YOU, ACCORDING TO THE RICHES OF HIS GLORY, TO BE STRENGTHENED WITH MIGHT THROUGH HIS SPIRIT IN THE INNER MAN

TG-22

TRUTH GEM

DECISIONS VS. OTSGWFIADI

A. MK 14:36b ... NEVERTHELESS, NOT WHAT I WILL, BUT WHAT YOU WILL.

B. WHEN WHAT IS CALLED A DECISION IS PRESENTED TO YOU (LIKE TO START STUDYING ON DO SOMETHING ELSE - OR - TO KEEP STUDYING LONGER OR STOP). WHAT IS THE BASIS FOR YOUR DECISION?... YOUR OWN UNDERSTANDING? YOUR PLEASURES?

C. PR 3:5-6 TRUST IN THE LORD WITH ALL YOUR HEART AND LEAN NOT ON YOUR OWN UNDERSTANDING; IN ALL YOUR WAYS ACKNOWLEDGE HIM AND HE SHALL DIRECT YOUR PATHS

D. INSTEAD OF DECISION TIME ITS OTSGWFIADI TIME: OPPORTUNITY TO SEEK GOD'S WILL, FIND IT, AND DO IT.

E. PART OF MT 7:7 ... SEEK AND YOU WILL FIND...

TRUTH GEM

DAVID AND GOLIATH

A. BEFORE GOLIATH'S DEFEAT: PART I SAMUEL 17:11
... THEY WERE DISMAYED AND GREATLY AFRAID.

B. PROPER BATTLE ATTITUDE: I SAM. 17:48b
... DAVID HURRIED AND RAN TOWARD THE ARMY TO MEET THE PHILISTINE.

C. AFTER GOLIATH'S DEFEAT: I SAM. 17:51b
... AND WHEN THE PHILISTINES SAW THAT THEIR CHAMPION WAS DEAD, THEY FLED.

D. THE LOSS OF ONE SOLDIER IN THE PHILISTINE ARMY CAUSED A MAJOR REVERSAL - FROM TAUNTING TO FLEEING ... A REASON: YOU DEFEAT AN ENEMY'S MOST POWERFUL WEAPON, THEIR STRENGTH IS GONE; WHAT THEY TRUSTED IN WAS GONE; THEY FLEE.

E. BY NOW YOU HAVE HAD VICTORY OVER SOME OF THE MOST POWERFUL PROBLEMS IN THE COURSE. SOME OF INTIMIDATION'S STRONGEST WEAPONS - FEAR OF HARD MATH PROBLEMS AND PAST EXPERIENCE OF DIFFICULTY WITH HARD MATH PROBLEMS, HAS BEEN OVERCOME. THIS ENEMY IS FLEEING... PURSUE ... PLUNDER WITH CONTINUED DISCIPLINE & CLARITY

F. PARTS OF I SAM 17:52,53 NOW THE MEN OF ISRAEL AND JUDAH ... PURSUED ... AND THEY PLUNDERED.

TRUTH GEM

JOY IN RESPONSIBILITY

A. MT 25:21 HIS LORD SAID TO HIM, "WELL DONE, GOOD AND FAITHFUL SERVANT; YOU WERE FAITHFUL OVER A FEW THINGS, I WILL MAKE YOU RULER OVER MANY THINGS. ENTER INTO THE JOY OF YOUR LORD.

B. IT TAKES RESPONSIBILITY TO BE GOOD IN MATH.

C. RESPONSIBILITY IS A GOOD WORD.

D. WHEN GOD PROMOTES YOU AFTER LONG FAITHFUL SERVICE TO MORE RESPONSIBILITY, GOD IS JOYFUL AT HAVING SOMEONE GOOD AND FAITHFUL OVER WHAT GOD CARES VERY MUCH FOR.

1. WE ARE TO ENTER INTO HIS JOY
2. IT IS COMMANDED.
3. GOD MAKES YOU RULER, SO THROUGH HIM YOU ARE ABLE.

E. MANY OF YOU HAVE BEEN GOOD AND FAITHFUL STUDENTS FOR YEARS; YOU HAVE BEEN PROMOTED TO THIS CLASS; ENTER INTO HIS JOY. THIS CLASS IS ALSO A PROVING GROUND FOR FURTHER PROMOTION BY GOD.

TG-25

TRUTH GEM

HOLY SPIRIT BAPTISM IMPLIES BOLDNESS

A. ACTS 4:31b ... THEY WERE ALL FILLED WITH THE HOLY SPIRIT AND SPOKE THE WORD OF GOD WITH BOLDNESS

B. RECALL PR 28:1b .. THE RIGHTEOUS ARE AS BOLD AS A LION.

C. YOU NEED GREAT BOLDNESS TO WORK SOME MATH PROBLEMS.

RIGHTEOUSNESS BOLDNESS + HOLY SPIRIT POWER BOLDNESS = SOLVED PROBLEMS IN THIS COURSE.

D. JESUS TOLD HIS DISCIPLES TO WAIT UNTIL THE RECEIVED POWER FROM ON HIGH UNTIL THEY WENT OUT TO WITNESS

E. WHATEVER WE DO IN HIS WILL THIS BOLDNESS IS AVAILABLE.

F. LK 11:13b ... HOW MUCH MORE WILL YOUR HEAVENLY FATHER GIVE THE HOLY SPIRIT TO THOSE WHO ASK HIM.

TRUTH GEM

FOR EVERY PROBLEM, THERE IS A PROBLEM OBLITERATING REVELATION THAT OBLITERATES THE PROBLEM

A. ACTS 9:3 AS HE JOURNEYED HE CAME NEAR DAMASCUS, AND SUDDENLY A LIGHT SHONE AROUND HIM FROM HEAVEN... GAL 1:23b HE WHO FORMERLY PERSECUTED US NOW PREACHES THE FAITH WHICH HE ONCE TRIED TO DESTROY.

B. PAUL HAD A GREAT PROBLEM. HE WANTED CHRISTIANS JAILED, EVEN HAD A PART IN A STONING. HE GOT A REVELATION. THAT PROBLEM WAS OBLITERATED.

C. FOR EVERY PROBLEM A PERSON HAS, THERE IS A PROMISE OF GOD THAT OBLITERATES THE PROBLEM.

D. II PET 1:4 ...BY WHICH HAVE BEEN GIVEN TO US EXCEEDINGLY GREAT AND PRECIOUS PROMISES, THAT THROUGH THESE YOU MAY BE PARTAKERS OF THE DIVINE NATURE, HAVING ESCAPED THE CORRUPTION THAT IS IN THE WORLD THROUGH LUST.

TRUTH GEM

LEARN FROM THE CLEAR CLEARLY DO

A. JOSH. 11:15 a AS THE LORD COMMANDED MOSES HIS SERVANT, SO MOSES COMMANDED JOSHUA, AND SO JOSHUA DID.

B. WHEN HIRED, THE FIRST THING DONE USUALLY IS TRAINING.

C. THERE IS A MAJOR NEED FOR PEOPLE TO BE TAUGHT CLEARLY FROM THOSE WHO SEE CLEARLY AND FOR THE TAUGHT ONES TO FAITHFULLY DO IT IN TUNE AND IN FOCUS.

D. ROM 13:10 a LOVE DOES NO HARM... (HARM GENERALLY COMES FROM NOT DOING A JOB WELL, THE WAY YOU HAVE BEEN TRAINED)

E. LIFE IS NOT ALL SUBJECTIVE WORD PROBLEMS. MUCH OF IT IS LEARNING FROM THOSE WHO SEE CLEARLY AND DOING IT.

F. SEEK THE CLEAR ONES AND LEARN.

CALLING & DESTINY

A. EPH 1:18 ... the eyes of your understanding enlightened that you may know what is the hope (i.e. <u>destiny</u>) of His <u>calling</u>

B. (Jack Shoup) The <u>calling</u> is the <u>office</u>. The destiny is to be fulfilled in that office. <u>Grace</u> is the provision to fulfill your destiny within and by being in your calling

C. <u>EXAMPLE</u>: Part of my calling is to be a math teacher. Part of my des<u>tiny</u> is to "make it plain". Since I have answered the call - there is grace to fulfill it.

Similar example: <u>call</u>: college student destiny - x = major grade = y

D. You can answer the call and not fulfill your destiny (Jack Shoup)

E. PLP 3:14 I press toward the goal for the prize (<u>destiny</u>) of the upward <u>call</u> of God in Christ Jesus

TG-29

TRUTH GEM

ESTABLISHED 2

A. I PET 5:10 But may the God of all grace who called us to His eternal glory by Christ Jesus, after you have suffered a while, perfect, ESTABLISH, strengthen, and settle you.

B. Establish - brought into incontrovertible existence

C. Be established in ending things well

1. semesters - you have spent a semester getting math understanding - use it to make\redeem your grade.
2. projects
3. moving

D. We go through so much to get to the end - the goal. Do not lose it there (Do not throw away a letter grade just to leave a few minutes early going home at end of semester - you can pack after exams!)

E. 2 Jn :8 Look to yourselves, that we do not lose those things that we worked for, but that we may receive a full reward.

TG-[redacted]30

TRUTH GEM

BE STRONG AND OF GOOD COURAGE X ≈ 4

A. "...be strong and of good courage" (Josh 1:6,9,18, ≈7)

B. There are things that some view as very hard to do, that with God being with you, you can do, but you have to be
<u>STRONG AND OF GOOD COURAGE</u>!

C. You do not look at the opponent, but <u>begin</u>, doing the best you can do, not relenting on what you are in the will of God for.

D. What is <u>a</u> way to get strength & Courage? You can be commanded to have it!!!

So I command you: BE STRONG AND OF GOOD COURAGE!

NOT DIS-<u>COURAGED</u>
BUT EN-<u>COURAGED</u>

TRUTH GEM

"JUST AS" TEACHING\LEARNING

A. I JN 2:27 BUT THE ANOINTING WHICH YOU HAVE RECEIVED FROM HIM ABIDES IN YOU, AND YOU DO NOT NEED THAT ANYONE TEACH YOU; BUT AS THE SAME ANOINTING TEACHES YOU CONCERNING ALL THINGS, AND IS TRUE, AND IS NOT A LIE, AND JUST AS IT HAS TAUGHT YOU, YOU WILL ABIDE IN HIM.

B. AS (AT THE TIME) YOU ARE TAUGHT SOMETHING GOOD, YOU IMMEDIATELY DO IT JUST AS (IN LIKE MANNER) TO THE WAY YOU WERE TAUGHT. YOU WILL ABIDE (DWELL) IN THE FLOW OF THE TEACHER.

C. TO DELAY DOING HOMEWORK OR TO ATTEMPT DOING IT DIFFERENTLY THAN TAUGHT CAN KEEP YOU FROM ABIDING IN THE COURSE FLOW AND TRUTH.

TG-32

TRUTH GEM

WHAT A TEST TESTS

A. (DEUT. 8:2) AND YOU SHALL REMEMBER THAT THE LORD YOUR GOD LED YOU ALL THE WAY THESE FORTY YEARS IN THE WILDERNESS, TO HUMBLE YOU AND TEST YOU TO KNOW WHAT WAS IN YOUR HEART, WHETHER YOU WOULD KEEP HIS COMMANDMENTS OR NOT.

B. SOME CONSIDER A TEST A BRAIN DUMP TO REVEAL WHAT WAS IN YOUR BRAIN AT THE TIME. BUT FOR THOSE WHO HAVE EYES TO SEE A TEST REVEALS WHAT IS IN THE HEART. A TEST WILL REVEAL

1. A HEART THAT IS ON FIRE TO BE ESTABLISHED IN WHAT IS IN THE WILL OF GOD TO BE LEARNED

   OR

2. A HEART THAT IS REGULARLY UNFAITHFUL IN STUDYING WISELY, ONLY SPURTS RIGHT BEFORE A TEST.

C. HEARTS CAN CHANGE. (PS 51:10) CREATE IN ME A CLEAN HEART, O GOD, AND RENEW A STEADFAST SPIRIT WITHIN ME.

TRUTH GEM

USE YOUR IRREVOCABLE GIFT

A. I PET 4:10 AS EACH ONE HAS RECEIVED A GIFT, MINISTER IT TO ONE ANOTHER AS GOOD STEWARDS OF THE MANIFOLD GRACE OF GOD

B. ROM 11:29 FOR THE GIFTS AND CALLING OF GOD ARE IRREVOCABLE.

C. EACH OF US HAS AN IRREVOCABLE GIFT

1. IT IS TO BE USED TO HELP US AND OTHERS
2. IT IS IRREVOCABLE SO THAT YOU CAN HAVE GREAT CONFIDENCE AT TIME OF NEED, THE GIFT WILL BE THERE TO BLESS

D. WHEN YOU DISCERN YOUR CALLING AND GIFTS

1. HONOR THE CALL AND GIFTS – DON'T DESIRE SOME OTHER.
2. USE EACH GIFT BY GRACE TO FULFILL THE CALL

E. BEING IN THE WILL OF GOD FOR BEING HERE BE SURE THERE IS A GIFT YOU HAVE THAT GOD WILL GRACE FOR YOU TO USE TO SUCCEED WITH JOY.

ONE

NOT DIS-INTEGRATED

A. I THESS 5:23 NOW MAY THE GOD OF PEACE HIMSELF SANCTIFY YOU COMPLETELY; AND MAY YOUR WHOLE SPIRIT, SOUL, AND BODY BE PRESERVED BLAMELESS AT THE COMING OF OUR LORD JESUS CHRIST.

B. MK 12:29 "... THE FIRST OF ALL THE COMMANDMENT IS: HEAR, O ISRAEL, THE LORD OUR GOD, THE LORD IS ONE.

C. YOU ARE ONE WHEN OUT OF LOVE OF GOD YOU FOCUS SPIRIT, SOUL, AND BODY ON THE GOD-LED TASK AT HAND ... THIS CLASS, IN THIS COURSE, NOW

NOT BODY HERE AND MIND ELSEWHERE
NOT BODY AND MIND HERE, BUT HEART NOT IN IT
NOT ABSENT IN THE BODY BUT WITH US IN SPIRIT

DIS-INTEGRATED

D. DESIRE AND ADMIRE BEING ONE.

TRUTH GEM

STRONGHOLD OF RIGHTEOUSNESS IN YOUR MIND

A. 2 COR 10:4-5 FOR THE WEAPONS OF OUR WARFARE ARE NOT CARNAL, BUT MIGHTY IN GOD FOR PULLING DOWN STRONGHOLDS, CASTING DOWN ARGUMENTS AND EVERY HIGH THING THAT EXALTS ITSELF AGAINST THE KNOWLEDGE OF GOD, BRINGING EVERY THOUGHT INTO CAPTIVITY TO THE OBEDIENCE OF CHRIST,

B. 2 COR 5:21 FOR HE MADE HIM WHO KNEW NO SIN TO BE SIN FOR US, THAT WE MIGHT BECOME THE RIGHTEOUSNESS OF GOD IN HIM.

C. SOME HAVE STRONGHOLDS IN THEIR MIND THAT THEY ARE MATH FAILURES, RESPONSIBILITY FAILURES, A MISTAKE, A SOCIAL REJECT... ETC. WHEN A CHALLENGE COMES UP THEIR MIND REPLAYS THAT TAPE STRONGLY, THEY YIELD TO THAT AND ARE BOUND — YET THOSE IMAGES IN THE MIND EXALT THEMSELVES AGAINTS THE KNOWLEDGE OF GOD THAT CHRISTIANS ARE THE RIGHTEOUSNESS OF GOD IN CHRIST JESUS, NEW CREATIONS, OLD THINGS HAVE PASSED AWAY, ALL THINGS HAVE BECOME NEW 2 COR 5:17

D. CAST DOWN THE OLD IMAGES!! IN A CHALLENGE HAVE YOUR MIND STRONGLY REPLAYING THE BOLD, RIGHTEOUS, OVERCOMING NATURE YOU HAVE BEEN MADE.

NECESSITY OF UNDERSTANDING

A. MT 13:19 WHEN ANYONE HEARS THE WORD OF THE KINGDOM AND DOES NOT UNDERSTAND IT, THEN THE WICKED ONE COMES AND SNATCHES AWAY WHAT WAS SOWN IN HIS HEART...

B. TO KEEP PATH OF LIFE KNOWLEDGE FROM BEING SNATCHED AWAY, GET UNDERSTANDING OF THAT KNOWLEDGE BY

1. PRAYING FOR WISDOM
2. DOING THE HOMEWORK

→ JAMES 1:22-24 BUT BE DOERS OF THE WORD AND NOT HEARERS ONLY. FOR IF ANYONE IS A HEARER OF THE WORD AND NOT A DOER, HE IS LIKE A MAN OBSERVING HIS NATURAL FACE IN A MIRROR; FOR HE OBSERVES HIMSELF, GOES AWAY, AND IMMEDIATELY FORGETS WHAT KIND OF MAN HE WAS.

TG-37

TRUTH GEM

PROTECTING THE PRECIOUS:
A HEART THAT DESIRES TO STUDY RIGHT THINGS

A. PR 4:23 KEEP YOUR HEART WITH ALL DILIGENCE, FOR OUT OF IT SPRING THE ISSUES OF LIFE.

B. BEING IN THE WILL OF GOD FOR TAKING THIS CLASS, THAT DESIRE TO LEARN EXCELLENTLY BY FLOWING IN STUDYING GRACE IS A PRECIOUS NATURE OF YOUR HEART; GUARD IT; KEEP IT.

C. IN FUTURE TRUTH GEMS WE WILL LEARN WAYS TO PROTECT YOUR HEART'S DESIRE TO BE GENUINE, STUDY RIGHT THINGS, AND EXCEL. IT IS A START TO KNOW IT IS PRECIOUS AND CAN BE KEPT.

D. I JN 5:18b ... BUT HE WHO HAS BEEN BORN OF GOD KEEPS HIMSELF, AND THE WICKED ONE DOES NOT TOUCH HIM.

TG-38

TRUTH GEM

PROTECTING THE PRECIOUS: A HEART THAT DESIRES TO STUDY RIGHT THINGS (PART 2)
FEED YOUR HEART THE RIGHT TEACHINGS

A. HEB 13:9 DO NOT BE CARRIED ABOUT BY VARIOUS AND STRANGE DOCTRINES. FOR IT IS GOOD THAT THE HEART BE ESTABLISHED BY GRACE, NOT WITH FOODS WHICH HAVE NOT PROFITED THOSE WHO HAVE BEEN OCCUPIED WITH THEM.

B. TEACHINGS ARE FOOD FOR THE HEART. WE ARE TO A CERTAIN EXTENT WALKING TEACHINGS.

C. TO KEEP A PRECIOUS DILIGENT HEART, FEED IT THE RIGHT FOODS (I.E. TEACHINGS):
GRACE, RIGHTEOUSNESS, STEADFASTNESS

D. DO NOT LET YOUR HEART EAT JUNK FOOD (DECEPTIVE, UNPROFITABLE TEACHINGS)

E. PR 4:23 KEEP YOUR HEART WITH ALL DILIGENCE; FOR OUT OF IT SPRING THE ISSUES OF LIFE.

TG-~~3~~9

TRUTH GEM

HIGHER NATURE EDUCATION

A. JOHN 8:23 AND HE (JESUS) SAID TO THEM, "YOU ARE FROM BENEATH; I AM FROM ABOVE. YOU ARE OF THIS WORLD; I AM NOT OF THIS WORLD.

B. JESUS WAS NOT SPEAKING ABOUT DIRECTIONALLY UP OR DOWN, BUT ABOUT NATURES.

C. MANY EDUCATIONAL IDEAS COME FROM THE REALM OF MAN'S WISDOM AND APPEAL TO THE LOWER NATURE (EXCUSES, WHY YOU CAN'T LEARN, BLAMING OTHERS, LOWERED EXPECTATIONS)

D. TRUE HIGHER EDUCATION APPEALS TO THE HIGHER NATURE WAYS OF THE MOST HIGH. (FAITHFULNESS, DILLIGENCE, WISDOM, NO EXCUSES, SUCCESS, VICTORY, OVERCOMING, COURAGE, CHALLENGE, DOMINION OF MATERIAL TO BE LEARNED, WISDOM IN STUDY HABITS)

TRUTH GEM

LOVE IS NOT PROVOKED

A. PART OF I COR 13:5 (LOVE)... IS NOT PROVOKED.

B. I USED TO THINK THAT JUST MEANT DO NOT LET PEOPLE PROVOKE YOU. ONE DAY AFTER GETTING PROVOKED AT A COMPUTER, I SAW LOVE IS NOT TO GET PROVOKED PERIOD.

C. LOVE IS AN INNER STATE OF BEING THAT CANNOT BE PROVOKED BY PEOPLE, THINGS, CIRCUMSTANCES... ANYTHING

D. DO NOT GET FRUSTRATED BY ONE HOMEWORK PROBLEM YOU HAVE NOT BEEN ABLE TO WORK YET. DO NOT LET THAT ROB YOU OF WORKING MANY OTHERS. DO NOT LET IT ROB YOU OF JOY OF VICTORY OVER OTHER PROBLEMS

E. THERE ARE NO VICTORIES WITHOUT BATTLES. DO NOT GET FRUSTRATED OVER A PROBLEM. IT IS A VICTORY IN THE MAKING.

GRACE-BOOSTED CARRYING YOUR OWN LOAD
VS.
PARALYZED DEPENDENCE

A. JN 5:7-8 Parts SIR, I HAVE NO ONE TO PUT ME IN THE POOL... JESUS SAID TO HIM, RISE TAKE UP YOUR BED AND WALK.

B. (K. HAGIN) SOME CONSIDER THEMSELVES HELPLESS DEPENDENT ON OTHER PEOPLE AND DO NOT REALIZE THERE IS TRANSFORMATIONAL WORD THAT CAN TRANSFORM THEM TO WHERE THEY CAN DO IT. THEY THINK THEY CANNOT LEARN WITHOUT AN INDIVIDUAL TUTOR OR GROUP WORK. WHAT IS NEEDED IS TO HEAR, BELIEVE AND FOLLOW THE RIGHT WISDOM WORD AND EMPOWERED BY GRACE THEY DO IT THEMSELF.

C. GAL. 6:2,5 BEAR ONE ANOTHER'S BURDENS... EACH ONE SHALL BEAR HIS OWN LOAD.

GIVE THE RIGHT WISDOM WORD THAT FREES OF HARMING DEPENDENCE, WHEN YOU ARE TO DO IT GRACE BOOSTED, NOT DEPENDENT ON OTHERS,

THEN THERE IS WORK YOU ARE MADE TO DO AS A SOURCE, NOT A DRAIN; DO IT HELPING OTHERS.

TG-42

GOD-PACED LEARNING
NOT SELF-PACED LEARNING

A. (Mt 16:24) IF ANYONE DESIRES TO COME AFTER ME, LET HIM DENY HIMSELF, AND TAKE UP HIS CROSS AND FOLLOW ME.

B. SELF WILL WANT TO STOP AND HAVE PIZZA

C. SELF WILL DECEIVE ITSELF DUE TO HUNGER FOR SELF-ESTEEM

IN ONE STUDY THE U.S. CAME IN LAST IN MATH SCORES, BUT FIRST IN HOW THEY FELT ABOUT MATH

D. HUNGER TO BE GOD-ESTEEMED, GOD-PACED EMPOWERED BY GRACE.