NO NOTE-TAKING DISCRETE MATHEMATICS VIA OVERHEADS*

* If you think you can learn better by taking notes, you can still do that!

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DEDICATION:

To the One Who makes things clear

SPECIAL THANKS:

To Janet Blankenship: For the excellent job done in typing this book and for the patience of Job required to do such an excellent job.

PART I

A normal style for a book
Foundations for logic and proof

PART II

Overheads for the teacher and notes for the student over PART I

PART III

Overheads for the teacher and notes for the student over material <u>not</u> in PART I. This is all there is. It should be enough for competent students that have a competent teacher.

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MIGHTY MICROPEDIA

FROM THE

GENRE OF

MUSTARDSEEDAPEDIAS

MIGHTY MICROPEDIAS
BY R.E.A.L. EDUCATION
(See www.arealeducation.com)

TRUTH GEMS FOR TEACHER AND STUDENT by Dr. J. Austin French. This micropedia consist of 53 Truth Gems from the Word of God directed at teaching and learning. Each Truth Gem and its explanation take one page. Since God is the Most High, this means His teachings are the most high teachings. No one knows better than the Creator how man was made, what he needs, what is the best way to teach man, and what is the best way for man to learn. Many of these truth gems start out each teaching session in the Math by Heart trilogy described below.

ALGEBRA II BY HEART by Dr. J. Austin French. This collection of two mighty micropedias consists of a micropedia text, a micropedia of supplementary materials, and DVDs of 53 teaching sessions. Both micropedias are free on the web or can be ordered on the web in paper copy or on one CD. The DVDs can be purchased from the web. This is a College Algebra course, which means it is a strong Algebra II course for high school. This is not what is called Intermediate Algebra (=Algebra I in high school) in some colleges.

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LOGIC FOR UNDERSTANDING MATHEMATICS by Dr. J. Austin French and Dr. Earl Dennis. This collection of two mighty micropedias consists of a micropedia text, a micropedia of supplementary materials, and DVDs of 31 detailed teaching sessions. The mystery of how to do proofs is revealed. Logic is taught and then that connection to math proof is made plain. Proofs are illustrated in the area of elementary set theory. It is for the advanced high school student through college. Math maturity to have done excellently in Algebra II is the only recommended prerequisite background. Both micropedias are free on the web or can be ordered on the web in paper copy or on one CD. The DVDs can be purchased from the web.

PART I

A NORMAL STYLE FOR A BOOK FOUNDATIONS FOR LOGIC AND PROOF

0. MATH: Clarity or Confusion?

Much effort is expended by some to produce an ocean of big, confusing words. This sea of words is designed to 'explain' concepts or help the fountain of words prevail. Texts become larger, but does clarity or joy of learning increase?

One of life's joys can be to see clearly and simply that which formerly confused or intimidated. Math can be so clear and there can be joy in experiencing that clarity. Some goals of this book are for the student to clearly learn the material covered in this text, know that they know the material, and have joy in seeing and knowing.

The second part of this book is designed to be the lecture notes for the teacher. They are intended to be made into overheads and used with an overhead projector. The student is freed from having to make a choice between taking notes and not following what the teacher is saying versus following carefully the teacher and not getting a good set of course notes. Freed student, don't complain that the teacher is just going by the book; rejoice that you have time to focus to make the concepts clear. Teacher, the overhead foils are concise enough to allow you to teach.

The second part of the book is handwritten. Students have said those type of foils were easier to follow and study than similar computer generated foils!! For the technically dependent the computer generated version of the foils is also provided.

Chapters 0 through 16 are like a regular book. There are foils to teach these chapters in the back. From that point on the book abruptly changes from text-with-foils format to a book only of foils for teacher lectures and student notes..., hence, the name of the text.

This text is not designed to be an encyclopedia of discrete math, but the discrete math taught at one place for math, computer science, and information systems majors.

The first chapters study logic and proof. Once this backbone of math is clearly and joyfully learned, we then press on to other discrete math topics.

Oh yes, fellow human teacher, if you have not taught someone else's notes before, I have, I survived, and students learned. Now on to our first topic: logic and proof.

Have you ever wondered how a person proves a theorem? Where does he start; once he has started what is he supposed to do; and how does he know when he is finished? You have probably heard many arguments and advertisements where you felt like you were being fed verbal garbage rather than valid reasoning. This book tries to help you with these problems by telling you what valid reasoning is and by positively explaining what is expected in a proof.

Many times we try to teach what a proof is not, rather than positively explaining what a proof is. In your attempts at presenting a proof at the board, how many times have you heard, "No, that's not right; you need to put some more steps in," or " you need to leave some steps out." We teachers expect students to learn proof by example. That is, after four or five courses of watching the teacher put up proofs, the teacher then says, " Okay, prove this." The student is then expected to put up the standard abbreviated proof style most books and teachers use without ever having learned what it is he is supposed to be abbreviating.

Many math students are scared of proofs and theory in general, but these same students usually have no fear of adding and subtracting. Let it be remembered that for twelve years at least you are drilled on the rules of adding and subtracting and are called upon to use them repeatedly. Usually a person is never taught the rules for proof and seldom called upon to prove things. It is no wonder, then, that students feel uncomfortable with proof -- the unknown is a little discomforting. There are rules for proof just like there are rules for adding and subtracting, and we can get as familiar with them as with the rules for adding and subtracting, but we have to be taught the rules for proof. The purpose of this book is to teach rules for proof. Some may feel that this material is too easy to constitute a college course and hence leave it out of the college curriculum. As a result, the student does not learn the basic ideas of logic and how it is applied to proof (but we teachers still expect them to know it even though it has not been taught!). Upon learning the material in this book, students who have wrestled with proof previously feel like saying, "So that is what you have been expecting me to do. Why haven't you told me before now?" Others say, "Why not let this be the first math course in college; it sure would have helped our understanding of our previous math courses." Since much of this material is easy, the basics can be mastered 100%. (There is usually no such thing as 82% proof, it is either a proof or it isn't.) How would you like to fly in an airplane with a pilot who had a 95% average -- the 5% he missed being takeoffs and landings?

This book is designed to be a text for the standard course in discrete math. You learn what proof is by practicing on set theory theorems. So, in addition to learning proof, you learn some set theory as well. If you have had no previous courses in college math, this book is still for you. In fact many people not in mathematics take this course for its interest sake and the way it trains logical thinking.

Many students will work on a problem for a long time and then finally "see the light," but they cannot logically communicate the idea. This happens when logic is the master rather than the servant. Logic is meant to be the servant of mathematics rather than the master. There are usually two hurdles involved in working any problem: (1) seeing the light and (2) logically communicating the idea once seen. The purpose of this book is to eliminate the second hurdle. After completing this book you should be free to work the math involved in a problem rather than be terrorized by the logic. (You teachers who know logic will think we are overstating the case, but the students who are at logic's mercy will not.)

One final word -- when this book is faithfully followed, it works with amazing results. This book gives you a good taste of what theorem proving is all about. One who completes this book is much better prepared for the advanced math courses than one who completes the standard set theory texts. It needs to be reemphasized that this book needs to be followed faithfully for best results. The abbreviated proof style most teachers and books use should not even be attempted until the chapter, "ABBREVIATIONS OF PROOFS" You earn the right to abbreviate proofs by being master of what is to be abbreviated.

We give our main suggestions for best results.

1) Try to follow the book faithfully.
2) Try to learn the basic philosophy behind the rules for proof that we state.
3) Do not abbreviate proofs until the chapter, "ABBREVIATIONS OF PROOFS"

1. STATEMENT

One of our goals in this course is to be able to tell what constitutes a proof in mathematics. If there is no standard as to what is valid reasoning, then one person's opinion is just as good as the next in determining whether a given argument is a proof. This would be fine if everyone were of the same opinion, but this is obviously not the case. Therefore, there should be a standard. One standard we will use for valid reasoning will be the propositional calculus. Propositional calculus is the study of propositions. In this book, the word statement will be used for the word proposition. In order that we may have common agreement about the meaning of the word statement, we state the following rule:

R_1: An expression is a statement when
- (a) the expression is a sentence;
- (b) the expression is a declarative sentence; and
- (c) it is meaningful to assign only one of the values true or false.

1.1 Example: The expression "Three plus two equals seven" is a statement because it is a declarative sentence to which we can meaningfully assign only one of the values "true" or "false". In this case, we assign the value "false".#

1.2 Example: The expression "One is greater than zero" is a statement because it is a declarative sentence to which we can only meaningfully assign the value "true".#

1.3 Example: The expression "Four times five" is not a statement because it is not a sentence.#

1.4 Example: The expressions "Look at the theorem" and "Was Sam able to get a proof" are not statements because we cannot meaningfully assign one of the values "true" or "false".#

EXERCISE: Which of the following expressions are statements?

1.5 $2 + 3 = 5$

1.6 $6 < 3$

1.7 Sam put on the brakes.

1.8 The car stopped.

1.9 We have your size coat.

1.10 We ordered a coat for you.

1.11 Show Sam what a proof is.

1.12 Sam door what it she 5 Sue.

1.13 Why is Sam taking this course?

1.14 The set of positive integers

1.15 The set of positive integers is a subset of the set of real numbers.

1.16 If Sam put on the brakes, then the car stopped.

2. CONNECTIVES

From simple statements it is possible to build other statements using connecting words. The sentences:

1 is not less than 2.

$1 < 2$ or $6 < 3$.

$1 < 2$ and $6 < 3$.

$1 < 2$ implies $6 < 3$.

All satisfy our description given in R_1 for a statement. The two statements used to build these sentences are "$1 < 2$" and "$6 < 3$". Upon careful observation one should agree to the following rules:

R_2: The negation of a statement is a statement.
R_3: The expression formed by joining two statements with the word "or" is a statement.
R_4: The expression formed by joining two statements with the word "and" is a statement.
R_5: The expression formed by joining two statements with the word "implies" is a statement.

The four following statements have a common property:

2.1 $1 < 2$ or $2 < 3$.

2.2 $1 < 2$ or $6 < 5$.

2.3 $1 < 0$ or $2 < 3$.

2.4 $1 < 0$ or $6 < 5$.

The common property of these statements (that we are interested in) is that they all have the same pattern. The above sentences are instances of the statement pattern "p or q". That is, if p is replaced by "1 < 2" and q is replaced by "6 < 5", then we have statement 2.2, etc. The expression "p or q" is an example of a statement pattern. "p or q" is not a statement but becomes a statement when statements replace the letters "p" or "q". The letters "p" and "q" are called statement variables. A statement variable is a symbol that can be replaced by any statement: hence statements are used as replacements for statement variables. (Variables do not "vary"; they are simply place-holders.) We alluded previously that statements 2.1, 2.2, 2.3, and 2.4 are instances of the statement pattern "p or q." An instance of a statement pattern is a statement that is formed by replacing all the statement variables of the statement pattern with statements. A statement pattern is not a statement; an instance of a statement pattern is a statement.

2.5 EXAMPLE: Consider the statement pattern "p or q". If "7 is a positive integer" replaces "p" and "1 < 0" replaces "q", then "7 is a positive integer or 1 < 0" is yet another instance of the statement pattern "p or q".#

The following statements are instances of the statement pattern "p and q".

2.6 $1 < 2$ and $2 < 3$.

2.7 $1 < 2$ and $6 < 5$.

2.8 $1 < 0$ and $2 < 3$.

2.9 $1 < 0$ and $6 < 5$.

The following statements are instances of the statement pattern "p implies q".

2.10 1 < 2 implies 2 < 3.

2.11 1 < 2 implies 6 < 5.

2.12 1 < 0 implies 2 < 3.

2.13 1 < 0 implies 6 < 5.

The following statements are instances of what statement pattern?

2.14 1 is not less than 2.

2.15 6 is not less than 5.

The question really asked is, how do we denote the statement pattern whose every instance is the negation of a statement? "One is not less than two" and "It is false that one is less than two" are just two ways that the statement "One is less than two" may be negated. In order that our logic be clear, we say that all negations are instances of the statement pattern "not p." If "1 < 2" replaces the statement variable "p", then the resulting instance is literally "not 1 < 2"; however, this instance can be rephrased "One is not less than two" to read more clearly.

One of our desired goals is to make certain that every statement is an instance of some statement pattern. Therefore,"1 < 2" is an instance of what statement pattern? The answer is so easy that it is hard. "1 < 2" is an instance of the statement pattern "p". (Note that we could use any other letter for p.) The preceding discussion on statement variables and statement patterns should help us see the reason for the following rules:

R_6:	A statement variable is a statement pattern.
R'_2:	The negation of a statement pattern is a statement pattern.
R'_3:	The expression formed by joining two statement patterns with the word "or" is a statement pattern.
R'_4:	The expression formed by joining two statement patterns with the word "and" is a statement pattern.
R'_5:	The expression formed by joining two statement patterns with the word "implies" is a statement pattern.
R_7:	An expression is not a statement pattern unless it can be formed according to rules R_6, R'_2, R'_3, R'_4, and R'_5. (Note that R'_2, R'_3, R'_4, and R'_5 were derived from R_2, R_3, R_4, and R_5 by replacing the words "statement pattern" for the word "statement".)

<u>EXAMPLE 2.16</u> It has previously been stated that "p or q" is a statement pattern; it will now be shown why "p or q" is a statement pattern according to rules for forming statement patterns. R_6 tells us that each of "p" and "q" is a statement pattern since each of "p" and "q" is a statement variable. R'_3 tells us that "p or q" is a statement pattern since "p" and "q" are statement patterns.#

Thus far we have stated that "p", "not p", "p or q", "p and q", and "p implies q" are statement patterns. The words "not", "or", "and", and "implies" are called connectives. For convenience and clarity, symbols will be used to denote the connectives as shown on the following chart:

Chart 2.17

not	~
or	V
and	Λ
implies	→

Hence "not p" means the same thing as "~p"; "p or q" means the same thing as "p V q"; etc.

2.18 "~p" and each of its instances are called negations.

2.19 "p V q" and each of its instances are called disjunctions.

2.20 "p Λ q" and each of its instances are called conjunctions.

2.21 "p → q" and each of its instances are called implications.

For the implication "p → q", p (or any replacement for p) is called the hypothesis and q (or any replacement for q) is called the conclusion.*

There were two main reasons for introducing the concepts of a statement pattern, statement variable, and instance. (1) It will be quicker, easier, and clearer studying logic symbolically as opposed to laboriously writing out statements. (2) One of our main concerns of logic will be the structure (pattern) of statements rather than the content or meaning of the statement.

In everyday language the words "if..., then" are used in the same sense as the word "implies". The reason for the following definition is therefore seen:

2.22 DEFINITION: "p implies q" means the same thing as "if p, then q".

Since "p → q", "p implies q", and "if p, then q" all mean the same thing, they will be used interchangeably.

*Other words used for implication, hypothesis, and conclusion are conditional, antecedent, and consequent respectively.

EXERCISES:

2.23 What was our description of a statement variable?

2.24 What was our description of an instance of a statement pattern?

2.25 Is a statement pattern a statement?

2.26 Is an instance of a statement pattern a statement?

2.27 Using the statement variable "p" and "q" and our symbols for connectives, write down the five statement patterns we have studied.

2.28 Do all connectives "connect" two statements?

2.29 Assume "p" and "q" are statement variables. Let "Sam had a 90 average" be a replacement for "p". Let "Sam made an A" be a replacement for "q". What is the resulting instance in each of the following statement patterns?

(a) p (b) q (c) $\sim p$ (d) $\sim q$ (e) $p \vee q$

(f) $q \vee p$ (g) $p \wedge q$ (h) $q \wedge p$

(i) $p \rightarrow q$ (j) $q \rightarrow p$

2.30 For each of the following statements, name a statement pattern for which the statement is an instance.

(a) One is not less than two.

(b) One is less than two implies six is less than five.

(c) $6 < 5$ and $1 < 2$

(d) $1 < 2$ or $6 < 5$

2.31 Are each of the statements in 2.30 instances of the statement pattern "p"?

2.32 Sentences 2.1 - 2.4, 2.6 - 2.15 are statements according to R_2, R_3, R_4, and R_5. It is therefore meaningful to assign one and only one of the values "true" and "false" to each sentence. Using your own intuition, assign a truth value to each of the sentences.

3. TRUTH VALUES OF STATEMENTS CONTAINING CONNECTIVES

While working 2.32, the student probably came to the conclusion that "1 < 0 or 2 < 3" and "1 < 0 or 6 < 5" have different truth values. "1 < 0 or 2 < 3" and "1 < 0 or 6 < 5" are instances of the same statement pattern, yet they have different truth values. One purpose in assigning 2.32 was to convince the student the truth value for an instance of the disjunction "p $\vee$ q" depends entirely on the truth values of the replacement for "p" and "q". A similar statement could be made for the negation, conjunction, and implication.

We now introduce the concept of the truth table. A truth table is a chart used to show the truth values for every possible instance of a statement pattern. We define the truth values for all possible instances of ~p, p V q, p $\wedge$ q, p $\rightarrow$ q by means of the following truth table.

3.1 TRUTH TABLE*

	p	q	~p	p V q	p $\wedge$ q	p $\rightarrow$ q
Case 1	T	T	F	T	T	T
Case 2	T	F	F	T	F	F
Case 3	F	T	T	T	F	T
Case 4	F	F	T	F	F	T

(T stand for TRUE... F stands for FALSE)

*From now on when we ask for a truth value for any statement, it is understood we are asking for the truth value assigned by truth table 3.1.

3.2 EXAMPLE: How to read truth table 3.1: Find the truth value for "1 < 2 or 6 < 5". "1 < 2 or 6 < 5" is an instance of "p V q". Since the true (T) statement "1 < 2" replaced the statement variable "p" and the false (F) statement "6 < 5" replaced the statement variable "q", we look at case 2. Since "1 < 2 or 6 < 5" is an instance of "p V q", we look at case 2 of the column for "p V q" and find that "1 < 2 or 6 < 5" is true.#

Truth table 3.1 had to have four cases in order to have a case for every possible combination of truth values. The following charts will show all the possible combinations of truth values when either one, two, or three statement variables are under consideration.

3.3

p
T
F

p	q
T	T
T	F
F	T
F	F

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

It is felt that after careful examination of 3.3 one would see the pattern developing and be able to write down all the possible combinations of truth values when more than three statement variables are under consideration.

One probably wonders why truth table 3.1 was filled in the way it was. Why didn't we define case 1 of the column for "p $\wedge$ q" to be F instead of T? Since we are trying to develop a standard for valid reasoning, we defined the various possibilities in a way that would not contradict our everyday usage of statements. (We may have introduced some cases that are not used in everyday language. In situations like this we do not have to worry if our definition contradicted everyday language simply because it is not a part of everyday language!) A moments's consideration should convince the reader that we would like an instance of "p $\wedge$ q" to be true only when both of the replacements of "p" and "q" are true and false in the other three cases.

Also, in everyday usage an instance of "p $\vee$ q" is true if the replacement for "p" is true and replacement for "q" is false or if the replacement for "p" is false and the replacement for "q" is true. If both the replacements for "p" and "q" are false, we say that the resulting instance of "p $\vee$ q" is false. In case 1 of the column for "p $\vee$ q", we chose to define this instance of "p $\vee$ q" to be true. This means that we are thinking of our connective "or" as the "inclusive or" that we sometimes use in everyday conversations.

(There is an "exclusive or" used in everyday conversation.)

The truth values in the column for "~p" are easily seen to fit our intuitive notions about statements. If we negate a true statement, we get a false statement. If we negate a false statement, we get a true statement.

The truth values in the column for "p → q" may be seen to be more arbitrary than the ones we have discussed. It seems reasonable that statement 2.10 ($1 < 2 \rightarrow 2 < 3$) should be considered true and statement 2.11 ($1 < 2 \rightarrow 6 < 5$) should be considered false. The question arises over implications with a false hypothesis. Truth table 3.1 defined any implication with a false hypotheses to be true regardless of the truth value of the conclusion. "If the moon is made of green cheese, then $1 > 0$" and "if the moon is made of green cheese, then $1 < 0$" are both examples of implications whose hypotheses are false; they are according to truth table 3.1 <u>true</u>. Defining these statements to be true may seem <u>strange</u>, but it does not <u>contradict</u> our everyday language. (Refer back to our previous statements.)

All of this discussion about truth table 3.1 was to help convince the reader that we have defined the truth values "correctly". The logic we are developing will be useless as a standard for valid reasoning if the initial definitions <u>contradicted</u> our common usage. Later on, other reasons will be given to convince the reader that our definitions are "correct". Now it is only claimed that our definitions do not <u>contradict</u> everyday usage.

An easy way to remember truth table 3.1 is as follows: The only way "p $\vee$ q" can be false is for both "p" and "q" to be false. The only way "p $\wedge$ q" can be true is for both "p" and "q" to be true. The only way "p $\rightarrow$ q" can be false is for "p" to be true and "q" to be false.

<u>EXERCISES</u>:

3.5 Compare the truth values the reader assigned to 2.1 - 2.4, 2.6 - 2.15 in exercise 2.32 with the truth values truth table 3.1 assigns to these statements.

3.6 What are the truth values of the following statements? Do they have the same truth values as 2.1 - 2.4? What is the difference between 2.1 - 2.4 and these statements?

(a) $2 < 3$ or $1 < 2$.

(b) $6 < 5$ or $1 <2$.

(c) $2 < 3$ or $1 < 0$.

(d) $6 < 5$ or $1 < 0$.

3.7 What are the truth values for the following statements? Do they have the same truth values as 2.6 - 2.9? What is the difference between 2.6 - 2.9 and these statements?

(a) $2 < 3$ and $1 < 2$.

(b) $6 < 5$ and $1 < 2$.

(c) $2 < 3$ and $1 < 0$.

(d) $6 < 5$ and $1 < 0$.

3.8 What are the truth values for the following statements? Do they have the same truth values as

2.1- - 2.13? What is the difference between 2.10 - 2.13 and these statements?

(a) 2 < 3 implies 1 < 2. (b) $6<5 \rightarrow 1<2$

(c) 2 < 3 implies 1 < 0.

(d) 6 < 5 implies 1 < 0.

3.9 List all the possible combinations of truth values when the four statement variables p, q, r and s are considered. (See 3.3)

4. TRUTH TABLES

This section involves learning a way to find the truth value of every instance of any statement pattern. So far, the main statement patterns we have considered are "~p", "p $\vee$ q", "p $\wedge$ q", and "p $\rightarrow$ q". However, rules R_6, R'_2, R'_3, R'_4, R'_5, and R_7 assure the existence of many more statement patterns as we will now illustrate. Since each of "p" and "q" is a statement variable, then each of "p" and "q" is a statement pattern, according to R_6. R'_2 tells us that the negation of any statement pattern is a statement pattern, so "~q" is a statement pattern. R'_3 tells us that the expression formed by joining two statement patterns with the word "or" is a statement pattern, so "p $\vee$ ~q" is a statement pattern. By a similar procedure we see that it is possible to build other statement patterns some of which are: "p $\wedge$ ~q", "p $\rightarrow$ ~q", "q $\rightarrow$ p", and "q $\rightarrow$ ~p".

The notion of component parts of statements and statement patterns will now be introduced to aid us in our study of statement patters.

4.1 <u>COMPONENT PARTS</u>: The component parts for either a disjunction, conjunction, or implication are the two statements (or statement patterns) joined by the connective. The only component part of a negation is the statement (or statement pattern) that is negated.

4.2 <u>EXAMPLE</u>: <u>Component parts of statement patterns</u>
The component parts of the statement pattern "p $\vee$ q" and "p" and "q". The component parts of the statement

pattern "p → q" are "p" and "q". Thus the component parts of any implication are the hypothesis and conclusion. The component parts of the statement pattern "p V ~q" are "p" and "~q". The component parts of the statement pattern "p → ~q" are "p" and "~q". The statement pattern "~q" has only one component part, and it is "q".#

4.3 EXAMPLE: Component parts of statements

The component parts of the statement "One is less than two or six is not less than five" are "one is less than two" and "six is not less than five". The component parts of " one is less than two implies six is not less than five" are again "one is less than two" and "six is not less than five". The only component part of "six is not less than five" is "six is less than five".#

Since one of the stated goals of this section is to learn a way to find the truth values for every instance of any statement pattern, we will begin with the statement pattern "p V ~q" and find the truth values for every one of its instances.

Truth values of a particular instance of "p V ~q". Let "one is less than two" be a replacement for "p". Let "six is less than five" be a replacement for "q". Using these replacements, "one is less than two or six is not less than five" is the resulting instance of "p V ~q". "Six is less than five" is false therefore by truth table 3.1, its negation, "Six is not less than five" is true. Since "one is less than two" is true, then "one is less than two or six is not less than five" is a disjunction with both of its component parts true. Since case 1 of the column for "p V q" in truth table 3.1 tells that any disjunction with both of its component parts true is a true statement, "one is less than two or six is not less than five" is true.

Truth values for every instance of "p V ~q". Since we have previously stated that a truth table is a chart used to show the truth values for every possible instance of a statement pattern, it is natural that we would make a truth table for "p V ~q". We will make a truth table for "p V ~q" (and all other statement patterns) by following the following five guidelines:

RULES FOR MAKING A TRUTH TABLE FOR A STATEMENT PATTERN:

4.4 Have a column for each of the different statement variable in the statement pattern.

4.5 Fill in the rows with all the possible combinations of truth values.

4.6 Have a column for the given statement pattern.

4.7 Every time there is a column for a statement pattern that is either a negation, disjunction, conjunction, or implication, there must be a column for its component parts.

4.8 Use truth table 3.1 to fill in every case of every remaining column.

It will now be shown how the rules are applied to "p V ~q". We first put a column under every statement variable and every connective.

p	V	~	q
1	2	3	4

Column 1 will tell us the truth values for replacements for "p". Column 4 will tell us the truth values for replacements for "q". Column 3 will tell us the truth values for instances of "~q", (It is incorrect to say that column 3 tells us the truth values for instances of "~"). Since column 2 is under the "V" sign, column 2 will tell us the truth values for instances of "p V ~q".

Since we have only 2 different statement variables in "p V ~q", there are only 4 possible combinations of truth values. We consider the 4 possible combinations with the 4 cases below:

	p	V	~	q
Case 1	T			T
Case 2	T			F
Case 3	F			T
Case 4	F			F

The truth table for "p V ~q" can be completed using truth table 3.1.

Truth table for "p V ~q".

	p	V	~	q
Case 1	T	T	F	T
Case 2	T	T	T	F
Case 3	F	F	F	T
Case 4	F	T	T	F

We will now explain how all of case 3 was completed. We first fill in column 3 of case 3, the column for "~q". Column 4 of case 3 tells us we are considering a true replacement for "q", so its negation is false. We therefore put an F in column 3 of case 3. We now know from columns 1 and 3 of case 3 that we have a disjunction with both component parts false. The disjunction must be false, so we put an F in column 2 of case 3. Cases 1,2, and 4 can be filled in similarly.

Let us make sure we know how to read the truth table for "p V ~q". Column 2 tells us the truth values for the various instances of "p V ~q". Case 1 tells us if the replacement for p is true and the replacement for q is true, then the resulting instance of "p V ~q" is true. For example, let "1 < 2" be a true statement to replace "p" and "5 < 6" be a true statement to replace "q". So by case 1, column 2, "1 < 2 or 5 is not less than 6" is a true instance of "p V ~q". Case

3 tells us if the replacement for "p" is false and the replacement for "q" is true, then the resulting instance of "p V ~q" is false. We now conclude our discussion on how to read the completed truth table for "p V ~q".

GROUPING SYMBOLS: Now that the truth table for "p V ~q" is completed, we would like to look at a difficulty that arises when considering the statement pattern "~p V q". The difficulty is that "~p V q" can have two meanings: Is "~p V q" a disjunction whose left component part is "~p" and whose right component part is "q", or is "~p V q" the negation of "p V q"? Since there can be two meanings to "~p V q", we introduce the use of grouping symbols to make it perfectly clear what is the intended statement. The disjunction whose left component part is "~p" and whose right component part is "q" will be denoted by "(~p) V q". The negation of "p V q" will be denoted by "~(p V q)".

The truth tables for "(~p) V q" and "~(p V q)" are given below. Notice that case 1 of the two truth tables assign different truth values to instances of "(~p) V q" and "~(p V q)" when the replacements for both "p" and "q" are true. The preceding statement should illustrate the importance of grouping symbols. (The reader should not worry if he does not completely understand how the truth tables are made and read; more discussion will be given on this matter.)

$(\sim$	$p)$	$\vee$	q
F	T	T	T
F	T	F	F
T	F	T	T
T	F	T	F

$\sim$	$(p$	$\vee$	$q)$
F	T	T	T
F	T	T	F
F	F	T	T
T	F	F	F

4.9 EXAMPLE: Let us see how everyday language might distinguish between an instance of "$(\sim p) \vee q$" and "$\sim(p \vee q)$". Let "$1 < 2$" be a replacement for "p". Let "$4 < 5$" be a replacement for "q". The instance of "$(\sim p) \vee q$" could be phrased "one is not less than two or $4 < 5$". The instance of "$\sim(p \vee q)$" in order to be clear could be phrased "the following statement is false: One is less than two or four is less than five." For this example we have chosen the replacements for "p" and "q" to both be true. Case 1 of the above truth tables tells us that these statements have opposite truth values.#

We observe that the only grouping symbols used so far have been parentheses. Many times for clarity several different types of grouping symbols will be used on one statement pattern as will now be shown.

4.10 EXAMPLE: Use of different types of grouping symbols Each of "p" and "$p \rightarrow q$" is a statement pattern, so by R'_4, "$p \wedge (p \rightarrow q)$" is a statement pattern. By R'_5 "$[p \wedge (p \rightarrow q)] \rightarrow q$" is a statement pattern (it is an implication whose hypothesis is "$p \wedge$

(p → q)" and whose conclusion is "q"). By R'_2 ~{[p ∧ (p → q)] → q} is a statement pattern.#

To make sure we understand grouping symbols and how to make a truth table we will make a truth table for "[p ∨ (~q)] ∧ (q → p)".

First, we put a column under each statement variable and each connective. (Do not have a column only under a grouping symbol.)

[p	∨	(~	q)]	∧	(q	→	p)
1	2	3	4	5	6	7	8

Column 1 is for "p".
Column 2 is for "p ∨ (~q)".
Column 3 is for "~q".
Column 4 is for "q".
Column 5 is for "[p ∨ (~q)] ∧(q → p)".
Column 6 is for "q".
Column 7 is for "q → p".
Column 8 is for "p".

We now fill in as many rows as there are possible combinations of true values. Since "p" and "q" are the only two statement variables, we have four cases as seen below:

[p	V	(~	q)]	Λ	(q	→	p)
T			T		T		T
T			F		F		T
F			T		T		F
F			F		F		F

The following is the completed truth table:

[p	V	(~	q)]	Λ	(q	→	p)
T	T	F	T	T	T	T	T
T	T	T	F	T	F	T	T
F	F	F	T	F	T	F	F
F	T	T	F	T	F	T	F
1	2	3	4	5	6	7	8

We first filled in column 3 looking at column 4. Next we filled in column 2 looking at columns 1 and 3. Next we filled in column 7 looking at columns 6 and 8. Last we filled in the main column, column 5, looking at columns 2 and 7. (While filling out a truth table it is easy to carelessly look in the wrong column -- so be careful.)

EXERCISES

4.11

[(p	→	q)	∧	(~	p)]	→	(~	q)
1	2	3	4	5	6	7	8	9

Column 1 is for ___________

Column 2 is for ___________

Column 3 is for ___________

Column 4 is for ___________

Column 5 is for ___________

Column 6 is for ___________

Column 7 is for ___________

Column 8 is for ___________

Column 9 is for ___________

To fill in column 2 you look at ___________

To fill in column 4 you look at ___________

To fill in column 5 you look at ___________

To fill in column 7 you look at ___________

To fill in column 8 you look at ___________

4.12 Are the truth tables for "p ∧ q" and "q ∧ p" the same in every case?

4.13 Are the truth tables for "p ∨ q" and q ∨ p" the same in every case?

4.14 Are the truth tables for "p → q" and "q → p" the same in every case?

4.15 Make a truth table for " p ∨ (~p)".

4.16 Make a truth table for "[(p → q) ∧ p] → q".

4.17 Make a truth table for "[(p → q) ∧ (~p)] → (~q)".

4.18 Make a truth table for "(p → q) ∧ (q → p)".

4.19 How many different statement patterns can be made from "$p \wedge p \rightarrow q \rightarrow q$" by inserting grouping symbols in different places?

4.20 Tell whether each of the following is a negation, disjunction, conjunction, or implication. For each of the following that is an implication name the hypothesis and conclusion.

(a) $(\sim p) \vee q$

(b) $\sim(p \vee q)$

(c) $[(p \rightarrow q) \wedge (\sim p)] \rightarrow (\sim q)$

(d) $(p \wedge q) \rightarrow p$

(e) $p \wedge (q \rightarrow p)$

(f) $[\sim(p \wedge q)] \rightarrow [(\sim p) \vee (\sim q)]$

(g) $[(\sim p) \vee (\sim q)] \rightarrow [\sim(p \wedge q)]$

(h) $\sim[p \wedge (p \rightarrow q)]$

4.21 Let $1 < 2$ be a replacement for "p". Let $6 < 3$ be a replacement for "q".

(a) Write out the resulting instance of each of these statement patterns.

(1) $p \vee (\sim p)$

(2) $(\sim p) \rightarrow q$

(3) $q \rightarrow (\sim p)$

(4) $\sim[p \vee (\sim q)]$

(b) Each of the following is an instance of what statement pattern?

(1) If $6 \nless 3$, then $1 < 2$.

(2) If $1 \nless 2$, then $6 \nless 3$.

(3) $6 \nless 3$ implies either $1 \nless 2$ or $6 < 3$.

5. TAUTOLOGY

Algebra, as well as logic, uses variables. The algebraic expression "2x + y = 0" involves the real number variables "x" and "y". "2x + y = 0" is not a statement, but becomes a statement when real numbers replace the "x" and "y" respectively. From this brief discussion one can see the similarity between an algebraic expression and a statement pattern. Our variables in logic are called statement variables because statements are replacements for the variables. The variables in algebra are called real number variables (or real variables) because real numbers are replacements for the variables. We have noted that an instance of a statement pattern is a statement formed by replacing all the statement variables with statements. Algebraic expressions also have instances. An instance of an algebraic expression is a statement that is formed by replacing all of the real number variables of the algebraic expression with real numbers. Therefore, an algebraic expression is not a statement but becomes a statement when real numbers are substituted for all the variables. Similarly, a statement pattern is not a statement but becomes a statement when statements are substituted for all the variables.

CORRESPONDENCE BETWEEN "ALGEBRAIC IDENTITY" AND "TAUTOLOGY"

To solve an algebraic expression means to find all the real numbers that produce true instances when replaced for all the variables. For example, (1, -2) is a solution for the algebraic expression "$2x + y = 0$" because the instance that results when "1" and "-2" replace "x" and "y" respectively is a true statement. "$3x + y = 3(x + y) - 2y$" is an algebraic expression whose every instance is a true statement; all such algebraic expression are called algebraic identities. The counterpart in logic to the algebraic identity is the tautology. A tautology is a statement pattern whose every instance is true. Two tautologies were encountered in the last set of exercises. They were the statement patterns "$p \vee (\sim p)$" and "$[p \wedge (p \rightarrow q)] \rightarrow p$". Tautologies play a central role in logic. For easy reference we display the definition of tautology.

5.1 DEFINITION: A tautology is a statement pattern whose every instance is true.

Truth tables are used to show the truth values for every instance of a statement pattern. To test whether a statement pattern is a tautology one merely has to make a truth table for the statement pattern. If the column for the statement pattern has a "T" in every case then it is a tautology. It must be emphasized that a tautology is a statement pattern, not a statement.

5.2 EXAMPLE: This example serves two purposes. First, it is an example of another tautology, and second, it is a truth table for a statement pattern with three statement variables.

A truth table will be made for "[(p → q) ∧ (q → r)] → (p → r)". Observe that guidelines 4.1, 4.2, 4.4, and 4.5 were followed in making the truth table.

[(p	→	q)	∧	(q	→	r)]	→	(p	→	r)
T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	F	T	T	F	F
T	F	F	F	F	T	T	T	T	T	T
T	F	F	F	F	T	F	T	T	F	F
F	T	T	T	T	T	T	T	F	T	T
F	T	T	F	T	F	F	T	F	T	F
F	T	F	T	F	T	T	T	F	T	T
F	T	F	T	F	T	F	T	F	T	F

Note that the column for [(p → q) ∧ (q → r)] → (p → r) has a "T" in every case, thus making it a tautology.#

THE CONNECTIVE "IF AND ONLY IF"

The new connective "if and only if" will now be defined in terms of the connectives " and" and "implies". The connective "if and only if" is denoted by "↔" and is defined as follows:

5.4 DEFINITION: "p ↔ q" means the same thing as "(p → q) ∧ (q → p)".

The reader made a truth table for "(p → q) ∧ (q → p)" in exercise 4.18 of the last section. Since "p ↔ q"* means the same thing as "(p → q) ∧ (q → p)", the column entitled "(p → q) ∧ (q → p)" could also be titled "p ↔ q". Therefore, the truth table for "p ↔ q" is:

p	↔	q
T	T	T
T	F	F
F	F	T
F	T	F

From now on the student needs only to refer to the preceding chart when filling in a truth table for a statement pattern that contains the connective "↔".

5.6 EXAMPLE: Truth table for p ↔ [~(~p)]

p	↔	[~	(~	p)]
T	T	T	F	T
F	T	F	T	F

*"p ↔ q" and each of its instances is called a biconditional.

5.7 Which of the following are tautologies? Make truth tables to find out.

(a) $[\sim(p \wedge q)] \leftrightarrow [(\sim p) \vee (\sim q)]$

(b) $[\sim(p \vee q)] \leftrightarrow [(\sim p) \wedge (\sim q)]$

(c) $[(p \rightarrow q) \wedge (\sim p)] \rightarrow (\sim p)$

(d) $[(p \rightarrow q) \wedge (q \rightarrow r)] \leftrightarrow (p \rightarrow r)$

(e) $(p \wedge q) \leftrightarrow (q \wedge p)$

(f) $(p \vee q) \leftrightarrow (q \vee p)$

(g) $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$

(h) $p \rightarrow (p \vee q)$

(i) $(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$

(j) $(p \wedge q) \rightarrow p$

6. EQUIVALENCE

Let us consider the following statement:
6.1 "Sam had a date with Jane Friday, and Sam had a date with Sue Saturday." Are any of the following statements the negation of the preceding statement?
6.2 "Sam did not have a date with Jane Friday, and Sam did not have a date with Sue Saturday."
6.3 "Sam did not have a date with Jane Friday, or Sam did not have a date with Sue Saturday."
In order to answer the question just asked, we introduce the concept of equivalence.

6.4 DEFINITION: The statement pattern "R" is equivalent to the statement pattern "Q" means "R ↔ Q" is a tautology.

Is "p" equivalent to "~(~p)"? To answer that question we look at definition 6.4; we replace "R" with "p" and "Q" with "~(~p)". According to definition 6.4, when we ask "Is 'p' equivalent to '~(~p)'?" we are really asking "Is 'p ↔ [~(~p)]' a tautology?" Since "p ↔ [~(~p)]" is a tautology, see example 5.6, "p" is equivalent to "~(~p)". To check if two statement patterns are equivalent, simply join the two statement patterns with the connective"↔". If the resulting statement pattern is a tautology, they are equivalent; if the resulting statement pattern is not a tautology, then the two statement patterns are not equivalent.

Is "$\sim(p \wedge q)$" equivalent to "$(\sim p) \vee (\sim q)$"? in other words, is "$[\sim(p \wedge q)] \leftrightarrow [(\sim p) \vee (\sim q)]$" a tautology? Yes, according to exercise 5.7 (a). Therefore, "$\sim(p \wedge q)$" is equivalent to "$(\sim p) \vee (\sim q)$".

Is "$(p \rightarrow q) \wedge (q \rightarrow r)$" equivalent to "$p \rightarrow r$"? In other words, is "$[(p \rightarrow q) \wedge (q \rightarrow r)] \leftrightarrow (p \rightarrow r)$" a tautology? No, according to exercise 5.7 (d). Therefore, "$(p \rightarrow q) \wedge (q \rightarrow r)$" is <u>not</u> equivalent to "$p \rightarrow r$".

It is easy to verify by a truth table that "$[\sim(\sim p)] \leftrightarrow p$" is a tautology. This means "$\sim(\sim p)$" is equivalent to "p". We previously noted that "p" is equivalent to "$\sim(\sim p)$" The order in which we say equivalence makes no difference. This is true for any statement patterns that are equivalent.

Definition 6.4 defined equivalence for <u>statement patterns</u>. We will now learn how to form equivalent <u>statements</u>.

6.5 By using the same replacements for the same statement variables in equivalent statement patterns, one forms <u>equivalent statements</u>.

"p" and "$\sim(\sim p)$" are equivalent statement patterns. By using "One is less than two" as a replacement for "p" in both statement patterns we see that "One is less than two" and "It is false that one is not less than two" are <u>equivalent statements</u>.

We have previously noted that "$\sim(p \wedge q)$" and "$(\sim p) \vee (\sim q)$" are equivalent statement patterns. Let "Sam had a date with Jane Friday" be a replacement for "p" in both the statement patterns "$\sim(p \wedge q)$" and "$(\sim p) \vee (\sim q)$" and let "Sam had a date with Sue Saturday" be a replacement for "q" in both statement patterns. Thus by 6.5, the two following statements are equivalent:

6.6 It is false that: Sam had a date with Jane Friday, and Sam had a date with Sue Saturday.

6.7 Sam did not have a date with Jane Friday, or Sam did not have a date with Sue Saturday.

Now 6.6 is the negation of 6.1. Since 6.6 is equivalent to 6.7, 6.7 can be considered the negation of 6.1. To answer the question we raised at the first of this section, the negation of 6.1 is 6.3 (6.3 and 6.7 are identical.)

We often use the two following expressions interchangeably:

The negation of "$p \wedge q$" is "$(\sim p) \vee (\sim q)$".

"$\sim(p \wedge q)$" is equivalent to "$(\sim p) \vee (\sim q)$".

The above lines tell us, <u>to negate a conjunction simply negate both component parts and change the "$\wedge$" to "$\vee$"</u>.

6.8 <u>EXAMPLE</u>: Negate "$(\sim A) \wedge B$". We negate the component parts "$\sim A$" and "B" and change the "$\wedge$" to "$\vee$" and get "$[\sim(\sim A)] \vee (\sim B)$" (which is equivalent to "$A \vee (\sim B)$".)#

6.9 <u>EXAMPLE</u>: What is "$\sim\{[\sim(p \vee q)] \wedge (s \rightarrow q)\}$" equivalent to? We are really asking, what is the negation of "$[\sim(p \vee q)] \wedge (s \rightarrow q)$"? We negate the component parts "$\sim(p \vee q)$" and "$s \rightarrow q$" and change the

"$\wedge$" to "$\vee$" to get "$\{\sim[\sim(p \vee q)]\} \vee [\sim(s \rightarrow q)]$" (which is equivalent to "$(p \vee q) \vee [\sim(s \rightarrow q)]$").#

Let us now see how we can negate any *disjunctions*. By working exercise 5.7 (b) one sees that "$[\sim(p \vee q)] \leftrightarrow [(\sim p) \wedge (\sim q)]$" is a tautology. This means "$\sim(p \vee q)$" is equivalent to "$(\sim p) \wedge (\sim q)$". Let "Sam made an A" be a replacement for "p" in both the statement patterns "$(p \vee q)$" and "$(\sim p) \wedge (\sim q)$" and let "Sue did not make a B" be a replacement for "q" in both statement patterns. According to 6.5, we have the two following statements equivalent.

6.10 It is false that: Sam made an A, or Sue did not make a B.

6.11 Sam did not make an A, and Sue made a B.[*]

So "Sam did not make an A, and Sue made a B" can be considered the negation of "Sam made an A, or Sue did not make a B."

We often use the two following expressions interchangeably:

The negation of "$p \vee q$" is "$(\sim p) \wedge (\sim q)$".

"$\sim(p \vee q)$" is equivalent to "$(\sim p) \wedge (\sim q)$".

The above line tells us, *to negate any disjunction simply negate both component parts and change the "$\vee$" to "$\wedge$"*.

6.12 *EXAMPLE*: Negate "$s \vee (\sim w)$". We negate the component parts "s" and "$\sim w$" and change the "$\vee$" to "$\wedge$" to get "$(\sim s) \wedge [\sim(\sim w)]$" (which is equivalent to

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[*]Recall: "Sue made a B" is equivalent to "It is false that Sue did not make a B".

"$(\sim s) \wedge w$".)#

6.13 <u>EXAMPLE</u>: What is "$\sim\{A \vee [\sim(p \wedge q)]\}$" equivalent to? We are really asking, what is the negation of "$A \vee [\sim(p \wedge q)]$"? We negate the component parts "A" and "$\sim(p \wedge q)$" and change the "$\vee$" to "$\wedge$" to get "$(\sim A) \wedge \{\sim[\sim(p \wedge q)]\}$" (which is equivalent to "$(\sim A) \wedge (p \wedge q)$").#

<u>CONVERSE, INVERSE, CONTRAPOSITIVE</u>

6.14 <u>DEFINITION</u>: The <u>converse</u> of the implication "$p \rightarrow q$" is the implication "$q \rightarrow p$".

6.15 <u>EXAMPLE</u>: The converse of "$p \rightarrow q$" is "$q \rightarrow p$". The converse of "$p \rightarrow (\sim q)$" is $(\sim q) \rightarrow p$". The converse of "$[\sim(p \vee q)] \rightarrow [s \wedge (\sim w)]$" is "$[s \wedge (\sim w)] \rightarrow [\sim(p \vee q)]$". The converse of "If Sam had a 90 average, then Sam made an A" is "If Sam made an A, then Sam had a 90 average.#

6.16 <u>DEFINITION</u>: The <u>inverse</u> of the implication "$p \rightarrow q$" is the implication "$(\sim p) \rightarrow (\sim q)$".

6.17 <u>EXAMPLE</u>: The inverse of "$p \rightarrow q$" is "$(\sim p) \rightarrow (\sim q)$". The inverse of "$p \rightarrow (\sim q)$" is "$(\sim p) \rightarrow [\sim(\sim q)]$", or equivalently "$(\sim p) \rightarrow q$". The inverse of "$[\sim(p \vee q)] \rightarrow [s \wedge (\sim w)]$" is "$\{\sim[\sim(p \vee q)]\} \rightarrow \{\sim[s \wedge (\sim w)]\}$", or equivalently "$(p \vee q) \rightarrow \{\sim[s \wedge (\sim w)]\}$". The inverse of "If Sam had a 90 average, then Sam made an A" is "If Sam did not have a 90 average, then Sam did not make an A."

6.18 <u>DEFINITION</u>: The <u>contrapositive</u> of the implication "$p \rightarrow q$" is the implication "$(\sim q) \rightarrow (\sim p)$".

6.19 <u>EXAMPLE</u>: The contrapositive of "$p \rightarrow q$" is "$(\sim q) \rightarrow (\sim p)$". The contrapositive of "$p \rightarrow (\sim q)$" is "$[\sim(\sim q)] \rightarrow (\sim p)$", or equivalently "$q \rightarrow (\sim p)$". The contrapositive of "$[\sim(p \vee q)] \rightarrow [s \wedge (\sim w)]$" is "$\{\sim[s \wedge (\sim w)]\} \rightarrow \{\sim[\sim(p \vee q)]\}$", or equivalently "$\{\sim[s \wedge (\sim w)]\} \rightarrow (p \vee q)$". The contrapositive of "If Sam had a 90 average, then Sam made an A" is "If Sam did not make an A, then Sam did not have a 90 average."#

The chart below will be a summary chart of the definitions of converse, inverse, and of contrapositive for the implication "$p \rightarrow q$".

6.20

Given implication $p \rightarrow q$	Converse $q \rightarrow p$
Inverse $(\sim p) \rightarrow (\sim q)$	Contrapositive $(\sim q) \rightarrow (\sim p)$

It is important to know if an implication is equivalent to either its converse, inverse, or contrapositive; however, it will be left as an exercise for the reader to find out. Just as a check on our reader's intuition, we will list a statement followed by its converse, inverse, and contrapositive: the reader is invited to decide if any of 6.22, 6.23, and 6.24 "sound" equivalent to 6.21.

6.21 <u>Given implication</u>: If Sam had a 90 average, then Sam made an A.

6.22 <u>Converse</u>: If Sam made an A, then Sam had a 90 average.

6.23 <u>Inverse</u>: If Sam did not have a 90 average, then Sam did not make an A.

6.24 <u>Contrapositive</u>: If Sam did not make an A, then Sam did not have a 90 average.

<u>EXERCISES</u>:

6.25 Which of the following pairs of statement patterns are equivalent? (Check by means of truth table.)

(a) "$p \leftrightarrow q$"; "$(p \rightarrow q) \wedge (q \rightarrow p)$"
(b) "$p \rightarrow q$"; "$(\sim p) \vee q$"
(c) "$p \rightarrow q$"; "$p \vee q$"
(d) "$\sim(p \rightarrow q)$"; "$p \rightarrow (\sim q)$"
(e) "$\sim(p \rightarrow q)$"; "$p \wedge (\sim q)$"
(f) "$p \wedge (q \vee r)$"; "$(p \wedge q) \vee (p \wedge r)$"
(g) "$p \wedge (q \vee r)$"; "$(p \vee q) \wedge (p \vee r)$"

6.26 Use the results of 6.25 to <u>negate</u> each of the following statements:

(a) If Sam had a 90 average, then Sam made an A.
(b) If Sam went to town, then Sam went to the grocery store.

6.27 Is "$p \rightarrow q$" equivalent to its converse? Its inverse? Its contrapositive?

6.28 Is the converse of "$p \rightarrow q$" equivalent to the inverse of "$p \rightarrow q$"? The contrapositive of "$p \rightarrow q$"?

6.29 Is the inverse of "$p \rightarrow q$" equivalent to the contrapositive of "$p \rightarrow q$"?

6.30 Write the converse, inverse, and contrapositive of each of the following:

(a) $q \rightarrow p$

(b) $(\sim p) \rightarrow (\sim q)$

(c) $(\sim w) \rightarrow [\sim(p \vee q)]$

(d) If Sam did not have a positive skin test, then Sam did not have TB.

6.31 Negate each of the following:

(a) A is a subset of B and B is a subset of A.

(b) Sam's house is the third or fourth house on the left.

(c) I am going to pass logic or die trying.

(d) The car is in the garage or we will not be going to town.

(e) $(\sim p) \vee (\sim r)$

(f) $(\sim p) \wedge q$

6.32 From working 6.25 (b) it is seen that "$p \rightarrow q$" and "$(\sim p) \vee q$" are equivalent. Therefore, "$(p \rightarrow q) \leftrightarrow [(\sim p) \vee q]$" is a tautology, so every case of its truth table is true. Below is an incomplete truth table for "$(p \rightarrow q) \leftrightarrow [(\sim p) \vee q]$". Since we know it is a tautology, every case is filled in as true. Fill in the blank column looking only at the columns indicated by the arrows.

(p	→	q)	↔	[(~	p	V	q]
T	T	T	T	X	X		X
T	F	F	T	X	X		X
F	T	T	T	X	X		X
F	T	F	T	X	X		X
	↑		↑				

This illustrates that the truth tables for equivalent statement patterns must have the same truth values in every case. This is sometimes used as a definition for two statement patterns being equivalent.

6.34 In working 6.27, the reader discovers some important facts that must be remembered.

(a) An implication is equivalent to its contrapositive.

(b) An implication is not equivalent to its converse.

(c) An implication is not equivalent to its inverse.

6.35 Equivalences applied to algebra: By working 6.25 (f) one finds that "$p \wedge (q \vee r)$" is equivalent to "$(p \wedge q) \vee (p \wedge r)$". We will use a few properties of absolute values and inequalities along with the fact that "$p \wedge (q \vee r)$" is equivalent to "$(p \wedge q) \vee (p \wedge r)$" to solve the following inequality:

$$3 < |x-5| < 4$$

$$(|x-5| < 4) \text{ and } (|x-5| > 3)$$

$$(-4 < x - 5 < 4) \text{ and } (x - 5 > 3 \text{ or } x - 5 < -3)$$
$$(1 < x < 9) \text{ and } (x > 8 \text{ or } x < 2)$$
$$p \quad \wedge \quad (q \vee r)$$
$$(p \wedge q) \quad \vee \quad (p \wedge r)$$
$$(1 < x < 9 \text{ and } x > 8) \text{ or } (1 < x < 9 \text{ and } x < 2)$$
$$(8 < x < 9) \text{ or } (1 < x < 2)$$

6.36 Solve $1 < |x-3| < 2$.

7. IMPLICATION

One of our stated goals is to provide a standard as to what constitutes valid reasoning. To apply the logic we have learned so far, one has to translate everyday language into statements joined only by the connectives "not", "or", "and", "implies", and "if and only if". The implication appears in everyday language in various forms, some of which only scarcely resemble our familiar "if..., then" and "implies" form. The following chart will show a list of the different ways an implication can be phrased:

7.1

Different forms of the implication
$p \rightarrow q$
p implies q
if p, then q
q, if p
p only if q
p is sufficient for q
q is necessary for p

7.2 EXAMPLE: "A sufficient condition for Jones to win the election is for Jones to carry district 2." means the same thing as "If Jones carries district 2, then Jones will win the election."#

7.3 EXAMPLE: "Sam had TB only if Sam had a positive skin test." means the same thing as "If Sam had TB, then Sam had a positive skin test."#

7.4 EXAMPLE: "Having 12 cents is a necessary condition for having 13 cents." means the same thing as "If you have 13 cents, then you have 12 cents."

7.5 EXAMPLE: "Sam got in the football game if and only if Sam had a ticket." means "Sam got in the football game if Sam had a ticket and Sam got in the football game only if Sam had a ticket." which means "If Sam had a ticket, then Sam got in the football game and if Sam got in the football game, then Sam got a ticket." This example illustrates that "If and only if" is expressed in a way that is faithful to the way it was defined.#

7.6 EXAMPLE: "A necessary and sufficient condition for one to be less than two is for four to be less than five." means "A necessary condition for one to be less than two is for four to be less than five and a sufficient condition for one to be less than two is for four to be less than five." which means "If one is less than two, then four is less than five and if four is less than five, then one is less than two." which also can be said by definition, "One is less than two if and only if four is less than five."#

A helpful memory device for "necessary conditions" and "sufficient conditions" is the following. On any map the arrow always points north(N) $\overset{N}{\uparrow}$. On any implication the arrow always points toward the necessary(N) condition, $\rightarrow$ N. So the conclusion of an implication is always the necessary condition, and the hypothesis of an implication is always the sufficient condition.

TWO ARGUMENTS TO SHOW THAT THE TRUTH TABLE FOR THE IMPLICATION IS DEFINED "CORRECTLY"

First argument: Consider the following statements:

7.7 I am going to town or I would give you a ride.

7.8 If I were going to town, then I would give you a ride.

7.9 Sam is not finished with the hammer or you could use it.

7.10 If Sam was finished with the hammer, then you could use it.

7.11 Sam did not eat the cookies or there would be crumbs on the table.

7.12 If Sam ate the cookies, then there would be crumbs on the table.

In everyday language 7.7 and 7.8 are used in an equivalent fashion; 7.9 and 7.10 are used in an equivalent fashion; and 7.11 and 7.12 are used in an equivalent fashion. 7.7 and 7.8 are similar instances of "$(\sim p) \vee q$" and "$p \rightarrow q$" respectively. 7.9 and 7.10 are similar instances of "$(\sim p) \vee q$" and "$p \rightarrow q$" respectively. 7.11 and 7.12 are similar instances of "$(\sim p) \vee q$" and "$p \rightarrow q$" respectively. Therefore, our everyday language seems to demand that "$(\sim p) \vee q$" and "$p \rightarrow q$" be equivalent. Since "$(\sim p) \vee q$" and "$p \rightarrow q$" should be equivalent, they should have the same truth values in every case of their truth table (see 6.32). The truth table for the disjunction was defined in an agreeable way. Therefore, there should be no argument when the truth values for "p → q" are defined to be the same as the truth values for "(~p) V q" in every case. (Note: in working 6.25 (b) the reader verified that "(~p) V q" and "p→ q" are equivalent.)

7.13

(~	p)	V	q
F	T	**T**	T
F	T	**F**	F
T	F	**T**	T
T	F	**T**	F

p	→	q
T	**T**	T
T	**F**	F
F	**T**	T
F	**T**	F

Second argument: It is clear that an implication with a true hypothesis and a true conclusion should be defined to be a true statement. It is clear that an implication with a true hypothesis and a false conclusion should be defined to be a false statement. In other words, it is clear the way the first two cases of the truth table for "p → q" should be defined, but it is not clear how the last two cases should be defined. Since there is agreement on the first two cases, there are only four possibilities for the way the truth table for "p → q" should be defined. They are as follows:

7.14

A

p	$\rightarrow$	q
T	T	T
T	F	F
F	F	T
F	F	F

B

p	$\rightarrow$	q
T	T	T
T	F	F
F	T	T
F	F	F

C

p	$\rightarrow$	q
T	T	T
T	F	F
F	F	T
F	T	F

D

p	$\rightarrow$	q
T	T	T
T	F	F
F	T	T
F	T	F

If A was chosen to be the truth table for "$p \rightarrow q$", then the truth values for "$p \rightarrow q$" and "$p \wedge q$" would be the same in every case, making "$p \rightarrow q$" and "$p \wedge q$" equivalent (see 6.32). This would produce some strange reasoning if it was allowed to stand as the definition, so it is not allowed. If B was chosen to be the truth table for "$p \rightarrow q$", then the truth values for "$p \rightarrow q$" and "q" would be the same in every case, making "$p \rightarrow q$" and "q" equivalent (see 6.32). This is definitely undesirable and so it is not allowed as the definition. If C was chosen to be the truth table for "$p \rightarrow q$", then the truth values for "$p \rightarrow q$" and "$p \leftrightarrow q$" would be the same in every case, making "$p \rightarrow q$" and "$p \leftrightarrow q$" equivalent (see 6.32). This too would produce strange reasoning and is therefore not allowed. Therefore, by the process of elimination D must be the "correct" truth table for "$p \rightarrow q$".

<u>UNDERSTOOD GROUPING SYMBOLS</u>

Most of us would agree that $6-5+7 = 8$. A few people might think $6-5+7 = -6$, because they might think $6-5+7$ means $6-(5+7)$ instead of $6+(-5)+7$. Here in this arithmetic example we see the use of understood grouping symbols. The only number subtracted from 6 is the number immediately following the minus sign, which is 5. We will use a similar rule for omitting grouping symbols involving the negation sign in logic (~). We will observe the following rule:

Anytime grouping symbols are omitted involving a negation sign, it is understood that only the statement (or statement pattern) immediately following the negation sign is negated.

7.15 EXAMPLE: Fill in the understood grouping symbols on "~p V q". Since the grouping symbols involving the negation are omitted, the intended statement pattern is "(~p) V q". One might say that the intended statement pattern could just as well be "~(p V q)". This is incorrect; "p" is the statement immediately following the negation sign so according to our rule only "p" is to be negated. Anytime a different grouping arrangement is intended all the grouping symbols must be put in. There is no way to write "~(p V q)" omitting grouping symbols without violating our rule for understood grouping symbols.#

We will now give two columns of statement patterns. The statement patterns in the left column will have understood grouping symbols. The statement patterns in the right column will have the understood grouping symbols filled in.

7.16 (a) ~p V ~q	(b) (~p) V (~q)
7.17 (a) p → ~q	(b) p → (~q)

7.18 (a) $\sim(p \to q) \wedge r$ (b) $[\sim(p \to q)] \wedge r$

7.19 (a) $\sim(p \vee \sim q) \wedge r$ (b) $\{\sim(p \vee [\sim q])\} \wedge r$

In 7.18 (a), a left parenthesis immediately follows the negation sign. This means we are to negate the statement inside the parenthesis, "$p \to q$".

EXERCISES:

7.20 Write each of the following in "if..., then" form.

(a) $1 > 0$ only if 1 is positive.

(b) For the set A to equal the set B it is necessary that A be a subset of B.

(c) Sam put up the umbrella, if it started raining.

(d) Sam put up the umbrella only if it started raining.

(e) A sufficient condition for Sam to fill in line 10-b on the income tax return was for Sam to have made over $10,000.

(f) A sufficient condition for you to get an A is for you to have a 90 average.

(g) A necessary condition for Sam to be able to prove a theorem is for Sam to know logic.

(h) Sam will be able to prove a theorem only if he knows logic.

(i) A necessary and sufficient condition for 0 to be less than 1 is for 1 to be greater than 0.

(j) A necessary condition for Sam to be a good mathematician is for Sam to work hard.

7.21 Fill in the understood grouping symbols.

(a) $\sim p \rightarrow q$

(b) $\sim(p \rightarrow q) \rightarrow \sim r$

(c) $\sim p \wedge \sim q$

(d) $p \rightarrow [\sim(r \rightarrow w) \wedge t]$

(e) $p \rightarrow [(\sim r \rightarrow w) \wedge t]$

(f) $\sim p \wedge \sim(w \rightarrow \sim q)$

(g) $\sim(p \wedge [\sim w \rightarrow \sim q])$

8. LIST OF TAUTOLOGIES

Many tautologies have been encountered in the reading and the homework. The important ones will be listed here. A necessary condition for the reader to excel in the art of proof is for the reader to have this list thoroughly learned.

T_1	$[(p \to q) \wedge p] \to q$	Modus Ponens
T_2	$[(p \to q) \wedge (\sim q)] \to (\sim p)$	Modus Tollens
T_3	$[\sim(p \wedge q)] \leftrightarrow [(\sim p) \vee (\sim q)]$	DeMorgan's Laws
T_4	$[\sim(p \vee q)] \leftrightarrow [(\sim p) \wedge (\sim q)]$	
T_5	$[\sim(p \to q)] \leftrightarrow [p \wedge (\sim q)]$	Law of negating an implication
T_6	$(p \to q) \leftrightarrow [(\sim p) \vee q]$	Implication Equivalence
T_7	$(p \to q) \leftrightarrow [(\sim q) \to (\sim p)]$	Contrapositive
T_8	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Syllogism
T_9	$(p \leftrightarrow q) \leftrightarrow [(p \to q) \wedge (q \to p)]$	
T_{10}	$p \leftrightarrow [\sim(\sim p)]$	Double negation
T_{11}	$(p \wedge q) \to p$	Law of simplification
T_{12}	$p \to (p \vee q)$	
T_{13}	$(p \wedge q) \leftrightarrow (q \wedge p)$	Commutative Laws
T_{14}	$(p \vee q) \leftrightarrow (q \vee p)$	
T_{15}	$(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$	
T_{16}	$[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$	Distributive Laws
T_{17}	$[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$	
T_{18}	$[p \wedge (q \wedge r)] \leftrightarrow [(p \wedge q) \wedge r]$	
T_{19}	$[p \vee (q \vee r)] \leftrightarrow [(p \vee q) \vee r]$	

9. VALID ARGUMENT

The reader may not have understood the purpose of the sections we have studied; all the previous sections have been the foundation for this and the next two sections.

9.1 An argument pattern consists of a set of statement patterns called the hypotheses of the argument pattern along with a statement pattern called the conclusion of the argument pattern.

9.2 An argument consists of a set of statements called the hypotheses of the argument along with a statement called the conclusion of the argument.

The reader should note that an argument pattern consists of statement patterns while an argument consists of statements. The reader should also note that we are now using the words hypothesis* and conclusion in a new way. It is felt that the context will make it clear whether we are talking about the hypothesis and conclusion of an implication or an argument (or argument pattern).

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*The word premise is sometimes used for the word hypothesis.

9.3 EXAMPLE OF AN ARGUMENT PATTERN AND THE FORM FOR AN ARGUMENT PATTERN. The argument pattern with hypotheses "$p \rightarrow q$", "$q \rightarrow r$", and "$\sim p$" and with conclusion "$\sim r$" will be written in the following form:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \underline{\sim p \qquad} \\ \therefore \sim r \end{array}$$

Three dots stand for the word "therefore". The argument pattern just diagrammed is read "p implies q, q implies r, not p, therefore not r". The hypotheses of the argument pattern are written above the line.#

9.4 EXAMPLE OF AN ARGUMENT AND THE FORM FOR AN ARGUMENT. The argument with "If Sam had a 90 average, then Sam made an A," "If Sam made an A, then Sam made the honor roll," and "Sam did not have a 90 average" as hypotheses and "Sam did not make the honor roll" as conclusion will be written in the following form:

If Sam had a 90 average, then Sam made an A.
If Sam made an A, then Sam made the honor roll.
<u>Sam did not have a 90 average.</u>
∴ Sam did not make the honor roll.

The diagram is read in a similar fashion to the diagram in example 9.3. Again let it be noted that the hypotheses of the argument are written above the line and conclusion of the argument is written below the line.#

(Note: The <u>argument</u> given in 9.4 is an example of an <u>instance of the argument pattern</u> given in 9.3. "Sam had a 90 average" is a replacement for the statement variable "p", "Sam made an A" is a replacement for the statement variable "q", and "Sam made the honor roll" is a replacement for the statement variable "r".)

Our goal is to be able to decide if an argument (or argument pattern) is valid or not. Valid argument and valid argument pattern have not been defined yet, but the reader probably has some intuitive ideas about valid argument. Before valid argument and valid argument pattern are defined, we would like to check how good the reader's intuition is. We will list several argument patterns along with an instance of each argument pattern. The reader is invited to guess which are valid and which are not valid.

9.5 (a) $p \rightarrow q$

$\underline{p\quad\quad}$

$\therefore q$

(b) If Sam had a 90 average, then Sam made an A.

<u>Sam had a 90 average.</u>

∴ Sam made an A.

9.6 (a) $p \rightarrow q$

$\underline{q\quad\quad}$

$\therefore p$

(b) If Sam had a 90 average, then Sam made an A.

<u>Sam made an A.</u>

∴ Sam had a 90 average.

9.7 (a) $p \rightarrow q$

$\underline{\sim p\quad\quad}$

$\therefore \sim q$

(b) If Sam had a 90 average, then Sam made an A.
<u>Sam did not have a 90 average.</u>
∴ Sam did not make an A.

9.8 (a) $p \rightarrow q$
$\underline{\sim q}$
$\therefore \sim p$

(b) If Sam had a 90 average, then Sam made an A.
<u>Sam did not make an A.</u>
∴ Sam did not have a 90 average.

9.9 (a) $p \rightarrow q$
$\underline{q \rightarrow r}$
$\therefore p \rightarrow r$

(b) If Sam had a 90 average, then Sam made an A.
<u>If Sam made an A, then Sam made the honor roll.</u>
∴ If Sam had a 90 average, then Sam made the honor roll.

It will be easier to define what is meant by a <u>valid argument pattern</u> than it will be to define what is meant by a <u>valid argument</u>. We therefore will delay momentarily our definition of <u>valid argument</u>, and we will now define the easier concept of a <u>valid argument pattern</u>.

9.10 <u>DEFINITION</u>: The argument pattern with hypotheses $H_1, H_2, H_3, \ldots, H_n$ and conclusion C is valid if and only if $(H_1 \wedge H_2 \wedge H_3 \wedge \ldots \wedge H_n) \rightarrow C$ is a tautology.

Let us examine argument pattern 9.5 (a) to see if it is valid. Argument pattern 9.5 (a) has two hypotheses. In order to apply definition 9.10, we will replace H_1 with the hypothesis that "p → q", and we will replace H_2 with the hypothesis "p". The conclusion of argument pattern 9.5 (a) is "q". So "q" will replace C in our definition. We have to check to see if "[(p → q) ∧ p] → q" is a tautology. "[(p → q) ∧ p] → q" is tautology T_1 of section 8. Thus, argument pattern 9.5 (a) is a valid argument pattern. Argument pattern 9.5 (a) is an example of the <u>modus ponens</u> argument pattern.

Let us examine argument pattern 9.6 (a) for validity. By definition 9.10. argument pattern 9.6 (a) is valid if and only if "[(p → q) ∧ q] → p" is a tautology. Truth table 9.11 shows "[(p → q) ∧ q] → p" not to be a tautology.

9.11

[(p	→	q)	∧	q]	→	p
T	T	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	T	T	**F**	F
F	T	F	F	F	T	F

Argument pattern 9.6 (a) is therefore invalid.*
Argument pattern 9.6 (a) is an example of <u>converse</u>

*"Invalid" means "not valid".

reasoning. Any argument pattern that has the form of converse reasoning is therefore invalid.

9.7 (a) is not a valid argument pattern. A truth table could be made to show "$[(p \to q) \wedge (\sim p)] \to (\sim q)$" not to be a tautology. Argument pattern 9.7 (a) is an example of inverse reasoning. All argument patterns that have the form of inverse reasoning are therefore invalid.

9.8 (a) is a valid argument pattern because "$[(p \to q) \wedge (\sim q)] \to (\sim p)$" is a tautology ($T_2$, section 8). 9.8 is an example of the modus tollens argument pattern. Any modus tollens argument pattern is therefore valid.

9.9 (a) is a valid argument pattern because "$[(p \to q) \wedge (p \to r)] \to (p \to r)$" is a tautology ($T_8$, section 8). 9.9 (a) is an example of a syllogism argument pattern. Any syllogism argument pattern is therefore valid.

The modus ponens, modus tollens, and syllogism argument patterns are the three valid argument patterns we have seen. To emphasize their importance we will list them below.

9.12	MODUS PONENS	MODUS TOLLENS	SYLLOGISM
	$p \to q$	$p \to q$	$p \to q$
	$\underline{p\quad}$	$\underline{\sim q\quad}$	$\underline{q \to r}$
	$\therefore q$	$\therefore \sim p$	$\therefore p \to r$

We are now going to discuss what is meant by a valid argument. The reader may think that in order to decide whether an argument is valid, one simply finds an argument pattern for which the given argument is an instance. If the argument pattern is valid, then the argument is valid. If the argument pattern is invalid, then the argument is invalid. The trouble with this idea is that the same argument can be an instance of several argument patterns, some valid and some invalid.

9.13 (a) $\begin{array}{l} p \\ \underline{q} \\ \therefore r \end{array}$ (b) $\begin{array}{l} p \to q \\ \underline{r\quad\quad} \\ \therefore q \end{array}$ (c) $\begin{array}{l} p \to q \\ \underline{p\quad\quad} \\ \therefore q \end{array}$

For instance argument 9.5 (b) is an instance of 9.13 (a), 9.13 (b), and 9.13 (c). Argument pattern 9.13 (a) is invalid. Argument pattern 9.13 (b) is invalid, but argument pattern 9.13 (c) is valid. Therefore, it is difficult to decide whether our argument is valid or invalid. Our goal is to find for any given argument an argument pattern with these two properties: (1) if the argument pattern is valid, then the argument is valid; (2) if the argument pattern is invalid, then the argument is invalid. To do this, we introduce the concept of prime statement.

9.14 DEFINITION: A prime statement is a statement that is neither a negation, disjunction, conjunction, implication, or biconditional.

9.15 EXAMPLE: "1 is less than 2" is a prime statement. "1 is not less than 2" is not a prime statement because it is a negation. "Sam went home or to the gym," is not a prime statement because it is a disjunction. "Sam went home" is a prime statement. "Sam went to the gym" is a prime statement. "If Sam had a 90 average, then Sam made an A" is not a prime statement because it is an implication. "Sam had a 90 average" is a prime statement. "Sam made an A" is a prime statement.

We are now going to see how we can test an argument for validity.

9.16 TO TEST A GIVEN ARGUMENT FOR VALIDITY:

(1) Find an argument pattern such that:

(a) the given argument is an instance of the argument pattern;

(b) only prime statements were substituted for the statement variables to make the instance that is the given argument;

(c) the same prime statement was substituted for the same variable throughout the argument pattern to make the instance that is the given argument.

(2) The argument is valid if and only if the argument pattern described in (1) is a valid argument pattern.

Let us now apply 9.16 to argument 9.5 (b) and see if it is a valid argument or not. 9.13 (a) is not the argument pattern we are after because the statement "If Sam had a 90 average, then Sam made an A" had to be substituted for the statement variable "p". So we substituted a statement that was not a prime statement for a statement variable. 9.13 (b) is not the statement pattern we are after because we substituted the same prime statement for the statement variables "p" and "r" in order to produce 9.5 (b) as an instance of 9.13 (b). 9.13 (c) is the desired statement pattern. If we replace the prime statement "Sam had a 90 average" for the statement variable "p" throughout the argument pattern 9.13 (c) and if we substitute the prime statement "Sam made an A" for the statement variable "q" throughout the argument pattern 9.13 (c), then the resulting instance will be the argument 9.5 (b). So it is seen that argument pattern 9.13 (c) does satisfy requirements set forth in 9.16. We have previously seen that 9.13 (c) is a valid argument pattern because it is argument pattern 9.5 (a). Therefore, 9.5 (b) is a valid argument.

Let us now look at argument 9.6 (b) A careful examination of argument pattern 9.6 (a) will show that argument pattern meets the requirements set forth in 9.16. Since argument pattern 9.6 (a) is an invalid argument pattern, then argument 9.6 (b) is an invalid argument.

We will now test the validity of argument 9.7 (b). A look at argument pattern 9.7 (a) should convince the reader that it meets the conditions set forth in 9.16. Since 9.7 (a) is an invalid argument pattern, then argument 9.7 (b) is an invalid argument.

We will now test the validity of argument 9.8 (b). A look at argument pattern 9.8 (a) should convince the reader that it meets the requirements set forth in 9.16. Therefore, argument 9.8 (b) is a valid argument since argument pattern 9.8 (a) is a valid argument pattern.

Now let us look at argument 9.9 (b). Argument pattern 9.9 (a) meets the requirements set forth in 9.16. Argument pattern 9.9 (a) is a valid argument pattern. Therefore argument 9.9 (b) is a valid argument.

Since argument pattern 9.5 (a) was an example of a modus ponens argument pattern, it will follow that argument 9.5 (b) is an example of a modus ponens argument. using similar reasoning 9.6 (b) is an example of converse reasoning; 9.7 (b) is an example of inverse reasoning; 9.8 (b) is an example of a modus tollens argument; and 9.9 (b) is an example of the syllogism argument.

All the argument patterns we have checked for validity have had only two hypotheses. The argument pattern given in 9.3 had three hypotheses. According to definition 9.10, argument pattern 9.3 is valid if and only if "$[(p \rightarrow q) \wedge (q \rightarrow r) \wedge (\sim p)] \rightarrow (\sim r)$" is a tautology. The hypothesis of the implication is "$(p \rightarrow q) \wedge (q \rightarrow r) \wedge (\sim p)$", which is the conjunction of <u>THREE</u> statement patterns. Since our definitions only deal with the truth values for the conjunction of two statements, we do not even know how to determine the truth values for the conjunction of three statements, much less make a truth table for the conjunction of three statement patterns.

9.17 <u>DEFINITION</u>: The conjunction of any number of statements is true if and only if all of the statements are true. The conjunction of any number of statements is false if and only if at least one of the statements is false.

9.18 EXAMPLE OF A TRUTH TABLE FOR THE CONJUNCTION OF THREE STATEMENT PATTERNS. Make a truth table for "$(p \to q) \land (q \to r) \land (\sim p)$". Our first attempt might look like this:

9.19	(p	→	q)	∧	(q	→	r)	∧	(~	p)
Case 1	T	T	T		T	T	T		F	T
Case 2	T	T	T		T	F	F		F	T
Case 3	T	F	F		F	T	T		F	T
Case 4	T	F	F		F	T	F		F	T
Case 5	F	T	T		T	T	T		T	F
Case 6	F	T	T		T	F	F		T	F
Case 7	F	T	F		F	T	T		T	F
Case 8	F	T	F		F	T	F		T	F
	1	2	3	4	5	6	7	8	9	10

Since we have no grouping symbols we would not know how to fill in the two remaining columns and we would not know what column contained the truth values for the whole statement pattern. We will therefore complete the truth table no further. We will just keep definition 9.17 in mind to read the truth values on truth table 9.19. In order to tell the truth value for case 1 we look at columns 2, 6, and 9 of case 1. We have the conjunction of true, true, and false statements respectively. Since at least one of the statements we are conjuncting is false, we see that case 1 will be <u>false</u>. Case 2 is false since we are conjuncting one true statement with two false statements. Case 5 is true since we are conjuncting 3 true statements.#

A truth table will now be made to check the validity of argument pattern 9.3.

9.20

[(p	→	q)	∧	(q	→	r)	∧	(~	p)]	→	(~	r)
T	T	T		T	T	T		F	T	T	F	T
T	T	T		T	F	F		F	T	T	T	F
T	F	F		F	T	T		F	T	T	F	T
T	F	F		F	T	F		F	T	T	T	F
F	T	T		T	T	T		T	F	F	F	T
F	T	T		T	F	F		T	F	T	T	F
F	T	F		F	T	T		T	F	**F**	F	T
F	T	F		F	T	F		T	F	T	T	F
1	2	3	4	5	6	7	8	9	10	11	12	13

(In filling out truth table 9.20 remember there is no column for truth values of the hypotheses of the implication. You have to look at columns 2, 6, and 9 in order to determine the truth value of the hypotheses of the implication.)

A look at truth table 9.20 will show that the statement pattern under consideration is not a tautology; therefore, argument pattern 9.3 is invalid. (This also means argument 9.4 is invalid since its argument pattern is invalid).

We have been given the argument pattern for every argument that we have checked for validity. On most occasions an argument will be given with no accompanying argument pattern. Therefore, one will have to find its argument pattern according to 9.16 and check it for validity. We will look at one such example now.

9.21 EXAMPLE: Check the following argument for validity:

(a) Sam did not run or Joe walked

Sam ran.

If Joe did not walk, then Sue got mad.

$\therefore$ Sue did not get mad.

Let "Sam did run" be a replacement of the statement variable "p". Let "Joe walked" be a replacement for the statement variable "q". Let "Sue got mad" be a replacement for the statement variable "r". Thus, the argument under consideration has as its argument pattern the following argument pattern (which satisfies 9.16).

(b) $(\sim p) \vee q$

p

$\underline{(\sim q) \rightarrow r}$

$\therefore \sim r$

This argument pattern is valid if and only if "$\{[(\sim p) \vee q] \wedge p \wedge [(\sim q) \rightarrow r]\} \rightarrow (r)$" is a tautology.

(c)

[(~	p)	V	q]	Λ	p	Λ	[(~	q)	→	r]	→	(~	r)
F	T	T	T		T		F	T	T	T	**F**	F	T
F	T	T	T		T		F	T	T	F	T	T	F
F	T	F	F		T		T	F	T	T	T	F	T
F	T	F	F		T		T	F	F	F	T	T	F
T	F	T	T		F		F	T	T	T	T	F	T
T	F	T	T		F		F	T	T	F	T	T	F
T	F	T	F		F		T	F	T	T	T	F	T
T	F	T	F		F		T	F	F	F	T	T	F
1	2	3	4	5	6	7	8	9	10	11	12	13	14

A look at the preceding truth table will show that "$\{[(\sim p) \vee q] \wedge p \wedge [(\sim q) \rightarrow r]\} \rightarrow (r)$" is not a tautology. Therefore, argument pattern 9.21 (b) is invalid. So argument 9.21 (a) is invalid since its argument pattern is invalid.#

The reader may probably wonder *why* valid argument (and valid argument pattern) was defined the way it was. The reason will be discussed in the next section. For now it is of prime importance for the reader to understand *how* the definition for validity is applied.

EXERCISES:

9.22 Tell whether each of the following is modus ponens, modus tollens, syllogism, converse reasoning, or inverse reasoning. Tell whether each is valid or invalid. Do not make truth tables; use the information you have already learned in this section.

(a) If $0 < 1$, then $4 < 6$
$0 < 1$
$\therefore 4 < 6$

(b) If $0 < 1$, then $4 < 6$
4 is not less than 6
$\therefore$ 0 is not less than 1

(c) If Sam put on the brakes, then the car stopped.
Sam did not put on the brakes.
$\therefore$ The car did not stop

(d) If Sam put on the brakes, then the car stopped.
The car stopped.
$\therefore$ Sam put on the brakes.

(e) If Sam did not sleep, then Sam will be tired.
Sam did not sleep.
$\therefore$ Sam will be tired.

(f) If Sam worked and $1 < 0$, then Joe ran.
Sam worked and $1 < 0$.
$\therefore$ Joe ran.

(g) $$\begin{array}{l} [(\sim w) \vee s] \rightarrow (\sim q) \\ (\sim q) \\ \hline \therefore (\sim w) \vee s \end{array}$$

(h) $$\begin{array}{l} [(\sim w) \vee s] \rightarrow (\sim q) \\ (\sim w) \vee s \\ \hline \therefore \sim q \end{array}$$

(i) If A is a set and A = B, then B is a set.
B is a set.
―――――――――――――――――――――――――――
∴ A is a set and A = B.

(j) $$\begin{array}{l} (p \vee q) \rightarrow (\sim w) \\ \sim(\sim w) \\ \hline \therefore \sim(p \vee q) \end{array}$$

(k) $$\begin{array}{l} [(\sim p) \vee q] \rightarrow (r \vee s) \\ \sim(r \vee s) \\ \hline \therefore \sim[(\sim p) \vee q] \end{array}$$

(l) If 1 < 0, then Sam is not tired.
If Sam is not tired, then 2 < 3.
―――――――――――――――――――――――――――
∴ If 1 < 0, then 2 < 3.

9.23 Which of the following argument patterns are valid?

(a) $$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

(b) $$\begin{array}{l} p \vee q \\ p \rightarrow r \\ q \rightarrow r \\ \hline \therefore r \end{array}$$

(c) $$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

(d) $$\begin{array}{l} p \\ \sim p \\ \hline \therefore q \end{array}$$

(e) $$\begin{array}{l} p \rightarrow q \\ p \wedge (\sim q) \\ \hline \therefore r \end{array}$$

(f) $$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

(g) $$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

(h) $$\begin{array}{l} \sim(p \wedge q) \\ (\sim p) \rightarrow r \\ (\sim q) \rightarrow r \\ \hline \therefore r \end{array}$$

(i) $p \vee q$

$p \to r$

$\underline{q \to s}$

$\therefore r \wedge s$

9.24 Which of the following arguments are valid?

(a) If Sam gets elected, then labor will not be all powerful.

If labor is not all powerful, then democracy will survive.

∴ If democracy did not survive, then Sam did not get elected.

(b) Sam did not go to town or Joe's house is termite proof.

If the horse did not eat Joe's grass, then Sam went to town.

If Joe did not spray for termites, then Joe's house is not termite proof.

∴ The horse ate Joe's grass or Joe did spray for termites.

(c) If you take this insurance policy then (you will feel secure and your family will have money in case you die.)

If your family will not have money in case you die, then you do not love your family.

∴ If you do love your family, then you will take this insurance policy.

(d) If Nashville is in Tennessee, then Sam made an A.

∴ Sam made an A.

9.25 Complete the following argument pattern so that it is (a) modus ponens, (b) converse reasoning, (c) inverse reasoning, (d) modus tollens, and (e) syllogism.

$$A \rightarrow (\sim B)$$

____

$\therefore$

9.26 Complete the following argument so that it is (a) modus ponens, (b) converse reasoning, (c) inverse reasoning, (d) modus tollens, and (e) syllogism.

If Sam had TB, then Sam had a positive skin test.

____

$\therefore$

10. DIRECT PROOF

Validity depends on structure, not content. It may surprise the reader that 9.22 (b) is a valid argument. 9.22 (b) is valid because it is an example of the modus tollens argument. The reader may be equally surprised to find that 9.24 (d) is an invalid argument. 9.24 (d) is invalid simply because it does not satisfy the definition for being a valid argument, as we will now see. 9.24 (d) has the following argument pattern, that satisfies 9.16.

10.1 $\underline{p \to q}$

$\therefore q$

10.1 is an invalid argument pattern since "$(p \to q) \to q$" is not a tautology. Therefore, 9.24 (d) is invalid. If "Nashville is in Tennessee" was added as an additional hypothesis for 9.24 (d), then it would be a valid argument (modus ponens). To say that an argument is valid does not necessarily mean that the conclusion is true and it does not necessarily mean any of the hypotheses are true. The validity of an argument is entirely independent of the meaning of the statements that make up the argument. The validity of an argument is entirely dependent on the structure (pattern) of the argument. Therefore when one reads an argument, if he wants to decide on its validity, he should not be distracted by the meaning of the statements that make up the argument; he should try to discover the structure of the argument.

Since these unusual arguments turned out to be valid according to our definition, the reader may begin to wonder if our definition is "correct" and may be especially curious as to why we defined it the way we did. We will tell our reasons after some brief comments on symbolization.

SYMBOLIZATION: Statements are abstract ideas. "One is less than two," "Une est moins que deux," and "Eins ist weniger als zwei" are all different symbolizations for the same statement.* How do symbols get started? A group of people will agree to let a certain symbol represent a certain abstract idea. They have it within their power to choose whatever symbols they want, to denote whatever ideas they want; there is not right or wrong choice of symbols. The purpose of symbols is to communicate ideas. In different languages it takes different amounts of symbolization to symbolize the same idea. The mathematician wants to communicate his ideas as clearly and economically as possible. Therefore, the mathematician has it within his power to symbolize any way he chooses in order to achieve clarity and economy. For example, the mathematician can say let "p'" denote the statement "One is less than two."** Thus "one is less than two," "Une est moins que

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*When we say "Consider the statement 'One is less than two,'" what we really mean is "Consider the statement denoted by (symbolized by) 'One is less than two.'"

**More precisely, let "p'" denote the statement denoted by "One is less than two."

deux," "Eins ist weniger als zwei," and "p'" are different denotations for the same statement in English, French, German, and the mathematician's language respectively. The mathematician's sole responsibility is to make it clear which symbols denote what, so the reader can translate from his language to the mathematician's, and vice versa.

Now let "p'" denote the statement "Sam had a 90 average." Let "q'" denote the statement "Sam made an A." Let r' denote the statement "Sam made the honor roll." Thus "p'", "q'" and "r'" are statements,* not statement variables, and since they are statements they can be replacements for statement variables.

"p → q" is a statement pattern. "p' → q'" is a statement; moreover, "p' → q'" is the instance of the statement pattern "p → q" formed by letting "p'" be a replacement for "p" and letting "q'" be a replacement for "q". Consider the two following diagrams:

10.2 (a)
$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \underline{\sim p \qquad} \\ \therefore \sim r \end{array}$$

(b)
$$\begin{array}{l} p' \rightarrow q' \\ q' \rightarrow r' \\ \underline{\sim p' \qquad} \\ \therefore \sim r' \end{array}$$

10.2 (a) is an argument pattern, not an argument. 10.2 (b) is an argument, not an argument pattern. Furthermore, it is clear that 10.2 (b) is an instance

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*When we say "p'" is a statement, we mean that same thing as when we say "'Sam had a 90 average' is a statement." This will be the last time it will be mentioned, but in actuality "p'" and "Sam had a 90 average" are both symbolizations of statements.

of the argument pattern 10.2 (a). The reader may recall that 10.2 (b) is argument 9.4, which was found to be invalid.

WHY WAS A VALID ARGUMENT DEFINED THE WAY IT WAS? We wanted a valid argument to have the property that whenever the hypotheses of a valid argument were accepted as true, one would necessarily have to accept the conclusion as being true. We maintain that the definition given for valid argument achieves this goal, and we will now explain why, using the following valid argument (modus tollens) with two hypotheses:

10.3 If Sam had a 90 average, then Sam made an A.
Sam did not make an A.
$\therefore$ Sam did not have a 90 average.

We will let "p'" denote "Sam had a 90 average". We let "q'" denote "Sam made an A." Now note that 10.4 (a) is another way of writing argument 10.3. Note also, that 10.4 (b) is the argument pattern for 10.4 (a) that satisfies 9.16.

10.4 (a) $p' \rightarrow q'$ $\quad$ (b) $p \rightarrow q$
$\underline{\sim q'}$ $\quad$ $\underline{\sim q}$
$\therefore \sim p'$ $\quad$ $\therefore \sim p$

Keep in mind that we are trying to show that if the hypotheses of the valid argument 10.4 (a) are true, then the conclusion is forced to be true. So let us assume the hypotheses "$p' \rightarrow q'$" and "$\sim q'$" are true and show that "$\sim p'$" must be true. Since 10.4 (a) is a valid argument, 10.4 (b) is a valid argument pattern. Therefore, "$[(p \rightarrow q) \wedge (\sim q)] \rightarrow (\sim p)$" must be a tautology. "$[(p' \rightarrow q') \wedge (\sim q')] \rightarrow (\sim p')$" is an instance of "$[(p \to q) \wedge (\sim q)] \to (\sim p)$" and is true since every instance of a tautology is true. Since "$p' \to q'$" and "$\sim q'$" are both true, "$(p' \to q') \wedge (\sim q')$" is true. Now "$[(p' \to q') \wedge (\sim q')] \to (\sim p')$" is a true implication whose hypothesis "$(p' \to q') \wedge (\sim q')$" is true; so "$\sim p'$" has to be true (if "$\sim p'$" were false, the implication "$[(p' \to q') \wedge (\sim q')] \to (\sim p')$" would be false). It is hoped that this argument has convinced the reader that a valid argument with true hypotheses must have a true conclusion.

10.5 <u>EXAMPLE</u>: Assume "$p' \to q'$" and "$q' \to r'$" are true statements. Under this assumption we are forced to accept that "$p' \to r'$" is a true statement since "$p' \to r'$" is the conclusion of a valid (syllogism) argument whose hypotheses ("$p' \to q'$" and "$q' \to r'$") are true.#

<u>Direct proof</u>. In Section 9 we learned that by making the proper truth tables we could tell whether an argument pattern was valid or invalid. This "truth table method" was good because one could prove every argument to be either valid or invalid, but for argument patterns with may hypotheses the "truth table method" can become very cumbersome and time consuming. We are now going to learn a different way to prove an argument pattern valid; this method of proof is called the direct proof. <u>Unlike the "truth table method" we will not be able to prove an argument invalid using the direct proof method</u>. The direct proof method for showing an argument pattern valid will make considerable use of the list of tautologies (Section 8) and our already known valid argument patterns. We will

now tell how one proves an argument pattern valid by direct proof; next we will give a specific example of an argument pattern proven valid by direct proof; and then we will explain why a direct proof does indeed prove an argument pattern to be valid.

10.6 TO PROVE AN ARGUMENT PATTERN VALID BY DIRECT PROOF

(a) Assume you are given an arbitrary instance of the argument pattern where all the hypotheses are true.

(b) By means of equivalent statements, known valid arguments, tautologies, and truth table 3.1, show that the conclusion of the argument must be true.

(Note: We may use the fact that a valid argument with true hypotheses must have a true conclusion.)

10.7 <u>EXAMPLE</u>: Prove the following argument pattern valid by direct proof.

$$\begin{array}{l} p \rightarrow q \\ (\sim r) \rightarrow (\sim q) \\ \underline{p \qquad\qquad\quad} \\ \therefore\ r \end{array}$$

0. Assume "p'", "q'", and "r'" are arbitrary statements that are replacements for "p", "q", and "r" respectively.
1. $p' \rightarrow q'$ Assumed for direct proof
2. $(\sim r') \rightarrow (\sim q')$ Assumed for direct proof

3. p' Assumed for direct proof
4. $q' \rightarrow r'$ 2, Contrapositive
5. $p' \rightarrow r'$ 1, 4, Syllogism
6. r' 5, 3, Modus ponens

Every direct proof will be written in the above form. The column on the left is a list of true statements. The column on the right tells the reason why the statements are true. Let us note how directions 10.6 were followed. Lines 0, 1, 2, and 3 satisfy 10.6 (a); we are assuming "p'", "q'", and "r'" are arbitrary statements that are replacements for "p", "q", and "r" respectively, that make all the hypotheses true. Our goal according to 10.6 (b) is to show that the conclusion must be true. Since "r" is the conclusion of the argument pattern and "r'" is a replacement for "r", then "r'" is the conclusion of the argument, and our goal is to show that "r'" must be true. line 2 is the contrapositive of line 4. A statement that is an implication and its contrapositive are equivalent statements, so they have the same truth values. Since statement 2 is true, statement 4 must be true and is added to the column of true statements ("2, T_7" could also be used as a reason for 4 being true, since T_7, tautology 7, tells us that an implication and its contrapositive are equivalent). Statements 1 and 4 are true statements that are hypotheses for a syllogism argument. Statement 5 is the conclusion for this particular syllogism argument. Statement 5 is therefore true and is added to our list of true statements. Statements 5 and 3 are true statements that are hypotheses for a modus ponens argument. Statement 6 is the conclusion for this modus ponens argument. Statement 6 is, therefore, true and is added to our list of true statements. Statement 6 is the statement we were trying to show must be true, so our proof is concluded.

The reader may wonder how do you know what statement to put next in a direct proof? You have to "see the light"! The best way known to "see the light" is first, to have the tautologies, known valid arguments, and truth table 3.1 <u>learned, memorized, and absorbed</u> -- this cannot be over-emphasized. Second, on a sheet of scratch paper write down the hypotheses that you have assumed true. Third, as you <u>diligently think</u> on the problem, write down on the scratch paper other statements that have to be true because of the tautologies, known valid arguments, and truth table 3.1, and try to get the conclusion to be one of them. Fourth, once the conclusion is added to the list of known truths, <u>then</u> write down your direct proof leaving out all the statements on your scratch paper that have no part in showing the conclusion to be true. So you "see the light" after hard work.

But the reader may wonder what happens if you try all of this and are still unable to show the conclusion true; does that mean the argument pattern is invalid? No, not necessarily. Just because the reader is unable to get a direct proof does not mean that it is impossible to get a direct proof. The only way we know to prove an argument pattern invalid is by the truth table method.

<u>The reason why a direct proof proves an argument pattern to be valid</u>: The only way a statement that is an implication can be false is for the hypothesis to be true and the conclusion to be false.

10.8 A way to show a <u>statement</u> that is an implication is true is first assume the hypothesis true and second, show the conclusion must be true.

10.9 (a) $p \rightarrow q$ (b) $p' \rightarrow q'$

$$\begin{array}{ll} (\sim r) \rightarrow (\sim q) & \quad (\sim r') \rightarrow (\sim q') \\ \underline{p \qquad\qquad} & \quad \underline{p' \qquad\qquad} \\ \therefore\ r & \quad \therefore\ r' \end{array}$$

In example 10.7 we claimed we had proven 10.9 (a) valid because we had followed the procedure outlined in 10.6; we will now see <u>why</u> procedure 10.6 proves 10.9 (a) valid. If we can show "$[(p \rightarrow q) \wedge (\sim r \rightarrow \sim q) \wedge p] \rightarrow r$" to be a tautology, then we have proven 10.9 (a) valid. "$[(p \rightarrow q) \wedge (\sim r \rightarrow \sim q) \wedge p] \rightarrow r$" is a tautology if and only if everyone of its instances is true. Let us assume "$[(p' \rightarrow q') \wedge (\sim r' \rightarrow \sim q') \wedge p'] \rightarrow r'$" is an arbitrary instance of "$[(p \rightarrow q) \wedge (\sim r \rightarrow \sim q) \wedge p] \rightarrow r$". Since "$[(p' \rightarrow q') \wedge (\sim r' \rightarrow \sim q') \wedge p'] \rightarrow r'$" is arbitrary, if we show it to be true, then we have shown that every instance of "$[(p \rightarrow q) \wedge (\sim r \rightarrow \sim q) \wedge p] \rightarrow r$" is true. According to 10.8, if we assume "$(p' \rightarrow q') \wedge (\sim r' \rightarrow \sim q') \wedge p'$" true and show "$r'$" true, then "$[(p' \rightarrow$

q') ∧ (~r' → ~q') ∧ p'] → r'" must be true, making "[(p → q) ∧ (~r → ~q) ∧ p] → r" a tautology, making 10.9 (a) valid. So when we assume 10.9 (b) to be an arbitrary instance of 10.9 (a) whose hypotheses "p' → q'", "~r' → ~q'", and "p'" are true and show "r'" true, we are not only following directions 10.6, we are also following the procedure described in the previous sentence that proves an argument pattern valid. (If the reader did not follow the preceding discussion, it is unimportant. it is important for the reader to be able to prove an argument pattern valid by direct proof.)

10.10 <u>EXAMPLE</u>: Prove the following argument pattern valid by direct proof.

$$\begin{array}{l} p \to r \\ q \to s \\ \underline{(\sim r) \wedge (\sim s)} \\ \therefore \sim(p \vee q) \end{array}$$

0. Assume each of "p'", "q'", "r'", and "s'" are arbitrary statements that are replacements for "p", "q", "r", and "s" respectively.

1.	p' → r'	Assumed for direct proof
2.	q' → s'	Assumed for direct proof
3.	(~r') ∧ (~s')	Assumed for direct proof
4.	~r'	3, Truth table definition of conjunction
5.	~p'	1, 4, modus tollens
6.	~s'	3, Truth table definition of conjunction
7.	~q'	2, 6, modus tollens

8. $(\sim p') \wedge (\sim q')$ — 5, 7, Truth table definition of conjunction

9. $\sim(p' \vee q')$ — 8, T_4 (DeMorgan's Law)

Statement 4 and statement 6 are true because they are the component parts of the <u>true</u> conjunction statement 3.

10.11 <u>EXAMPLE</u>: Prove the following statement pattern valid by direct proof:

$$\begin{array}{l} \sim[p \rightarrow (\sim q)] \\ \underline{(p \wedge q) \rightarrow [(\sim r) \vee s]} \\ \therefore r \rightarrow s \end{array}$$

0. Assume "p'", "q'", "r'", and "s'" are arbitrary statements that are replacements for "p", "q", "r", and "s" respectively.

1.	$\sim[p' \rightarrow (\sim q')]$	Assumed for direct proof
2.	$(p' \wedge q') \rightarrow [(\sim r') \vee s']$	Assumed for direct proof
3.	$p' \wedge [\sim(\sim q')]$	1, T_5
4.	$p' \wedge q'$	3, T_{10}
5.	$(\sim r') \vee s'$	2, 4, modus ponens
6.	$r' \rightarrow s'$	5, T_6

<u>EXERCISES</u>:

10.12 Everyone of the following argument patterns that is valid try to prove valid by means of direct proof. Everyone of the following argument patterns that is invalid is to be proven invalid by means of a truth table.

(a) $$\begin{array}{l} (\sim p) \to (\sim q) \\ (\sim r) \to (\sim p) \\ q \\ \hline \therefore r \end{array}$$

(b) $$\begin{array}{l} (\sim p) \vee (\sim q) \\ r \to (p \wedge q) \\ \hline \therefore \sim r \end{array}$$

(c) $$\begin{array}{l} p \to q \\ \sim(p \to r) \\ (q \vee s) \to w \\ \hline \therefore w \end{array}$$

(d) $$\begin{array}{l} (p \to q) \to w \\ \sim(r \to s) \\ r \to [(\sim p) \vee q] \\ \hline \therefore w \end{array}$$

(e) $$\begin{array}{l} p \to [(\sim q) \vee r] \\ (\sim q) \to w \\ p \\ \hline \therefore w \end{array}$$

(f) $$\begin{array}{l} (p \wedge q) \to r \\ \sim q \\ \hline \therefore \sim r \end{array}$$

(g) $$\begin{array}{l} p \to (\sim q) \\ r \to q \\ \hline \therefore \sim(p \wedge r) \end{array}$$

(h) $$\begin{array}{l} \sim[p \vee (\sim r)] \\ q \to p \\ \hline \therefore \sim(q \vee r) \end{array}$$

(i) $$\begin{array}{l} p \leftrightarrow q \\ (p \to q) \to w \\ (q \to p) \to s \\ \hline \therefore w \wedge s \end{array}$$

(j) $$\begin{array}{l} p \vee q \\ p \to r \\ q \to r \\ \hline \therefore r \end{array}$$

(k) $$\begin{array}{l} p \to q \\ \sim(p \to r) \\ \hline \therefore \sim(q \to r) \end{array}$$

(l) $$\begin{array}{l} q \vee (\sim p) \\ \sim[q \wedge (\sim r)] \\ \hline \therefore p \to r \end{array}$$

10.13 Prove each of 9.24 (a) and 9.24 (b) valid by means of direct proof.

11. INDIRECT PROOF

The indirect proof is a third way one can show an argument pattern valid. Just like the direct proof, the indirect proof cannot be used to prove an argument pattern invalid. The truth table method is still the only way we know to prove an argument pattern invalid.

One way to prove a statement true is to assume the statement false and by means of valid reasoning arrive at a contradiction. Since we do not allow contradictions in mathematics, if there is nothing wrong with the reasoning, then the assumption that the statement was false must be wrong. Since it is wrong that the statement is false, the statement must be true. This type of argument is the basis of all indirect proof.*

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*We probably have been using indirect proof without realizing it. We usually use it to answer this question: If p is true and $p \rightarrow q$ is true, then why can't q be false?

11.1 TO PROVE AN ARGUMENT PATTERN VALID BY INDIRECT PROOF

(a) Assume there is an instance of the argument pattern for which the hypotheses are true and the negation of the conclusion is false.

(b) By means of tautologies, known valid arguments, and truth table 3.1, arrive at a contradiction.

11.1 (a) is really saying "assume the argument pattern invalid." (We well convince the reader of this momentarily.) So when one arrives at a contradiction, it means the assumption that the argument was invalid is wrong. Therefore the argument must be valid.

Let us look at an arbitrary argument pattern** and show that when one assumes 11.1 (a), he is assuming the argument pattern to be invalid.

11.2 (a) H_1 (b) A_1

$\underline{H_2}$ $\underline{A_2}$

$\therefore C$ $\therefore B$

Let us assume 11.2 (b) is an instance of 11.2 (a) for which the hypotheses are true and the negation of the conclusion is true. "$(A_1 \wedge A_2) \wedge (\sim B)$" must therefore be a true statement. By T_5, "$(A_1 \wedge A_2) \wedge (\sim B)$" is equivalent to "$\sim[(A_1 \wedge A_2) \rightarrow B]$" is therefore true. This means "$(A_1 \wedge A_2) \rightarrow B$" is false. "$(A_1 \wedge A_2) \rightarrow B$" is

**For convenience sake, we will consider an argument pattern with only two hypotheses.

a false instance of "$(H_1 \wedge H_2) \to C$". Therefore, "$(H_1 \wedge H_2) \to C$" is not a tautology since every instance of a tautology must be true. Thus, 11.2 (a) is invalid since "$(H_1 \wedge H_2) \to C$" is not a tautology. By similar reasoning one can show that if 11.2 (a) is assumed invalid then there is an instance of 11.2 (a) for which the hypotheses are true and the negation of the conclusion is true. Therefore, when one assumes 11.1 (a), one is really assuming that the given argument pattern is invalid.

11.3 EXAMPLE: Prove 10.12 (j) valid by indirect proof. (This argument pattern is called "proof by cases". It plays a very important role in proof and should be remembered.)

$$\text{10.12 (j)} \qquad \begin{array}{l} p \vee q \\ p \to r \\ \underline{q \to r} \\ \therefore r \end{array}$$

0. Assume each of "p'", "q'", and "r'" is a replacement for "p", "q", and "r" respectively.

1.	$p' \vee q'$	Assumed for indirect proof
2.	$p' \to r'$	Assumed for indirect proof
3.	$q' \to r'$	Assumed for indirect proof
4.	$\sim r'$	Assumed for indirect proof
5.	$\sim p'$	2, 4, modus tollens
6.	$\sim q'$	3, 4, modus tollens
7.	$(\sim p') \wedge (\sim q')$	5, 6, truth table definition of conjunction
8.	$\sim(p' \vee q')$	Contradiction: 7, T_4
9.	$p' \vee q'$	1

Lines 0, 1, 2, 3, and 4 fulfill the requirements of 11.1 (a). Lines 0, 1, 2, 3, and 4 really say that

$$\begin{array}{l} p' \vee q' \\ p' \to r' \\ \underline{q' \to r'} \\ \therefore r' \end{array}$$

is an instance of the argument pattern for which the hypotheses are true and the negation of the conclusion is true. In all indirect proofs it is good to write the two contradictory statements together as we have done in statements 8 and 9 to make the contradiction clear.

This example of an indirect proof shows clearly the power of an indirect proof. The reader was probably unable to get a direct proof for 10.12 (j), but the indirect proof is relatively easy.

Note that the procedure for writing up an indirect proof is the same as that for a direct proof. The left column is a list of true statements and the right column tells the reason each statement is true.

11.4 EXAMPLE: Prove 10.12 (g) valid by means of an indirect proof.

$$\text{10.12 (g)} \qquad \begin{array}{l} p \to (\sim q) \\ \underline{r \to q} \\ \therefore \sim(p \wedge r) \end{array}$$

0. Assume each of "p'", "q'", and "r'" are statements that are replacements for "p", "q", and "r" respectively.

1.	$p' \to (\sim q')$	Assumed for indirect proof
2.	$r' \to q'$	Assumed for indirect proof

3.	$\sim[\sim(p' \wedge r')]$	Assumed for indirect proof
4.	$p' \wedge r'$	3, T_{10}
5.	p'	4, truth table definition for conjunction
6.	r'	4, truth table definition for conjunction
7.	$\left.\begin{array}{l}\sim q' \\ q'\end{array}\right\}$ Contradiction	1, 5, modus ponens
8.		2, 6, modus ponens

The reader might have gotten the wrong impression from example 11.3 that in order to get a contradiction for an indirect proof one had to prove the negation of one of the hypotheses to be true. In example 11.3 we proved by step 8 that the negation of "$p' \vee q'$" was true and "$p' \vee q'$" was one of the hypotheses. A purpose of example 11.4 is to show that <u>any contradiction whatsoever</u> will satisfy the requirements set forth in 11.1 for proving an argument pattern valid by indirect proof. In example 11.4 the contradiction is that both "q'" and $\sim q'$" are true; neither "q'" not "$\sim q'$" is one of the hypotheses.

During the discussion of direct proof the following question was asked; "If one is unable to get a direct proof that an argument is valid, then does that prove that the argument pattern is invalid?" The answer was "no". Let us consider a similar question for indirect proof. If one tries to prove an argument pattern valid by means of an indirect proof, but cannot arrive at a contradiction, then does that prove that the argument pattern is invalid? The answer again is "no". Just because one person cannot arrive at a contradiction while attempting an indirect proof does not mean that another person with different insights will not be able to arrive at a contradiction. But if a person with good insights into logic is unable to get either a direct or indirect proof for an argument pattern, then one should consider the possibility that the argument pattern is invalid.

Since the last three sections have been the main purpose of all we have been studying, then let us review some important things we have learned. We have learned three methods to prove an argument pattern valid. They are: (1) truth table, (2) direct proof, and (3) indirect proof. We have learned only one way to prove an argument pattern invalid and that is by means of a truth table.

11.5 EXAMPLE: Prove 9.24 (b) to be a valid argument by truth table, direct proof, and indirect proof.

Sam did not go to town or Joe's house is termite proof.
If the horse did not eat Joe's grass, then Sam went to town.
If Joe did not spray for termites, then Joe's house is not termite proof.

∴ The horse ate Joe's grass or Joe did spray for termites.

Let "Sam did go to town" be a replacement for the statement variable "p". Let "Joe's house is termite proof" be a replacement for the statement variable "q".

Let "The horse did eat Joe's grass" be a replacement for the statement variable "r". Let "Joe did spray for termites" be a replacement for the statement variable "s". Therefore, argument 9.24 (b) has as its argument pattern which satisfies 9.16 the following argument pattern:

$$\begin{array}{l} (\sim p) \vee q \\ (\sim r) \rightarrow p \\ \underline{(\sim s) \rightarrow (\sim q)} \\ \therefore r \vee s \end{array}$$

The preceding argument pattern will now be shown valid by each of the three designated methods. Argument 9.24 (b) is therefore valid since its argument pattern according to 9.16 is valid.

a) The argument pattern proven valid by truth table (shown on following page):

{[(~	p)	V	q]	∧	[(~	r)	→	p]	∧	[(~	s)	→	(~	q)]}	→	(r	V	s)
F	T	T	T		F	T	T	T		F	T	T	F	T	T	T	T	T
F	T	T	T		F	T	T	T		T	F	F	F	T	T	T	T	F
F	T	T	T		T	F	T	T		F	T	T	F	T	T	F	T	T
F	T	T	T		T	F	T	T		T	F	F	F	T	T	F	F	F
F	T	F	F		F	T	T	T		F	T	T	T	F	T	T	T	T
F	T	F	F		F	T	T	T		T	F	T	T	F	T	T	T	F
F	T	F	F		T	F	T	T		F	T	T	T	F	T	F	T	T
F	T	F	F		T	F	T	T		T	F	T	T	F	T	F	F	F
T	F	T	T		F	T	T	F		F	T	T	F	T	T	T	T	T
T	F	T	T		F	T	T	F		T	F	F	F	T	T	T	T	F
T	F	T	T		T	F	F	F		F	T	T	F	T	T	F	T	T
T	F	T	T		T	F	F	F		T	F	F	F	T	T	F	F	F
T	F	T	F		F	T	T	F		F	T	T	T	F	T	T	T	T
T	F	T	F		F	T	T	F		T	F	T	T	F	T	T	T	F
T	F	T	F		T	F	F	F		F	T	T	T	F	T	F	T	T
T	F	T	F		T	F	F	F		T	F	T	T	F	T	F	F	F

The above truth table shows that "{[(~p) V q)] ∧ [(~r) → p] ∧ [(~s) → (~q)]} → (r V s)" is a tautology.

Therefore our argument pattern under consideration is a valid argument pattern.

(b) The argument pattern proven valid by direct proof:

0. Assume each of "p'", "q'", "r'", and "s'" is a statement that is a replacement for "p", "q", "r", and "s" respectively.

1.	$(\sim p') \vee q'$	Assumed for direct proof
2.	$(\sim r') \rightarrow p'$	Assumed for direct proof
3.	$(\sim s') \rightarrow (\sim q')$	Assumed for direct proof
4.	$p' \rightarrow q'$	1, T_6
5.	$q' \rightarrow s'$	3, Contrapositive
6.	$(\sim r) \rightarrow q'$	2, 4, Syllogism
7.	$(\sim r') \rightarrow s'$	6, 5, Syllogism
8.	$[\sim(\sim r')] \vee s'$	7, T_6
9.	$r' \vee s'$	8, T_{10}

(c) The argument pattern shown valid by indirect proof:

0. Assume each of "p'", "q'", "r'", and "s'" is a statement that is a replacement for "p", "q", "r", and "s" respectively.

1.	$(\sim p') \vee q'$	Assumed for indirect proof	
2.	$(\sim r') \rightarrow p'$	Assumed for indirect proof	
3.	$(\sim s') \rightarrow (\sim q')$	Assumed for indirect proof	
3a.	$\sim(r' \vee s')$	Assumed for indirect proof	
4.	$p' \rightarrow q'$	1, T_6	
5.	$q' \rightarrow s'$	3, Contrapositive	
6.	$(\sim r') \rightarrow q'$	2, 4, Syllogism	
7.	$(\sim r') \rightarrow s'$	6, 5, Syllogism	
8.	$[\sim(\sim r')] \vee s'$	7, T_6	
9.	$r' \vee s'$	Contradiction	8, T_{10}
10.	$\sim(r' \vee s')$		3 (a)

11.5 (b) and 11.5 (c) illustrate that if one can get a direct proof, then one can always get an indirect proof. If one is able to get a direct proof, that means that he is able to prove that the conclusion is true. Since one of the initial assumptions of an indirect proof is that the negation of the conclusion is true, one would always be able to arrive at the contradiction that the conclusion and its negation are both true. Note that steps 1, 2, 3, 4, 5, 6, 7, 8, and 9 are identical in 11.5 (b) and 11.5 (c). The indirect proof 11.5 (c) contains the direct proof 11.5 (b) in order to prove the conclusion "$r' \vee s'$" true in step 9; this automatically contradicts that the negation of the conclusion "$\sim(r' \vee s')$" is true (step 3a).

11.6 <u>EXAMPLE</u>: The following argument is a valid argument.

Sam went to town.
<u>Sam did not go to town.</u>
∴ Jack has a car.

Let us examine the reason why this strange-looking argument is valid. This argument has the following argument pattern:

$$\begin{array}{l} p \\ \underline{\sim p} \\ \therefore q \end{array}$$

This argument is valid since "$[p \wedge (\sim p)] \rightarrow q$" is a tautology. "$[p \wedge (\sim p)] \rightarrow q$" is a tautology since every instance of this implication will have a false hypothesis, thereby making every instance a true implication.

This example illustrates that every argument with contradictory hypotheses is a valid argument.#

<u>EXERCISES</u>:

11.7 Attempt an indirect proof for every valid argument pattern in 10.12 for which the reader was unable to get a direct proof.

11.8 Every one of the following argument patterns that is valid is to be proven valid by means of an indirect proof. Every one of the following argument patterns that is invalid is to be proven invalid by means of a truth table.

(a)
$$\begin{array}{l} p \to q \\ \sim(p \to r) \\ \hline \therefore \sim(q \to r) \end{array}$$

(b)
$$\begin{array}{l} p \to [(\sim q) \wedge (\sim r)] \\ q \wedge r \\ \hline \therefore \sim p \end{array}$$

(c)
$$\begin{array}{l} (\sim p) \to [(\sim q) \vee (\sim r)] \\ (\sim q) \to (\sim r) \\ \hline \therefore (\sim p) \to (\sim r) \end{array}$$

(d)
$$\begin{array}{l} (\sim p) \vee q \\ \sim[(\sim q) \to r] \\ (\sim r) \to w \\ (\sim s) \to [p \vee (\sim w)] \\ \hline \therefore s \end{array}$$

(e)
$$\begin{array}{l} p \to q \\ p \to (\sim q) \\ p \\ \hline \therefore (\sim p) \to (q \vee p) \end{array}$$

12. OPEN SENTENCES

CONSTANTS, REPLACEMENTS, AND VARIABLES. During our discussion of the propositional calculus, we used variables very extensively. On one occasion we briefly mentioned real number variables, but for the most part we have talked about statement variables. We are now going to broaden our discussion to talk about variables in general. A variable is a symbol that is a place-holder for any element of a given class. We stated previously that a statement variable is a symbol that is a place-holder for any statement. We see that what we have been calling a statement variable fits our description for being a variable, because the given class whose elements can replace the variable is the class of all statements. A real number variable is a symbol that is a place-holder for any real number. We see that a real number variable fits our description for being a variable, because the given class whose elements can replace the variable is the class of all real numbers. By a replacement for a statement variable, we mean a statement. By a replacement for a real number variable, we mean a real number. In general a replacement for a variable is any element from the given class for which the variable is a place-holder. On some occasions in mathematics we will use two or more words to mean the same thing. For instance, "set" and "collection" are two words that mean the same thing. A reason for using two words to mean the same thing is to make sentences more readable that use the same concept over and over in one sentence. It seems more readable to talk about a <u>collection of sets</u> than a <u>set of sets</u>. The preceding reason is one of the reasons why we will use the word "constant" to mean the same thing as the word "replacement" throughout the remainder of this book.

12.1 <u>DEFINITION</u>: The word "constant" means the same thing as the word "replacement."

It seems clearer to say "constants replace variables" than to say "replacements replace variables." When we talk about letting a constant replace a statement variable, the context demands that the <u>constant</u> be a <u>statement</u> since statements are replacements for statement variables. When we talk about replacing a constant for a real variable (real number variable), the context demands that the <u>constant</u> be a <u>real number</u> since real numbers are replacements for real variables. Due to the reader's probable background in algebra and elementary calculus, he/she has probably encountered the word "constant" only in the sense of a constant being a real number. This because the only variables used in algebra and elementary calculus are real variables.

<u>OPEN SENTENCES</u>. The following expressions are examples of open sentences.

12.2 x is less than 2.

12.3 y had a 90 average.

12.4 p if and only if [~(~p)].

12.5 X is a set and y is an element of X.

12.6 $x + y < 10$

We can think of an <u>open sentence as an expression involving variables that is not a statement, but becomes a statement when constants replace all the variables. A statement that is formed by replacing constants for all the variables in an open sentence is called an instance of the open sentence</u>. We will take a closer look at open sentences 12.2 through 12.6 and some of their instances in the following examples:

12.7 <u>EXAMPLE</u>: Open sentence 12.2 is "x is less than 2." The only variable in this expression is "x". From the context we see that "x" is a real variable, thus "x" is a place-holder for any real number. If we replace "x" with the constant "1" we get "1 is less than 2" as a <u>true</u> instance of 12.2. If we replace "x" with the constant "3½" we get "3½ is less than 2" as a <u>false</u> instance of 12.2.#

12.8 <u>EXAMPLE</u>: Open sentence 12.3 is "y had a 90 average." The only variable in this expression is "y". From the context we see that "y" is a place-holder for any person. If we replace "y" with the constant "Tom Smith" we get "Tom Smith had a 90 average" as an instance of 12.3.#

12.9 <u>EXAMPLE</u>: Open sentence 12.4 is "p if and only if [~(~p)]." The only variable in this expression is "p". From the context we see that "p" is a statement variable; thus, "p" is a place-holder for any statement. If we replace "p" with the constant "Sam had a 90 average," we get "Sam had a 90 average if and only if it is false that Sam did not have a 90 average" as a true instance of 12.4. It should be clear to the reader that every instance of 12.4 will be true since 12.4 is a statement pattern that is a tautology.#

12.10 EXAMPLE: Open sentence 12.5 is "X is a set and y is an element of X." Open sentence 12.5 involves the two variables "X" and "y". If we replace "X" with the constant "{1,2}" and "y" with the constant "2" we get the true instance "{1,2} is a set and 2 is an element of {1,2}." If we replace "X" with the constant "{1,2,3}" and "y" with the constant "5" we get the false instance "{1,2,3} is a set and 5 is an element of {1,2,3}."#

12.11 EXAMPLE: Open sentence 12.6 is "x + y < 10." 12.6 involves the two real number variables "x" and "y". If we replace "x" with the constant "-3" and "y" with the constant "16" we get the false instance "-3 + 16 < 10."#

We have seen that one way to make a statement from an open sentence is to replace constants for all the variables in the open sentence. In the next section we will study yet another way to make a statement from an open sentence.

12.12 EXAMPLE: Let I = {1,2,3,4} and let "P(x)" denote the open sentence "x + 5 < 10." Does every replacement for "x" from "I" make a true instance of "P(x)"? If "1" replaces "x" then we get the true instance "1 + 5 < 10." If "2" replaces "x", then we get the true instance "2 + 5 < 10." If "3" replaces "x" then we get the true instance "3 + 5 < 10." If "4" replaces "x" then we get the <u>true</u> instance "4 + 5 < 10." Thus every replacement for "x" from "I" makes a true instance of "P(x)".#

<u>EXERCISES</u>:

12.13 Assume P(x) is the open sentence "x is an even positive integer and x is less than 10." Name 3 true instances of P(x) and name 3 false instances of P(x).

12.14 Assume P(x) is the open sentence "x is a positive integer implies x is a real number."

(a) Does P(x) have a true instance? If yes, then name one.

(b) Does P(x) have a false instance? If yes, then name one.

12.15 Let $A = \{3,5,7\}$ $\qquad B = \{5,6,7,8\}$

$C = \{1,2,3,4,5,6,7,8,9\}$

(a) List all the replacements for x that make true instances of "$x \in A$ or $x \in B$".

(b) List all the replacements for x that make true instances of "$x \in A$ and $x \in B$".

(c) List all the replacements for x that make true instances of "$x \in C$ and $x \notin B$".

12.16 Tell the truth value of each of the following statements. (Assume A, B, and C are the sets in 12.15.)

(a) Every replacement for x from A makes a true instance of "$x \in A$ implies $x \in B$".

(b) There is a replacement for x from A that makes a true instance of "$x \in A$ and $x \notin B$".

(c) Every replacement for x from C makes a true instance of "$x \in B$ implies $x \in C$".

(d) There is a replacement for x from C that makes a true instance of "$x \in B$ and $x \notin C$".

(e) Every replacement for x from C makes a true instance of "$x \notin A$ or $x \notin B$".

(f) There is a replacement for x from C that makes a true instance of "$x \in A$ and $x \in B$".

(g) Every replacement for x from C makes a true instance of "$x \in B$ and $x \notin C$".

(h) There is a replacement for x from C that makes a true instance of "$x \in B$ implies $x \in C$".

13. QUANTIFIERS

In Section 12 we learned that one way to make a statement out of an open sentence is to replace all the variables with constants. In this section we will learn another way. Exercise 12.16 is designed to introduce this section which we are now studying. Problems 12.16 (a), (c), (e), and (g) are designed to introduce the universal quantifier "for every". "$\forall$" is a symbol for "for every". The expressions "for each" and "for all" are alternate ways of writing the universal quantifier "for every".

13.1 DEFINITION: Assume $P(x)$ is an open sentence and I is a class of replacements for "x". "$\forall$ x $\in$ I, P(x)" (read "for all x belonging to I, P(x)") means "every replacement for x from I makes a true instance of P(x)."

13.2 EXAMPLE: This example refers to exercises 12.6 (a), (c), and (e). Recall that in exercise 12.15, A = {3,5,7}, B = {5,6,7,8}, and C = {1,2,3,4,5,6,7,8,9}.

12.16 (a) reads: Every replacement for x from A makes a true instance of "x $\in$ A implies x $\in$ B". Using the symbolism introduced in definition 13.1, 12.16 (a) could be written "$\forall$ x $\in$ A, x $\in$ A implies x $\in$ B". If we let R(x) denote the open sentence "x $\in$ A implies x $\in$ B", then 12.16 (a) could read "$\forall$ x $\in$ A, R(x)". It should be noted that A is our class of replacements for "x".

12.16 (c) reads: Every replacement for x from C makes a true instance of "$x \in B$ implies $x \in C$". Using the symbolism introduced in definition 13.1, 12.16 (c) could be written "$\forall\ x \in C$, $x \in B$ implies $x \in C$". If we let S(x) denote the open sentence "$x \in B$ implies $x \in C$", then 12.16 (c) could read "$\forall\ x \in C$, S(x)". It should be noted that C is our class of replacements for "x".

12.16 (e) reads: Every replacement for x from C makes a true instance of " $x \notin A$ or $x \notin B$". Using the symbolism introduced in definition 13.1, 12.16 (e) could be written "$\forall\ x \in C$, $x \notin A$ or $x \notin B$". If we let T(x) denote the open sentence "$x \notin A$ or $x \notin B$", then 12.16 (e) could read "$\forall\ x \in C$, T(x)". It should be noted that C is our class of replacements for "x".#

In our previous statements we said we were going to learn a new way to change an open sentence into a statement. We should note that in the preceding example, all that was needed to change the given open sentence into a statement was to prefix the open sentence with the universal quantifier. For example, in example 13.2, "$x \in A$ implies $x \in B$" is an open sentence; it is not a statement. When we prefixed it with the universal quantifier, we came up with "$\forall\ x \in A$, $x \in A$ implies $x \in B$" which is a statement (a false statement). In example 13.2 we were considering the open sentence "$x \in B$ implies $x \in C$." This is not a statement. But when we prefixed this open sentence with "$\forall$ x $\in$ C," we formed the statement "$\forall$ x $\in$ C, x $\in$ B implies x $\in$ C" (a true statement). In example 13.2 we were considering the open sentence "x $\notin$ A or x $\notin$ B." This is not a statement. But when we prefix this open sentence with "$\forall$ x $\in$ C," we form the statement "$\forall$ x $\in$ C, x $\notin$ A or x $\notin$ B" (a false statement).

We are now going to study another quantifier: the existential quantifier. The existential quantifier is "there is". Another way of saying the existential quantifier is "there exists" or "for some". A symbol used to denote the existential quantifier is "$\exists$". Therefore, "$\exists$", "for some", "there is", and "there exists" may all be used interchangeably.

13.3 DEFINITION: Assume P(x) is an open sentence and I is a class of replacements for x. "$\exists$ x $\in$ I, P(x)" (read "there is an x belonging to I, P(x)") means "there is a replacement for x from I that makes a true instance of P(x)." An alternate way of writing "$\exists$ x $\in$ I, P(x)" is "$\exists$ x $\in$ I $\ni$ P(x)." The symbol "$\ni$" is read "such that". Therefore "$\exists$ x $\in$ I $\ni$ P(x)" is read "there is an x belonging to I such that P(x)."

13.4 EXAMPLE: 12.16 (b), (d), (f), and (h) are all used to introduce the existential quantifier; therefore, we will look at some of them in this example. The sets A, B, and C, are as defined in 12.15.

12.16 (b) reads: There is a replacement for x from A that makes a true instance of "$x \in A$ and $x \notin B$". Using the symbolism introduced in definition 13.3, 12.16 (b) could be written "$\exists\ x \in A \ni x \in A$ and $x \notin B$". If we let P(x) denote the open sentence "$x \in A$ and $x \notin B$", then 12.16 (b) could read "$\exists\ x \in A \ni P(x)$". Note that A is the class of replacements for "x".

12.16 (d) reads: There is a replacement for x from C that makes a true instance of "$x \in B$ and $x \notin C$". Using the symbolism introduced in definition 13.3, 12.16 (d) could be written "$\exists\ x \in C \ni x \in B$ and $x \notin C$". If we let Q(x) denote the open sentence "$x \in B$ and $x \notin C$", then 12.16 (d) could be written "$\exists\ x \in C \ni Q(x)$". Note that C is the class of replacements for the variable "x".

12.16 (f) reads: There is a replacement for x from C that makes a true instance of "$x \in A$ and $x \in B$". Using the symbolism introduced in 13.3, 12.16 (f) could be written "$\exists\ x \in C \ni x \in A$ and $x \in B$". If we let M(x) denote the open sentence "$x \in A$ and $x \in B$", then 12.16 (f) could be written "$\exists\ x \in C \ni M(x)$". Note that C is the class of replacements for the variable "x".#

Let us now notice how the existential quantifier is used to make statements out of open sentences in example 13.4. 12.16 (b) considers the open sentence "$x \in A$ and $x \notin B$"; it is not a statement. But when "$\exists\ x \in A$" is prefixed to the open sentence "$x \in A$ and $x \notin B$", we formed the statement "$\exists\ x \in A \ni x \in A$ and $x \notin B$" (a true statement). 12.16 (d) considers the open sentence "$x \in B$ and $x \notin C$"; it is not a statement. But when "$\exists$

$x \in C$" is prefixed to the open sentence "$x \in B$ and $x \notin C$", we formed the statement "$\exists\, x \in C \ni x \in B$ and $x \notin C$" (a false statement).

<u>The universe of replacements</u>: In definitions 13.1 and 13.3, the class I of replacements for x is called the <u>universe of replacements</u> for x. In example 13.2, we noted that the universe of replacements for 12.16 (a) is the set A. In example 13.2 we noted that the universe of replacements for 12.16 (c) is the set C, and we noted that the universe of replacements for 12.16 (e) again is the set C. In example 13.4 the universe of replacements first for 12.16 (b) is the set A. For 12.16 (d) and (f) the universe of replacements is the set C. On many occasions we will not specifically state the universe of replacements. It will be understood. For example, let's consider the statement "$\forall$ x, if x is a positive integer then x is a real number." Here it is understood that the universe of replacements for x contains at least all of the positive integers and at least all of the real numbers. In the statement "$\forall\, x, x \in B$ implies $x \in C$" the understood universe of replacements for x contains at least all the elements in B and it contains at least all of the elements in C. For the statement "$\forall$ x, x is a positive integer, if and only if x is a natural number" the understood universe of replacements for x contains at least every natural number. Two things should be noted in our discussion about understood universes. <u>First</u> of all, the only examples given where the universe of replacements for x is not specifically

stated following "$\forall$" are statements involving <u>implications</u> and <u>biconditionals</u>. Usually this will be the only time the universe of replacements for x will be understood. When the statement under consideration involves a conjunction, disjunction, or negation, the universe of replacements for x will almost always be specifically stated, if the quantifier under consideration is "$\forall$". <u>Secondly</u>, it should be noted that when we were naming the understood universe of replacements for x, we used the words "at least". When we talked about the understood universe of replacements for x for the statement "$\forall$ x, x $\in$ B implies x $\in$ C", we said that it contained at least every element in B and at least every element in C. We do not have to worry about replacements for x with elements other than those from B and C because they will make the implication "x $\in$ B implies x $\in$ C" true, because the implication would have a false hypothesis. This may sound confusing at first but it will be very helpful later on. So a general rule would be that any time there is an understood universe of replacements for x, the understood universe should contain at least every element of every set in the open sentence that is associated with the variable x. The context usually makes the understood universe clear.

There is no magic about the variable x. Any symbol may be used as a variable. "$\forall$ x, x $\in$ B implies x $\in$ C" and "$\forall$ y, y $\in$ B implies y $\in$ C" both mean the same thing. So in review "$\forall$ x, P(x)" means "every replacement for x from the understood universe makes a true instance of P(x)." Also "$\exists$ x $\ni$ P(x)" means "there is a replacement for x from the understood universe that makes a true instance of P(x)." Usually, when the quantifier "$\forall$" is used and the universe is understood, the open sentence following the quantifier "$\forall$" will be either an implication or biconditional. (This will not necessarily be the case when "$\exists$" is used and the universe is understood).

This still leaves the question unsettled as to what are all of the elements in an understood universe. We know "at least" which elements are in an understood universe. A good rule to go by is the following: <u>Unless the context demands differently, the universe of replacements for "$\forall$ x, P(x)" and "$\exists$ x, P(x)" are precisely those elements that are meaningful replacements for x</u>. The following example will illustrate this rule.

13.5 <u>EXAMPLE</u>: Let A = {3,5,7}.
Let C = {1,2,3,4,5,6,7,8,9}

(a) The universe of replacements for "$\exists$ x, x < 1" would be the set of real numbers, since any replacement for x other than a real number would be meaningless ({1} < 1 is meaningless. (2,7) < 1 is meaningless.)

(b) The universe of replacements for "$\forall$ x, x $\in$ A implies x < 10" would be the set of real numbers. Even though 20 $\notin$ A, it is meaningful to say "20 $\in$ A implies 20 < 10." So notice that the understood universe contains "at least" the elements of A.

(c) The universe of replacements for "$\forall$ x, x $\in$ A implies x $\in$ C" is the class of all elements that can be an element of some set. For instance "(2,7)" can be an element of a set, so "(2,7)" is an element of the understood universe. Notice that "(2,7) $\in$ A implies (2,7) $\in$ C" is a true instance of "x $\in$ A implies x $\in$ C" since "(2,7) $\in$ A" is a false hypothesis. Notice also that the understood universe contains at least all of the elements of A and at least all of the elements of C.

(d) The universe of replacements for "$\exists$ x, x $\in$ A and x $\notin$ C" is the same as the universe described in 13.5 (c).#

<u>EXERCISES</u>: Let N denote the set of positive integers and let R denote the set of real numbers. Tell whether each of the following statements is true or false. Write out what each statement means according to definitions 13.1 and 13.3. If the universe of replacements is an understood universe, then name an acceptable universe.

13.6(a) $\exists$ x $\in$ R $\ni$ x + 7 $\neq$ 10 (Answer: According to definition 13.3, $\exists$ x $\in$ R $\ni$ x + 7 $\neq$ 10 means: There is a replacement for x from R that makes a true instance of "x + 7 $\neq$ 10." This is true. 4 is one such replacement.)

(b) $\forall$ x, x $\in$ N implies x > 0. (Answer: This means: Every replacement for x from the understood universe makes a true instance of "x $\in$ N implies x > 0." This is true. The universe contains at least every element in N. The universe is the set of real numbers since real numbers are the only meaningful replacement for x in "x > 0".)

13.7 (a) $\exists\ x \in R,\ x + 7 = 10$

(b) $\forall\ x \in R,\ x + 7 \neq 10$

13.8 (a) $\exists\ x \in R \ni x < 5$

(b) $\forall\ x \in R,\ x \not< 5$

13.9 (a) $\forall\ x \in R,\ x \in N$

(b) $\exists\ x \in R,\ x \notin N$

13.10 (a) $\exists\ n \in N \ni n + 1 \notin N$

(b) $\forall\ n \in N,\ n + 1 \in N$

13.11 (a) $\forall\ x \in R,\ x + x = x^2$

(b) $\exists\ x \in R \ni x + x \neq x^2$

13.12 (a) $\exists\ x \in R \ni x + x = x^2$

(b) $\forall\ x \in R,\ x + x \neq x^2$

13.13 (a) $\forall\ x \in R$, $x \in R$ implies $x \in N$

(b) $\exists\ x \in R \ni x \in R$ and $x \notin N$

13.14 (a) $\forall\ x \in R$, $x \in N$ implies $x \in R$

(b) $\exists\ x \in R \ni x \in N$ and $x \notin R$

13.15 (a) $\forall\ n \in N,\ n - 1 \in N$

(b) $\exists\ n \in N,\ n - 1 \notin N$

13.16 (a) $\forall\ x$, $x \in N$ implies $x \in R$

(b) $\exists\ x$, $x \in N$ and $x \notin R$

13.17 (a) $\forall\ x$, $x \in N$ implies $x + 1 \in N$

(b) $\exists\ x$, $x \in N$ and $x + 1 \notin N$

13.18 Let $\varnothing$ denote the empty set.

(a) $\forall\ x$, $x \in \varnothing$ implies $x \in N$

(b) $\exists\ x$, $x \in \varnothing$ and $x \notin N$

13.19 (a) $\forall\ x$, $x \in \{1,2\}$ implies $x \in N$

(b) $\exists\ x \ni x \in \{1,2\}$ and $x \notin N$

13.20 Let E denote the set of even positive integers.

(a) $\forall\ n$, $n \in E$ if and only if ($n \in N$ and 2 is a factor of n)

(b) $\exists\ n \ni$ ($n \in E$ and [$n \notin N$ or 2 is not a factor of n]) or ([$n \in N$ and 2 is a factor of n] and $n \notin E$)

14. OPEN SENTENCES INVOLVING TWO OR MORE VARIABLES

We continue to let R denote the set of real numbers and N denote the set of positive integers. We have previously noted that "$x + y = 10$" is an open sentence. This open sentence involves two variables, x and y. One way to make a statement out of this open sentence is to replace constants for both variables. For instance, is x is replaced with 7 and y is replaced with 3, we get the true instance "$7 + 3 = 10$". When we were discussing open sentences involving only one variable, we noted that there were two ways we could make a statement out of that open sentence. The first method was to replace the variable with a constant, and the second method was to precede the open sentence with the quantifier "for every" or the quantifier "there exists". Let us note what happens when we prefix the open sentence "$x + y = 10$" with "$\exists\ y \in R$". The expression formed is "$\exists\ y \in R \ni x + y = 10$"; upon careful observation the reader should note that this is not a statement. "$\exists\ y \in R \ni x + y = 10$" is an open sentence; the variable y is "covered" but the variable x is still "free". Any time we replace the free variable x with a real number, we will make a statement. When x is replaced with 5, we get the instance "$\exists\ y \in R \ni 5 + y = 10$", which is true. When we replace x with the real number -1, we get the instance "$\exists\ y \in R \ni -1 + y = 10$", which is also true. The reader should note that no matter what real number we replace x with, we will make a true instance of "$\exists$ y

$\in R \ni x + y = 10$". It is therefore true that: Every replacement for x from the set R makes a true instance of "$\exists y \in R \ni x + y = 10$". According to definition 13.1, this means that "$\forall x \in R (\exists y \in R \ni x + y = 10)$" is true.

The question now comes up, does the order in which we prefixed the quantifiers make a difference? That is, does "$\forall x \in R (\exists y \in R \ni x + y = 10)$" mean the same thing as "$\exists y \in R \ni(\forall x \in R, x + y = 10)$"? "$\exists y \in R \ni(\forall x \in R, x + y = 10)$" means: There is a replacement for y from the set R that makes a true instance of "$\forall x \in R, x + y = 10$". Let us examine a few instances of "$\forall x \in R, x + y = 10$". If y is replaced with a constant 7, then we get the false instance "$\forall x \in R, x + 7 = 10$". If we replace y with the constant -10, then we will get another false instance, namely "$\forall x \in R, x + (-10) = 10$". The reader should note that no matter what replacement we make for y from the set of real numbers, we will always get a false instance. Therefore, there cannot be a replacement for y from the set of real numbers that makes a true instance of "$\forall x \in R, x + y = 10$". This means that "$\exists y \in R \ni(\forall x \in R, x + y = 10)$" is false. It is vital to notice, then, that the order in which we prefix the quantifiers makes the difference.

14.1 EXAMPLE: Notation for quantified open sentences involving two or more variables. Let us assume "$P(x,y)$" is an open sentence involving the free variables x and y. "$\forall x \in I, \exists y \in J \ni P(x,y)$" means the same thing as "$\forall x \in I(\exists y \in J \ni P(x,y))$".

Therefore, "$\forall$ x $\in$ I($\exists$ y $\in$ J $\ni$ x + y = 10)" could be written "$\forall$ x $\in$ I, $\exists$ y $\in$ J $\ni$ x + y = 10". Similarly, "$\exists$ x $\in$ I $\ni$ $\forall$ y $\in$ J, P(x,y)" means the same thing as "$\exists$ x $\in$ I $\ni$ ($\forall$ y $\in$ J $\ni$ P(x,y))". Thus, "$\exists$ y $\in$ R $\ni$ ($\forall$ x $\in$ R, x + y = 10)" could be written "$\exists$ y $\in$ R, $\forall$ x $\in$ R, x + y = 10". Also, "$\forall$ x $\in$ I, $\forall$ y $\in$ J, P(x,y)" means the same thing as "$\forall$ x $\in$ I, ($\forall$ y $\in$ J, P(x,y))". "$\exists$ x $\in$ I, $\exists$ y $\in$ J $\ni$ P(x,y)" means the same thing as "$\exists$ x $\in$ I, ($\exists$ y $\in$ J $\ni$ P(x,y))". Let us now assume "Q(x,y,z)" denotes an open sentence involving the three free variables x, y, and z. "$\forall$ x $\in$ I, $\exists$ y $\in$ J, $\forall$ z $\in$ K, Q(x,y,z)" means the same thing as "$\forall$ x $\in$ I, [$\exists$ y $\in$ J, ($\forall$ z $\in$ K, Q(x,y,z))]". It is felt that from these examples, the reader should be able to understand what is meant no matter how many quantifiers prefix an open sentence.#

INTUITION AND RIGOR. If we were to ask, "What is the limit of 2x - 1 as x approaches 5?", most people would correctly answer "9" even though they did not rigorously understand the limit concept. Once one's intuition says 9 is the answer, he can then see if 9 satisfies the rigorous definition of limit. In mathematics one needs a good intuition to lead you in the right direction and give you good ideas; then one must be able to rigorously defend or reject his intuitive ideas. For instance, most people who have never heard of the word quantifier would probably agree "For every x $\in$ R, x $\in$ N" is false (provided they knew what R and N were). It is important to be able to back up that intuition with rigor. "For every x $\in$ R, x $\in$ N" rigorously means "Every replacement for x from R makes a true instance of '$x \in N$'". This is obviously false since "$\frac{1}{2} \in N$" is an instance of "$x \in N$". Since we first think on the intuitive level and then examine our intuition with rigor, a good intuitive idea of what is meant by "$\forall\ x \in I,\ P(x)$" and "$\exists\ x \in I \ni P(x)$" is needed as well as knowing how it is rigorously defined.

<u>EXERCISES</u>: TRUE OR FALSE

14.2 (a) $\forall\ x \in R,\ \exists\ y \in R,\ x < y$

(b) $\exists\ x \in R \ni \forall\ y \in R\ x \geq y$

14.3 (a) $\exists\ x \in N \ni \forall\ y \in N,\ x \leq y$

(b) $\forall\ x \in N\ \exists\ y \in N \ni x > y$

14.4 (a) $\exists\ x \in R \ni \forall\ y \in N,\ x < y$

(b) $\forall\ x \in R,\ \exists\ y \in N \ni x \geq y$

14.5 (a) $\forall\ x \in R,\ \exists\ y \in R \ni xy = 1$

(b) $\exists\ x \in R,\ \forall\ y \in R\ xy = 0$

(c) $\forall\ y \in R,\ \exists\ x \in R \ni xy = 0$

(d) $\exists\ y \in R,\ \exists\ x \in R \ni xy = 0$

14.6 (a) $\forall\ x \in R,\ \exists\ y \in R \ni x + y = 0$

(b) $\exists\ x \in R,\ \forall\ y \in R \ni x + y = 0$

(c) $\exists\ x \in R,\ \exists\ y \in R \ni x + y = 0$

(d) $\forall\ x \in R,\ \forall\ y \in R\ x + y = 0$

14.7 (a) $\forall\ x \in R,\ \forall\ y \in R\ xy = yx$

(b) $\exists\ x \in R,\ \forall\ y \in R\ xy = yx$

14.8 (a) $\forall\ x \in R,\ \forall\ y \in R\ x - y = y - x$

(b) $\exists\ x \in R,\ \exists\ y \in R \ni x - y = y - x$

(c) $\exists\ x \in R,\ \forall\ y \in R\ x - y = y - x$

14.9 (a) $\forall\ x \in R,\ \exists\ n \in N \ni n \geq x$

(b) $\exists\ n \in N,\ \forall\ x \in R\ n \geq x$

14.10 (a) $\forall\ x \in R,\ \forall\ y \in R,\ \forall\ z \in R,\ x(y + z) = (xy) + (xz)$

(b) $\forall\ x \in R,\ \forall\ y \in R,\ \exists\ z \in R,\ x + (yz) = (x + y)\bullet(x + z)$

(c) $\forall\ x \in R,\ \exists\ y \in N,\ \forall\ z \in N\ (x + y) < z$

(d) $\forall\ x \in R,\ \forall\ y \in R,\ \forall\ z \in R,\ (x + y) - z = x + (y - z)$

15. TRANSLATIONS TO QUANTIFIERS

We have previously mentioned that one of our purposes is to find a standard for valid reasoning. If we want to apply everyday language to this standard, then we must translate everyday language into our logic language. The main reason for section 7 was to be able to translate everyday language into our logical language we had learned to that point. Everyday language uses quantifiers, but it is not used in the same form that we have been discussing. Therefore, this section will study how we translate everyday sentences involving quantifiers into properly quantified open sentences. It should be emphasized that this section is very important, so learn it well.

15.1 EXAMPLE: Translate: Every person secretly likes math. Let P denote the set of all people. A way this sentence could be translated is "$\forall$ x $\in$ P, x secretly likes math." Another way of translating this sentence is "$\forall$ x, if x is a person, then x secretly likes math."#

15.2 EXAMPLE: Translate: This course is easy for some students. Let P denote the set of all people. A way to translate this sentence is "$\exists$ x $\in$ P $\ni$ this course is easy for x."#

15.3 EXAMPLE: Translate: Every person should take some math course. Let P denote the set of all people. Let C denote the set of all math courses. A way to translate the sentence under consideration is "$\forall$ x $\in$ P, $\exists$ y $\in$ C $\ni$ x should take y." "$\forall$ x $\in$ P, x should take some math course" is not a preferred translation since the open sentence "x should take some math course" still contains a quantifier.

15.4 EXAMPLE: Translate: There is a math course that everyone should take. Let P denote the set of all people. Let C denote the set of all math courses. A way to translate this sentence is "$\exists$ y $\in$ C $\ni$ $\forall$ x $\in$ P, x should take y." (Observe the difference between the translation under consideration in this example and the translation in example 15.3.) "$\exists$ y $\in$ C $\ni$ everyone should take y" is not a preferred translation since the open sentence "everyone should take y" still contains a quantifier.#

EXERCISES: Translate to properly quantified open sentences.

15.5 Every element of A is an element of B.

15.6 Some dogs have fleas.

15.7 x + 2 = 2 + x is true for any real number.

15.8 All angles are right angles.

15.9 All married men have someone as a wife.

15.10 The square of any counting number is a counting number.

15.11 The sum of any two positive integers is a positive integer.

15.12 "$\emptyset$" is a subset of every set.

15.13 Someone likes math.

15.14 Everyone likes math.

15.15 No one likes math.

15.16 Someone does not like math.

15.17 Everyone likes someone.

15.18 Someone likes everyone.

15.19 Nobody likes everybody.

15.20 Someone likes someone.

15.21 0 times any number is 0.

15.22 The product of any two nonzero numbers is never zero.

15.23 1 times any number is that number.

15.24 The order in which two numbers are added makes no difference in their sum.

15.25 Every number has a number that can be added to it to get 0.

15.26 Every element of M is less than some element of K.

15.27 Some element of M is less than every element of K.

15.28 Every element of M is less than every element of K.

15.29 Some element of M is less than some element of K.

15.30 ~~There is a rational number between any two real numbers.~~ Between any two distinct real numbers, there is a rational number.

15.31 Every open set containing p contains a point of K that is not p.

15.32 Every open set containing p contains a point of K and a point not in K.

15.33 IMPORTANT: Negate 15.5, 15.8, 15.25, and 15.30

16. NEGATIONS

Most math students have a very poor understanding of how to negate statements. Many are at a loss as to how to negate relatively simple sentences and would have no hope of negating such a formidable-looking statement as "$\forall\ x \in R,\ \exists\ y \in N \ni \forall\ z \in N,\ x + y < z$."

We will now look at a simple example to give us some insight into negation. Let A = {3,5,7} and let B = {5,6,7,8}. It is not true that every element of A is an element of B. Thus, there must be an element of A that is not an element of B. We see that each of the two following statements is the negation of the other.

16.1 Every element of A is an element of B.

16.2 There is an element of A that is not an element of B. When we translate the two preceding statements into properly quantified open sentences, we get the two following statements.

16.3 $\forall\ x \in A,\ x \in B$

16.4 $\exists\ x \in A \ni x \notin B$

If we let P(x) denote "$x \in B$", then the preceding statements become:

16.5 $\forall\ x \in A,\ P(x)$

16.6 $\exists\ x \in A \ni \sim P(x)$

If we let Q(x) denote "$x \notin B$", then 16.3 and 16.4 become:

16.7 $\forall\ x \in A,\ \sim Q(x)$

16.8 $\exists\ x \in A \ni Q(x)$

Examples like 16.5 through 16.8 cause us to make the following rule:

16.9 A RULE FOR NEGATION

(a) "$\sim(\forall\ x \in I,\ P(x)\ )$" is equivalent to "$\exists\ x \in I \ni \sim P(x)$".

(b) "$\sim(\exists\ x \in I,\ Q(x)\ )$" is equivalent to "$\forall\ x \in I,\ \sim Q(x)$".

We see that the way to negate an expression prefixed by the quantifier "for every" is to change the quantifier to "there exists" and negate what follows. We see similarly that the way to negate an expression prefixed by the quantifier "there exists" is to change the quantifier to "for every" and to negate what follows. This simple technique will give us great power in being able to negate statements. We will now give a few examples to illustrate.

16.10 EXAMPLE: Negate "$\exists\ x \in R \ni x + 7 = 10$." We see that this is an expression prefixed by the quantifier "there exists". Therefore, we change the quantifier to "for every" and negate what follows. Thus, the negation is "$\forall\ x \in R,\ \sim(x + 7 = 10)$," which can be rewritten "$\forall\ x \in R,\ x + 7 \neq 10$."#

16.11 EXAMPLE: Negate "$\forall\ x \in R,\ x \nless 5$." We see that this is an expression prefixed by the quantifier "for every". Therefore, we change the quantifier to "there exists" and negate what follows. Thus, our negation is "$\exists\ x \in R \ni \sim(x \nless 5)$", which can be rewritten "$\exists\ x \in R \ni x < 5$."#

16.12 <u>EXAMPLE</u>: Negate "$\forall\ x \in R, \exists\ y \in R \ni x + y = 0$." This is an expression prefixed by the quantifier "for every". Therefore, we change the quantifier to "there exists" and negate what follows. Thus far we have: "$\exists\ x \in R \ni [\sim(\exists\ y \in R \ni x + y = 0)]$." The expression in the brackets consists of the negation of an expression prefixed by the quantifier "there exists". We therefore will change that quantifier to the quantifier "for every", and negate what follows. We get: "$\exists\ x \in R \ni \forall\ y \in R, \sim(x + y = 0)$", which can be rewritten "$\exists\ x \in R \ni \forall\ y \in R, x + y \neq 0$".#

16.13 <u>EXAMPLE</u>: We will see by this example that there is no difference in negating an expression where the universe is understood. Negate "$\forall\ x$, $x \in N$ implies $x + 1 \in N$." This is an expression prefixed by the quantifier "for every". We now change the quantifier to "there exists" and negate what follows. We have: "$\exists\ x \ni \sim(x \in N$ implies $x + 1 \in N)$" which can be rewritten "$\exists\ x \ni x \in N$ and $x + 1 \notin N$."#

16.14 <u>EXAMPLE</u>: We will now negate the formidable-looking sentence that we stated at the start of this section. Negate "$\forall\ x \in R, \exists\ y \in N \ni \forall\ z \in N, x + y < z$." We will now demonstrate a step-by-step demonstration of the process of changing the quantifier and negating what follows.

$$\sim(\forall\ x \in R, \exists\ y \in N \ni \forall\ z \in N, x + y < z)$$
$$\exists\ x \in R, \sim(\exists\ y \in N \ni \forall\ z \in N, x + y < z)$$
$$\exists\ x \in R, \forall\ y \in N, [\sim(\forall\ z \in N, x + y < z)]$$
$$\exists\ x \in R, \forall\ y \in N, \exists\ z \in N \ni [\sim(\ x + y < z)]$$
$$\exists\ x \in R, \forall\ y \in N, \exists\ z \in N \ni x + y \geq z \quad \#$$

Up until now every example that we have talked about has a nonempty universe of replacements. For instance, does the expression "$\forall$ x $\in \emptyset$, x $\in$ N" have any meaning? To some people such an expression may have a meaning yet to others it may have no meaning. We define that statement to be true. In general the expression "$\forall$ x $\in \emptyset$, P(x)" is defined to be true no matter what the open sentence P(x) is. Let us see why we define this to be true. According to our rules, its negation should be "$\exists$ x $\in \emptyset \ni$ ~P(x)." The assertion "$\exists$ x $\in \emptyset \ni$ ~P(x)" is obviously false since it implies that the empty set has an element, namely a replacement for x that makes ~P(x) true. Since "$\exists$ x $\in \emptyset \ni$ ~P(x)" is false, it is very reasonable to see why we defined its negation "$\forall$ x $\in \emptyset$, P(x)" to be true, no matter what open sentence P(x) denoted. So we must be careful to observe that "$\forall$ x $\in \emptyset$, ~P(x)" is not the negation of "$\forall$ x $\in \emptyset$, P(x)". "$\forall$ x $\in \emptyset$, ~P(x)" and "$\forall$ x $\in \emptyset$, P(x)" are both true. The negation of "$\forall$ x $\in \emptyset$, P(x)" is "$\exists$ x $\in \emptyset$ $\ni$ ~P(x)". For emphasis we will display clearly the important fact of this preceding paragraph.

16.15 "$\forall$ x $\in \emptyset$, P(x)" is defined to be true no matter what open sentence P(x) denotes.

EXERCISES: NEGATE EACH OF THE FOLLOWING STATEMENTS:

16.16 $\forall$ x $\in$ R, x $\in$ R implies x $\in$ N

16.17 $\forall$ x, x $\in$ N implies x $\in$ R

16.18 $\exists$ x, x $\in \emptyset$ and x $\in$ N

16.19 (E denotes even positive integers) $\forall$ n, n $\in$ E if and only if (n $\in$ N and 2 is a factor of n)

16.20 $\exists$ x $\in$ C $\ni$ x $\in$ A and x $\in$ B

16.21 $\forall$ x $\in$ C, x $\in$ A or x $\in$ B

16.22 $\forall$ x $\in$ R, $\exists$ y $\in$ R xy = 1

16.23 $\exists$ x $\in$ R $\ni$ $\forall$ y $\in$ R, xy = 0

16.24 $\forall$ x $\in$ R, $\forall$ y $\in$ R, $\forall$ z $\in$ R, x(y +z) = (xy) + (xz)

16.25 (Let R^+ denote the positive reals.) $\forall$ e $\in$ R^+, $\exists$ δ $\in$ R^+ $\ni$ $\forall$ x $\in$ R, if x is in the domain of f and $|x - c| < \delta$, then $|f(x) - f(c)| < e$.

16.26 $\forall$ M, if M is a subset of N and M $\neq$ $\varnothing$, then $\exists$ k $\in$ M $\ni$ $\forall$ x $\in$ M, k $\leq$ x.

16.27 $\forall$ e $\in$ R^+, $\exists$ n $\in$ N $\ni$ $\forall$ x $\in$ N, if x $\geq$ n, then 1/x < e.

16.28 $\forall$ e $\in$ R^+, $\exists$ δ $\in$ R^+ $\ni$ $\forall$ x $\in$ R, $\forall$ y $\in$ R, if x is in the domain of f and y is in the domains of f and $|x - y| < \delta$, then $|f(x) - f(y)| < e$.

PART II

OVERHEADS FOR THE TEACHER AND NOTES FOR THE STUDENT OVER PART I

J. GOAL: TO PRESENT IDEAS CLEARLY AND CONCISELY AND CLEANLY.

K. GOAL: THE STUDENT BE FREED FROM NOTE-TAKING TO BE ABLE TO FOCUS ON THE IDEAS AS THEY ARE PRESENTED.

L. GOAL: TO FIRST STUDY LOGIC AND THEN CONNECT THE REALM OF LOGIC WITH MATH PROOFS.

M. WE NOW START THE STUDY OF LOGIC:

THERE ARE NEGATIVE EFFORTS AT TEACHING PROOF:

1. "NO, THAT'S NOT RIGHT."
2. "YOU LEFT A LINE OUT."
3. "YOU PUT EXTRA LINES IN."

SOME STUDENTS HAVE FEAR WHEN THEY SEE THESE WORDS ON HOMEWORK OR A TEST:

"PROVE THIS"

FEAR IS TO BE ERADICATED

N. EXPECTED TO KNOW AN ABBREVIATED PROOF STYLE WITHOUT EVER LEARNING THE FULL THING TO BE ABBREVIATED.

O. THERE ARE RULES FOR PROOF JUST LIKE THERE ARE RULES FOR ADDING AND SUBTRACTING.

P. AN ARGUMENT IS EITHER A PROOF OR IT ISN'T.

1. SO LEARN THE BASICS 100%
2. BAD PILOT: 95% AVERAGE. ONLY FLUNKED TAKEOFFS AND LANDINGS.

Q. TWO HURDLES IN PROOF

1. SEEING THE LIGHT
2. LOGICALLY COMMUNICATING THE IDEA (GOAL: ELIMINATE THIS SECOND HURDLE)

R. LOGIC IS TO BE THE SERVANT, NOT THE MASTER.

S. FOR BEST RESULTS DO NOT ABBREVIATE PROOFS UNTIL THE CHAPTER ON ABBREVIATING PROOFS

UNDERSTANDING DEFINITIONS

A. DEFINITIONS: CLEARY, EXACTLY DESCRIBE THE CONCEPT, NO MORE NO LESS

B. ALL DEFINITIONS "IF AND ONLY IF"

12 IF AND ONLY IF DOZEN

C. A DEFINITION WRITES\SPEAKS THE CONCEPT INTO EXISTENCE IN THE COURSE

D. EXAMPLE: (DEFINITION) p IS A QUALOM IF AND ONLY IF p IS A POSITIVE INTEGER GREATER THAN 7

E. THEOREM: (FOLLOWS FROM A DEFINITION) EVERY QUALOM IS GREATER THAN 5

F. THERE IS ONLY ONE DEFINITION FOR A CONCEPT IN A COURSE ... THE EXACT WORDING WHEN ORIGINALLY DEFINED

G. THINGS EQUIVALENT TO A DEFINITION FALL UNDER THE THEOREM NAME

H. SO ON TESTS WHEN ASKED FOR A DEFINITION, GIVE THE EXACT WORDING, NOT SOMETHING EQUIVALENT (LIKE A POSITIVE INTEGER GREATER THAN $7\frac{1}{2}$).

[CHAPTER 1]
STATEMENT

A. STATEMENT = PROPOSITION

B. YOU ARE STUDYING THE PROPOSITIONAL CALCULUS.

C. <u>DEFINITION</u>: STATEMENT IFF
   1. SENTENCE
   2. DECLARATIVE SENTENCE
   3. MEANINGFUL TO ASSIGN ONLY ONE OF THE VALUES TRUE OR FALSE.

D. <u>EXAMPLES</u>:
   1. $1+1=3$ STATEMENT: FALSE
   2. $1+1$ NO STATEMENT
   3. THERE IS A PRIME NUMBER BETWEEN 10002134 AND 10002176. STATEMENT. THERE IS ONLY ONE VALUE TRUE OR FALSE THAT CAN BE ASSIGNED I DO NOT KNOW WHAT IT IS.

E. <u>HOMEWORK</u>: DO 1.5 THROUGH 1.16 AT THE END OF CHAPTER 1 IN PART I.

[CHAPTER 2]
CONNECTIVES

A. CONNECT STATEMENTS WITH "OR", "AND", "IMPLIES", "NOT" : YOU GET ANOTHER STATEMENT.

$1+1=2$ OR $3+4=5$

$1+1=2$ AND $3+4=5$

$1+1=2$ IMPLIES $3+4=5$

NOT $1+1=2$

B. STATEMENTS AND STATEMENT PATTERNS

STATEMENT PATTERN: p OR q

STATEMENT: $1+1=2$ OR $3+4=5$ INSTANCE OF

→ NOT A STATEMENT, BUT BECOMES A STATEMENT WHEN STATEMENTS REPLACE STATEMENT VARIABLES

C. INSTANCES OF STATEMENT PATTERNS ARE STATEMENTS. STATEMENT PATTERNS ARE <u>NOT</u> STATEMENTS.

DEFINITION:

D. STATEMENT VARIABLE: A SYMBOL THAT CAN BE REPLACED BY ANY STATEMENT.

E. EXAMPLES

1. STATEMENT VARIABLES: p, q
2. REPLACE p WITH $1+1=2$. REPLACE q WITH $3+4=5$ IN "p OR q" TO GET "$1+1=2$ OR $3+4=5$"

F. VARIABLES DO NOT VARY!! THE THINGS THAT REPLACE VARIABLES VARY

G. SYMBOLS FOR CONNECTIVES

OR $\vee$

AND $\wedge$

IMPLIES
IF..., THEN $\longrightarrow$

NOT $\sim$ ($\neg$ ALSO USED FOR NOT)

H. FORMING STATEMENT PATTERNS

1. STATEMENT VARIABLES ARE STATEMENT PATTERNS.
2. CONNECT STATEMENT PATTERNS WITH "OR", "AND", "IMPLIES", "NOT": YOU GET ANOTHER STATEMENT PATTERN

I. DISJUNCTION : $p \vee q$

J. CONJUNCTION: $p \wedge q$

K. NEGATION : $\sim p$

L. IMPLICATION: $p \rightarrow q$

p HYPOTHESIS

q CONCLUSION

M. CONDITIONAL = IMPLICATION

$p \rightarrow q$

p ANTECEDENT

q CONSEQUENT

N. ONLY IMPLICATIONS HAVE NAMES FOR COMPONENT PARTS p, q. DISJUNCTIONS, CONJUNCTIONS, AND NEGATIONS DO NOT HAVE NAMES FOR COMPONENT PARTS.

O. NAME STATEMENT TYPES

IF $1+1=2$, THEN $3+4=5$

$1+1=2$ OR $3+4=5$

P. <u>HOMEWORK</u>: DO 2.23 THROUGH 2.32 AT THE END OF CHAPTER 2 IN PART I

[CHAPTER 3] 3-137

TRUTH VALUES OF STATEMENTS CONTAINING CONNECTIVES

DEFINITIONS

A.

p	q	$\sim p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	F	T
F	F	T	F	F	T

$\vee$ IS THE "INCLUSIVE OR" NOT THE "EXCLUSIVE OR".

B. ALL POSSIBLE COMBINATIONS

p
T
F

p	q
T	T
T	F
F	T
F	F

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

C. <u>HOMEWORK</u>: DO 3.5 THROUGH 3.9 AT THE END OF CHAPTER 3 IN PART I. FOR 3.6, 3.7, 3.8 THE "what is the difference between..." PART OF THE QUESTION IS COMMUTATIVITY

[CHAPTER 4]
TRUTH TABLES

A. COMPONENT PARTS OF ... ARE

COMPONENT PARTS OF	ARE
$p \vee \sim q$	$p, \sim q$
$p \wedge q$	p, q
$p \rightarrow \sim r$	$p, \sim r$
$\sim p$	p
$1<2$ or $3>5$	$1<2,\ 3>5$
$\sim(p \wedge q)$	$p \wedge q$

B. DIFFERENT TRUTH VALUES FOR INSTANCES OF $p \vee q$

$1<2$	OR	$3>5$	T
T		F	
$-1>0$	OR	$5>6$	F
F		F	

TRUTH VALUES OF INSTANCES DEPEND ON THE TRUTH VALUES OF THE COMPONENT PARTS.

C. TRUTH VALUE OF INSTANCE OF $p \vee \sim q$

p: $1<2$ (i.e. REPLACE p WITH $1<2$)

q: $5>6$

INSTANCE: $1<2$ OR NOT$(5>6)$

$$\underbrace{\underset{T}{1<2} \quad \text{OR} \quad \underset{T}{\text{NOT}}\underset{F}{(5>6)}}_{T}$$

D. TRUTH VALUES FOR EVERY INSTANCE OF $p \vee \sim q$

p	$\vee$	$\sim$	q
1	2	3	4
T	T	F	T
T	T	T	F
F	F	F	T
F	T	T	F

COLUMNS UNDER VARIABLES AND UNDER CONNECTIVES

COL. 1: TRUTH VALUES FOR INSTANCES OF p

COL. 2: TRUTH VALUES FOR INSTANCES OF $p \vee \sim q$

COL. 3: TRUTH VALUES FOR INSTANCES OF $\sim q$

COL. 4: TRUTH VALUES FOR INSTANCES OF q

TO FILL IN COLUMN	LOOK IN COLUMN(S)
3	4
2	1, 3

E. THE NEED FOR GROUPING SYMBOLS

$p: 1 < 2$ $\quad$ $q: 4 < 5$

$(\sim p) \vee q$ $\quad$ $(\underset{F}{\text{NOT}}\ \underset{T}{1<2})\ \underset{T}{\text{OR}}\ \underset{T}{4<5}$ $\quad$ T

$\sim (p \vee q)$ $\quad$ $\underset{F}{\text{NOT}}\ (\underset{T}{1<2}\ \underset{T}{\text{OR}}\ \underset{T}{4<5})$ $\quad$ F

F. TRUTH TABLES FOR $(\sim p) \vee q$ AND $\sim(p \vee q)$

$(\sim p)$	p	$\vee$	q
F	T	T	T
F	T	F	F
T	F	T	T
T	F	T	F

$\sim$	p	$\vee$	q
F	T	T	T
F	T	T	F
F	F	T	T
T	F	F	F

$p: 1 < 2$ $\quad$ $q: 4 < 5$

LOOK IN ROW 1 FOR THIS INSTANCE

NOTE: THIS CONFIRMS PREVIOUS SECTION E.

G. USE OF DIFFERENT TYPES OF GROUPING SYMBOLS

$[(\sim p) \vee [q \rightarrow (\sim r)]] \wedge w$ CONJUNCTION

$[(\sim p) \vee q] \rightarrow [(\sim r) \wedge w]$ IMPLICATION

$(\sim p) \vee [q \rightarrow [(\sim r) \wedge w]]$ DISJUNCTION

$\sim\{p \vee (q \rightarrow [(\sim r) \wedge w])\}$ NEGATION

H. $[p \vee (\sim q)] \wedge (q \rightarrow p)$

1	2	3	4	5	6	7	8
p	$\vee$	$(\sim$	$q)]$	$\wedge$	$(q$	$\rightarrow$	$p)$

COLUMN NUMBER	TRUTH VALUES FOR INSTANCES OF
1	p
2	$p \vee (\sim q)$
3	$\sim q$
4	q
5	$[p \vee (\sim q)] \wedge (q \rightarrow p)$
6	q
7	$q \rightarrow p$
8	p

TO FILL IN COLUMN	LOOK IN COLUMN(S)
2	1, 3
3	4
5	2, 7
7	6, 8

I. COMPLETE TRUTH TABLE

$(p \vee \sim q) \wedge (q \rightarrow p)$

p	$\vee$	$\sim$	q	$\wedge$	q	$\rightarrow$	p
1	2	3	4	5	6	7	8
T	T	F	T	T	T	T	T
T	T	T	F	T	F	T	T
F	F	F	T	F	T	F	F
F	T	T	F	T	F	T	F

F
$p: -1 > 0$
$q: 3 < 7$
T

DO NOT MAKE A COLUMN FOR GROUPING SYMBOLS

J. <u>HOMEWORK</u>: DO 4.11 THROUGH 4.21 AT THE END OF CHAPTER 4 IN PART I OF THE BOOK.

[CHAPTER 5]

TAUTOLOGY

A. ALGEBRAIC EXPRESSION VS. STATEMENT PATTERN

1. $x + 1 = 3$
2. NEITHER T NOR F
3. x : REAL VARIABLE
4. REPLACE x WITH 1
   $1+1=3$ F INSTANCE
5. REPLACE x WITH 2
   $2+1=3$
   TRUE INSTANCE

1. $p \vee \sim q$
2. NEITHER T NOR F
3. p, q STATEMENT VARIABLES
4. REPLACE p WITH $1<0$, q WITH $2=1+1$.
   $1<0 \vee \sim(2=1+1)$
   FALSE INSTANCE
5. REPLACE p WITH $2>0$, q WITH $2=1+1$
   $2>0 \vee \sim(2=1+1)$
   TRUE INSTANCE

B. ALGEBRAIC IDENTITY VS. TAUTOLOGY

1. ALGEBRAIC EXPRESSION

$$x + x = 2x$$

2. REPLACE x WITH 1

$$1 + 1 = 2 \cdot 1 \quad T$$

3. REPLACE x WITH 3

$$3 + 3 = 2 \cdot 3 \quad T$$

4. REPLACE x WITH 5

$$5 + 5 = 2 \cdot 5 \quad T$$

5. NOTE: EVERY INSTANCE IS TRUE

6. DEF: ALGEBRAIC IDENTITY: AN ALGEBRAIC EXPRESSION WHOSE EVERY INSTANCE IS TRUE

1. STATEMENT PATTERN

$$p \vee \sim p$$

2. REPLACE p WITH $1 < 0$

$$1 < 0 \vee \sim(1 < 0) \quad T$$

3. REPLACE p WITH $5 > 1$

$$5 > 1 \vee \sim(5 > 1) \quad T$$

4. REPLACE p WITH $1 = 3$

$$1 = 3 \vee \sim(1 = 3) \quad T$$

5. NOTE: EVERY INSTANCE IS TRUE

6. DEF: TAUTOLOGY: A STATEMENT PATTERN WHOSE EVERY INSTANCE IS TRUE

C. TEST FOR TAUTOLOGY: MAKE A TRUTH TABLE AND SEE IF THE MAIN COLUMN IS ALL TRUE

D. $p \vee \sim p$ IS A TAUTOLOGY

p	$\vee$	$\sim$	p
T	T	F	T
F	T	T	F

E. $(p \rightarrow q) \rightarrow ((\sim p) \vee q)$ IS A TAUTOLOGY

p	$\rightarrow$	q	$\rightarrow$	$\sim$	p	$\vee$	q
T	T	T	T	F	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	F	T	T
F	T	F	T	T	F	T	F

F. $(p \rightarrow q) \wedge (q \rightarrow p)$ IS NOT A TAUTOLOGY

p	$\rightarrow$	q	$\wedge$	q	$\rightarrow$	p
T	T	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	F	T	F	F
F	T	F	T	F	T	F

G. BICONDITIONALS

1. $\longleftrightarrow$ IF AND ONLY IF IFF
2. $p \leftrightarrow q$ IS A BICONDITIONAL
3. DEF: $p \leftrightarrow q$ MEANS $(p \to q) \wedge (q \to p)$
4. 2 CONDITIONALS, HENCE THE NAME BICONDITIONAL.
5. TRUTH TABLE FOR $p \leftrightarrow q$ (SEE SECTION F PRECEDING).

p	$\leftrightarrow$	q
T	T	T
T	F	F
F	F	T
F	T	F

WHEN THE COMPONENT PARTS HAVE A COMMON VALUE, THE BICONDITIONAL IS TRUE.

H. EXAMPLE:

$(p$	$\to$	$q)$	$\leftrightarrow$	$((\sim$	$p)$	$\vee$	$q)$
T	T	T	T	F	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	F	T	T
F	T	F	T	T	F	T	F

I. IS $[(p \to q) \wedge (q \to r)] \leftrightarrow (p \to r)$ A TAUTOLOGY?

A TRUTH TABLE WITH 3 STATEMENT VARIABLES.

$[(p$	$\to$	$q)$	$\wedge$	$(q$	$\to$	$r)]$	$\leftrightarrow$	$(p$	$\to$	$r)$
T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	F	T	T	F	F
T	F	F	F	F	T	T	F	T	T	T
T	F	F	F	F	T	F	T	T	F	F
F	T	T	T	T	T	T	T	F	T	T
F	T	T	F	T	F	F	F	F	T	F
F	T	F	T	F	T	T	T	F	T	T
F	T	F	T	F	T	F	T	F	T	F

NOT A TAUTOLOGY!

J. <u>HOMEWORK</u>: DO 5.7 a, b, c, d, e, f g, h, i, j AT THE END OF CHAPTER 5 IN PART I OF THE BOOK.

[CHAPTER 6]

EQUIVALENCE

A. NOT (JOE RAN AND SUE HIT)

WHICH IS EQUIVALENT TO A ABOVE? B OR C BELOW.

B. JOE DID NOT RUN AND SUE DID NOT HIT.

C. JOE DID NOT RUN OR SUE DID NOT HIT.

WE STUDY EQUIVALENCES TO FIND OUT.

D. <u>DEF</u>: "STATEMENT PATTERN L IS EQUIVALENT TO STATEMENT PATTERN R" MEANS "$L \leftrightarrow R$ IS A TAUTOLOGY."

E. IS p EQUIVALENT TO $\sim(\sim p)$?

CHECK IF $p \leftrightarrow [\sim(\sim p)]$ IS A TAUTOLOGY.

p	$\leftrightarrow$	$[\sim$	$(\sim$	$p)]$
T	T	T	F	T
F	T	F	T	F

YES, IT IS A TAUTOLOGY

YES, p IS EQUIVALENT TO $\sim(\sim p)$.

F. IS $\sim(p \wedge q)$ EQUIVALENT TO $(\sim p) \vee (\sim q)$?

$$[\sim(p \wedge q)] \longleftrightarrow [(\sim p) \vee (\sim q)]$$

$\sim$	p	$\wedge$	q	$\longleftrightarrow$	$\sim$	p	$\vee$	$\sim$	q
F	T	T	T	T	F	T	F	F	T
T	T	F	F	T	F	T	T	T	F
T	F	F	T	T	T	F	T	F	T
T	F	F	F	T	T	F	T	T	F

YES, A TAUTOLOGY. YES, EQUIV.

G. IS $\sim(p \wedge q)$ EQUIVALENT TO $\sim p \wedge \sim q$?

$$[\sim(p \wedge q)] \longleftrightarrow [(\sim p) \wedge (\sim q)]$$

$\sim$	p	$\wedge$	q	$\longleftrightarrow$	$\sim$	p	$\wedge$	$\sim$	q
F	T	T	T	T	F	T	F	F	T
T	T	F	F	F	F	T	F	T	F
T	F	F	T	F	T	F	F	F	T
T	F	F	F	T	T	F	T	T	F

NOT A TAUTOLOGY.
NOT EQUIVALENT.

H. <u>EQUIVALENT</u> <u>STATEMENTS</u>: REPLACE THE SAME STATEMENT VARIABLES IN EQUIVALENT STATEMENT PATTERNS WITH THE SAME STATEMENTS.

$\sim(p \wedge q)$ IS EQUIVALENT TO $(\sim p) \vee (\sim q)$

p: JOE RAN q: SUE HIT

NOT (JOE RAN AND SUE HIT) IS equivalent to

JOE DID NOT RUN OR SUE DID NOT HIT

I. HOMEWORK SHOWED THE FOLLOWING PAIRS EQUIVALENT

$p \wedge q$	$q \wedge p$
$p \vee q$	$q \vee p$
$p \leftrightarrow q$	$q \leftrightarrow p$

J. NEXT WE STUDY THINGS POSSIBLY EQUIVALENT TO $p \rightarrow q$

K. CONVERSE, INVERSE, CONTRAPOSITIVE

GIVEN IMPLICATION

$p \rightarrow q$

CONVERSE

$q \rightarrow p$

INVERSE

$(\sim p) \rightarrow (\sim q)$

CONTRAPOSITIVE

$(\sim q) \rightarrow (\sim p)$

L. EXAMPLE:

GIVEN: IF SAM HAD 12¢, THEN SAM HAD 11¢.

CONVERSE: IF SAM HAD 11¢, THEN SAM HAD 12¢.

INVERSE: IF SAM DID NOT HAVE 12¢, THEN SAM DID NOT HAVE 11¢.

CONTRAPOSITIVE: IF SAM DID NOT HAVE 11¢, THEN SAM DID NOT HAVE 12¢.

M. IS AN IMPLICATION EQUIVALENT TO ITS CONVERSE?

$(p$	$\rightarrow$	$q)$	$\longleftrightarrow$	$(q$	$\rightarrow$	$p)$
T	T	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	F	T	F	F
F	T	F	T	F	T	F

NO

N. AN IMPLICATION IS *NOT* EQUIVALENT TO ITS INVERSE SINCE $(p \to q) \leftrightarrow [(\sim p) \to (\sim q)]$ IS *NOT* A TAUTOLOGY.

O. AN IMPLICATION *IS* EQUIVALENT TO ITS *CONTRAPOSITIVE* SINCE $(p \to q) \leftrightarrow [(\sim q) \to (\sim p)]$ *IS* A TAUTOLOGY

P. NOTE: $p \to q$ IS EQUIVALENT TO $(\sim p) \vee q$

$$(p \to q) \longleftrightarrow [(\sim p) \vee q]$$

p	$\to$	q	$\longleftrightarrow$	$\sim$	p	$\vee$	q
T	T	T	T	F	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	F	T	T
F	T	F	T	T	F	T	F

Q. EXAMPLE OF $p \to q$ IS EQUIV. TO $(\sim p) \vee q$

1. IF SAM ATE THE COOKIES, THEN THERE WOULD BE CRUMBS ON THE TABLE.
   SAM DID *NOT* EAT THE COOKIES *OR* THERE WOULD BE CRUMBS ON THE TABLE.
2. IF SOMETHING WAS WRONG WITH THIS CAR, THEN I WOULD TELL YOU.
   THERE IS NOTHING WRONG WITH THIS CAR, *OR* I WOULD TELL YOU.

R. 5 BIG EQUIVALENCES: LEARN WELL

1. DEMORGAN'S LAWS

$\sim(p \wedge q) \qquad (\sim p) \vee (\sim q)$

$\sim(p \vee q) \qquad (\sim p) \wedge (\sim q)$

2. CONTRAPOSITIVE

$p \rightarrow q \qquad (\sim q) \rightarrow (\sim p)$

3. IMPLICATION EQUIVALENCE

$p \rightarrow q \qquad (\sim p) \vee q$

4. NEGATION OF IMPLICATION

$\sim(p \rightarrow q) \qquad p \wedge (\sim q)$

5. WRITING IN DIFFERENT EQUIV. FORM

$\sim(A \wedge (\sim B)) \qquad \sim A \vee (\sim(\sim B))$

$\sim A \vee B$

$\sim((\sim A) \vee (\sim B)) \qquad A \wedge B$

$\sim((\sim A) \vee (E \wedge (\sim F))) \qquad A \wedge (\sim(E \wedge (\sim F)))$

$A \wedge ((\sim E) \vee F)$

DIFFERENT EQUIV. FORM CONTINUED

$B \to (\sim E)$ | $E \to (\sim B)$

$(\sim B) \vee (\sim E)$

$\sim (B \wedge E)$

$\sim [B \to (\sim E)]$ | $B \wedge E$

$\sim [(\sim B) \vee (E \to (\sim W))]$ | $B \wedge [\sim (E \to (\sim W))]$

$B \wedge (E \wedge W)$

T. HOMEWORK: IN PART I OF THIS BOOK DO 6.25 a, f, g, 6.30, 6.32, 6.35, AND 6.36. ALSO, EACH OF THE FOLLOWING IS IN ONE OF THE FORMS $\sim (p \wedge q)$, $\sim (p \vee q)$, $\sim (p \to q)$, OR $p \to q$. WRITE IN A DIFFERENT EQUIVALENT FORM.

1. $\sim (p \wedge q)$
2. $\sim (A \wedge B)$
3. $\sim (M \wedge B)$
4. $\sim (A \wedge (\sim B))$
5. $\sim [(\sim A) \wedge B]$
6. $\sim [M \wedge (\sim D)]$
7. $\sim [(\sim A) \wedge (\sim B)]$

8. $\sim[(\sim p)\wedge(\sim q)]$

9. $\sim[A\wedge(B\vee C)]$

10. $\sim[(\sim A)\wedge(B\vee C)]$

11. $\sim[A\wedge(B\wedge C)]$

12. $\sim\{(\sim B)\wedge[\sim(A\vee C)]\}$

13. $\sim(p\vee q)$

14. $\sim(A\vee B)$

15 $\sim(A\vee(\sim B))$

16. $\sim[M\vee(\sim D)]$

17. $\sim[(\sim A)\vee(\sim B)]$

18. $\sim[A\vee(B\wedge C)]$

19. $\sim[(\sim A)\vee(B\wedge C)]$

20. $\sim\{(\sim A)\vee[\sim(B\vee C)]\}$

21. $\sim(p\rightarrow q)$

22. $\sim(A\rightarrow B)$

23. $\sim(M\rightarrow B)$

24. $\sim[(\sim A)\rightarrow B]$

25. $\sim[M\rightarrow(\sim D)]$

26. $\sim[(\sim A)\rightarrow(\sim B)]$

27. $\sim[A \rightarrow (B \vee C)]$
28. $\sim[(\sim A) \rightarrow (B \wedge C)]$
29. $\sim[A \rightarrow (B \rightarrow C)]$
30. $p \rightarrow q$
31. $A \rightarrow B$
32. $A \rightarrow (\sim B)$
33. $(\sim A) \rightarrow (\sim B)$
34. $(\sim A) \rightarrow (B \vee C)$
35. $(\sim A) \rightarrow ((\sim D) \vee W)$

U. OBSERVATION FROM HOMEWORK PROBLEM 6.32: INSTANCES OF EQUIVALENT STATEMENT PATTERNS HAVE THE SAME TRUTH VALUE IN EVERY CASE.

$$(p \rightarrow q) \leftrightarrow [(\sim p) \vee q]$$

T	T	T	T	X	X		X
T	F	F	T	X	X		X
F	T	T	T	X	X		X
F	T	F	T	X	X		X

SO IN A PROOF YOU COULD SAY...

28 $x \in A \rightarrow x \in B$

29 $x \notin A \vee x \in B$ LINE 28, IMP. EQUIVALENCE

V. TWO OPPOSITE CONCEPTS

1. WRITE IN DIFFERENT EQUIVALENT FORM.

THE PROBLEM IS GIVEN IN ONE OF THE FORMS OF THE 5 BIG EQUIVALENCES WRITE IN ITS EQUIVALENT FORM.

GIVEN: $\sim(p \vee q)$

ANSWER: $(\sim p) \wedge (\sim q)$

2. NEGATING (THIS IS THE OPPOSITE OF WRITING IN DIFFERENT EQUIVALENT FORM!!!!) THE STEPS ARE:

YOU ARE GIVEN THE THING TO BE NEGATED: $p \vee q$

PUT A $\sim$ SIGN IN FRONT OF THE WHOLE THING:
THE TRIVIAL NEGATION $\sim(p \vee q)$

WRITE THIS IN DIFFERENT EQUIVALENT FORM: $(\sim p) \wedge (\sim q)$

THE MIDDLE STEP IS USUALLY DONE MENTALLY AND ABBREVIATED BY

NEGATE: $p \vee q$

ANSWER: $(\sim p) \wedge (\sim q)$

W. DRILL ON NEGATING:

THING TO BE NEGATED	THE TRIVIAL NEGATION THINK: WHAT IS EQUIVALENT TO	ANSWER
$p \vee q$	$\sim(p \vee q)$	$(\sim p) \wedge (\sim q)$
$A \to B$	$\sim(A \to B)$	$A \wedge (\sim B)$
$(\sim E) \wedge W$	$\sim[(\sim E) \wedge W]$	$E \vee (\sim W)$
$M \to (\sim E)$	$\sim[M \to (\sim E)]$	$M \wedge E$

X. DO NOT MAKE THE FOLLOWING MAJOR ERROR IN A PROOF:

⋮

28. $x \in A$ OR $x \in B$

29. $x \notin A$ AND $x \notin B$ LINE 28, DEMORGAN'S LAW

⋮

A PROOF IS TO BE A LIST OF <u>TRUE</u> STATEMENTS (SUBJECT TO ASSUMPTIONS). LINES 28 AND 29 CANNOT BOTH BE TRUE, SINCE THEY ARE THE NEGATION OF EACH OTHER.

Y. A WORD OF CAUTION ABOUT HOW $\leftrightarrow$ IS READ. $\leftrightarrow$ IS READ "IF AND ONLY IF". $\leftrightarrow$ IS <u>NOT</u> READ "IS EQUIVALENT TO"

"$[\sim(p \vee q)] \leftrightarrow [(\sim p) \wedge (\sim q)]$" IS READ "$\sim(p \vee q)$ IF AND ONLY IF $(\sim p) \wedge (\sim q)$". IT IS <u>NOT</u> A STATEMENT, BUT A STATEMENT PATTERN. INSTANCES OF IT ARE STATEMENTS (ALL TRUE, IN THIS CASE).

"$\sim(p \vee q)$ IS EQUIVALENT TO $(\sim p) \wedge (\sim q)$" <u>IS</u> A STATEMENT. IT MEANS "$[\sim(p \vee q)] \leftrightarrow [(\sim p) \wedge (\sim q)]$ IS A TAUTOLOGY"

SOME BOOKS USE SPECIAL SYMBOLS FOR "IS EQUIVALENT TO" LIKE $\equiv$ OR $\Leftrightarrow$. IN THAT CASE,

$\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$ WOULD

MEAN "$\sim(p \vee q) \leftrightarrow (\sim p) \wedge (\sim q)$ IS A TAUTOLOGY" AND IS READ "$\sim(p \vee q)$ IS EQUIVALENT TO $(\sim p) \wedge (\sim q)$". SIMILARLY, $p \Leftrightarrow [\sim(\sim p)]$ WOULD MEAN "$p \leftrightarrow [\sim(\sim p)]$ IS A TAUTOLOGY" AND IS READ "p IS EQUIVALENT TO $\sim(\sim p)$".

WE WILL NOT USE THE SYMBOLS $\equiv$ AND $\Leftrightarrow$ IN THIS BOOK, EXCEPT RARELY.

Z. HOMEWORK: DO 6.31 IN PART I OF THIS BOOK. ALSO, NEGATE EACH OF THE FOLLOWING:

1. $A \vee (\sim B)$
2. $(\sim E) \rightarrow Q$
3. $A \vee (B \rightarrow E)$
4. $(\sim B) \wedge E$
5. $(\sim A) \rightarrow (\sim B)$
6. $(\sim E) \vee (\sim P)$
7. $A \rightarrow [(\sim B) \wedge E]$

IMPLICATION

A. AN IMPLICATION $p \to q$ CAN BE EXPRESSED IN OUR LANGUAGE IN MANY WAYS. THE CHART BELOW CAN BE USED TO TRANSLATE BACK TO $p \to q$ FORM.

ALL IN THE FOLLOWING LIST MEAN THE SAME THING:

$p \to q$

p IMPLIES q

IF p, THEN q

q, IF p

p ONLY IF q

p IS SUFFICIENT FOR q

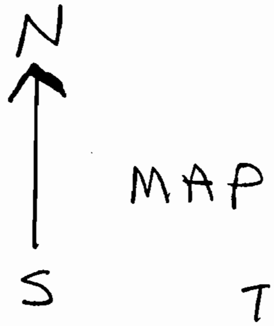
q IS NECESSARY FOR p

B. EXAMPLES OF TRANSLATING BACK TO "IF p, THEN q" FORM

1. A SUFFICIENT CONDITION FOR JOE TO RUN IS FOR SUE TO WIN.

   IF SUE WINS, THEN JOE RUNS.

A MEMORY DEVICE FOR NECESSARY AND SUFFICIENT LANGUAGE: ON A MAP THE ARROW ALWAYS POINTS <u>NORTH</u>



NECESSARY STARTS WITH N

$$S \longrightarrow N$$

THE ARROW POINTS TO THE <u>N</u>ECESSARY CONDITION

2. 5 BEING LESS THAN 6 IS SUFFICIENT FOR 2 TO BE AN ELEMENT OF E.

   IF $5 < 6$, THEN $2 \in E$.

3. A NECESSARY CONDITION FOR 5 TO BE AN ELEMENT OF E IS FOR 4 TO BE AN ELEMENT OF W.

   IF $5 \in E$, THEN $4 \in W$

4. SAM RUNNING IS NECESSARY FOR JOE TO WIN.

   IF JOE WINS, THEN SAM RUNS.

5. SAM WINS ONLY IF JOE RUNS.

IF SAM WINS, THEN JOE RUNS.

6. BOB LAUGHS, IF SUE SIGHS.

IF SUE SIGHS, THEN BOB LAUGHS.

7. A NECESSARY AND SUFFICIENT CONDITION FOR $4 \varepsilon W$ IS FOR $5 \varepsilon T$.

a. <u>STEP 1</u>: A NECESSARY CONDITION FOR $4 \varepsilon W$ IS FOR $5 \varepsilon T$

AND

A SUFFICIENT CONDITION FOR $4 \varepsilon W$ IS FOR $5 \varepsilon T$.

b. <u>STEP 2</u>: IF $4 \varepsilon W$, THEN $5 \varepsilon T$

AND

IF $5 \varepsilon T$, THEN $4 \varepsilon W$

c. <u>STEP 3</u>: $4 \varepsilon W$ IFF $5 \varepsilon T$

<u>MORAL</u>: NECESSARY AND SUFFICIENT CONDITIONS ARE IFF

C. FORMERLY YOU MIGHT HAVE BALKED AT THE TRUTH TABLE FOR $p \to q$ BEING DEFINED THE WAY IT WAS.

p	$\to$	q
T	T	T
T	F	F
F	T	T
F	T	F

ARGUMENT 1 THAT IT WAS DEFINED CORRECTLY.

IN EVERYDAY LANGUAGE WE USE $p \to q$ AND $(\sim p) \vee q$ EQUIVALENTLY:

IF JOE ATE THE COOKIES, THEN THERE WOULD BE CRUMBS ON THE TABLE.

JOE DID NOT EAT THE COOKIES OR THERE WOULD BE CRUMBS ON THE TABLE.

SINCE WE WANT $p \to q$ AND $(\sim p) \vee q$ TO BE EQUIVALENT, THE MAIN COLUMNS OF THEIR TRUTH TABLES SHOULD BE THE SAME. THERE WAS NO QUESTION ABOUT THE TRUTH TABLE FOR $(\sim p) \vee q$

$(\sim$	$p)$	$\vee$	q
F	T	T	T
F	T	F	F
T	F	T	T
T	F	T	F

SO THE MAIN COLUMNS SHOULD MATCH

p	$\to$	q
T	T	T
T	F	F
F	T	T
F	T	F

D. ARGUMENT 2 THAT $p \to q$ TRUTH TABLE IS CORRECT.

p	$\to$	q
T	T	T
T	F	F
F	?	T
F	?	F

IT IS PROBABLY CLEAR THAT THE FIRST TWO LINES ARE CORRECT. THAT LEAVES ONLY 4 CHOICES FOR THE MIDDLE COLUMN

1	2	3	4
T	T	T	T
F	F	F	F
F	T	F	T
F	F	T	T

1 IS BAD. THAT WOULD MEAN $p \to q$ WOULD BE EQUIVALENT TO $p \wedge q$.

2 IS BAD. THAT WOULD MEAN $p \to q$ WOULD BE EQUIVALENT TO q.

3 IS BAD. THAT WOULD MEAN $p \to q$ WOULD BE EQUIVALENT TO $p \leftrightarrow q$.

4 IS THE ONLY ONE LEFT. WE EMBRACE IT AS TRUTH.

E. UNDERSTOOD GROUPING SYMBOLS FOR $\sim$: WHEN GROUPING SYMBOLS ARE LEFT OFF INVOLVING $\sim$, IT IS UNDERSTOOD THAT THE ONLY THING NEGATED IS WHAT IS IMMEDIATELY TO THE RIGHT OF THE $\sim$ SIGN.

	UNDERSTOOD	FILLED IN
1.	$\sim p \vee \sim q$	$(\sim p) \vee (\sim q)$
2.	$p \rightarrow \sim q$	$p \rightarrow (\sim q)$
3.	$\sim (p \rightarrow q) \vee b$	$[\sim (p \rightarrow q)] \vee b$
4.	$\sim (p \vee \sim q) \wedge b$	$[\sim (p \vee [\sim q])] \wedge b$

NOTE: $\sim (p \rightarrow q) \vee b$ IS A DISJUNCTION

F. HOMEWORK: DO 7.20 AND 7.21 IN PART I OF THE BOOK

[CHAPTER 8]

LIST OF TAUTOLOGIES

KNOW THESE WELL

T_1 $[(p \rightarrow q) \wedge p] \rightarrow q$ MODUS PONENS

T_2 $[(p \rightarrow q) \wedge (\sim q)] \rightarrow \sim p$ MODUS TOLLENS

T_3 $\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$ DEMORGAN'S LAW

T_4 $\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$ DEMORGAN'S LAW

T_5 $\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$ NEGATION OF IMPLICATION

T_6 $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ IMPLICATION EQUIVALENCE

T_7 $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ CONTRAPOSITIVE

T_8 $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ SYLLOGISM

T_9 $(p \leftrightarrow q) \leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$

T_{10} $p \leftrightarrow \sim(\sim p)$ DOUBLE NEGATION

T_{11} $(p \wedge q) \rightarrow p$ SIMPLIFICATION

T_{12} $p \rightarrow (p \vee q)$

T_{13} $(p \wedge q) \leftrightarrow (q \wedge p)$ COMMUTATIVE

T_{14} $(p \vee q) \leftrightarrow (q \vee p)$ COMMUTATIVE

T_{15} $(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$ COMMUTATIVE

T_{16} $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$

DISTRIBUTIVE

T_{17} $[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$

DISTRIBUTIVE

T_{18} $[p \wedge (q \wedge r)] \leftrightarrow [(p \wedge q) \wedge r]$

ASSOCIATIVE

T_{19} $[p \vee (q \vee r)] \leftrightarrow [(p \vee q) \vee r]$

ASSOCIATIVE

[CHAPTER 9]

VALID ARGUMENT

A. ARGUMENT PATTERN

$p \rightarrow q$ PREMISE (OR HYPOTHESIS)

$\sim q$ PREMISE (OR HYPOTHESIS)

$\therefore \sim p$ CONCLUSION

THE THREE DOTS ARE READ "THEREFORE".

PREMISES = HYPOTHESES

B. ARGUMENT

IF SAM HAS 12¢, THEN SAM HAS 11¢.

SAM DOES NOT HAVE 11¢.

$\therefore$ SAM DOES NOT HAVE 12¢.

THIS ARGUMENT IS AN <u>INSTANCE</u> OF THE ARGUMENT PATTERN ABOVE.

C. DEFINITION OF <u>VALID</u> <u>ARGUMENT</u> <u>PATTERN</u>:

AN ARGUMENT PATTERN IS VALID IFF THE CONJUNCTION OF THE PREMISES IMPLYING THE CONCLUSION IS A TAUTOLOGY.

D. LATER YOU WILL BE TOLD WHY VALIDITY IS DEFINED THIS WAY.

E. $p \to q$

$\sim q$

$\therefore \sim p$

VALID OR INVALID

(INVALID = NOT VALID)

$[(p$	$\to$	$q)$	$\wedge$	$\sim$	$q]$	$\to$	$\sim$	p
T	T	T	F	F	T	T	F	T
T	F	F	F	T	F	T	F	T
F	T	T	F	F	T	T	T	F
F	T	F	T	T	F	T	T	F

TAUTOLOGY

YES : VALID

F. $p \vee \sim q$

q

$\therefore \sim p$

VALID OR INVALID

$[(p$	$\vee$	$\sim$	$q)$	$\wedge$	$q]$	$\to$	$\sim$	p
T	T	F	T	T	T	F	F	T
T	T	T	F	F	F	T	F	T
F	F	F	T	F	T	T	T	F
F	T	T	F	F	F	T	T	F

NOT A TAUTOLOGY.

INVALID

G. THE CONJUNCTION OF ANY NUMBER OF STATEMENTS IS TRUE IFF ALL THE STATEMENTS ARE TRUE (AT LEAST ONE FALSE MEANS THE WHOLE CONJUNCTION IS FALSE).

H. ARGUMENT PATTERN WITH 3 PREMISES.

$$\begin{array}{l} p \to q \\ q \to r \\ r \\ \hline \therefore p \end{array}$$

VALID OR INVALID

$$[(p \to q) \wedge (q \to r) \wedge r] \longrightarrow p$$

	p	$\to$	q	$\wedge$	q	$\to$	r	$\wedge$	r	$\longrightarrow$	p
1	T	T	T		T	T	T		T	T	T
2	T	T	T		T	F	F		F	T	T
3	T	F	F		F	T	T		T	T	T
4	T	F	F		F	T	F		F	T	T
5	F	T	T		T	T	T		T	F	F
6	F	T	T		T	F	F		F	T	F
7	F	T	F		F	T	T		T	F	F
8	F	T	F		F	T	F		F	T	F
	a	b	c	d	e	f	g	h	i	j	k

NO MAIN COLUMN FOR $[(p \to q) \wedge (q \to r) \wedge r]$
KEEP IT IN YOUR HEAD!

ROW 1: COLUMNS b, f, i ALL TRUE.
ANTECEDENT TRUE
TRUE $\to$ TRUE (COLUMN j) IS TRUE

ROW 2: AT LEAST ONE OF COLUMNS b, f, i IS FALSE. SO THE ANTECEDENT $[(p \rightarrow q) \wedge (q \rightarrow r) \wedge r]$ INSTANCE IS FALSE.
FALSE $\rightarrow$ TRUE (COLUMN j) IS TRUE.

I VALID ARGUMENT VS. VALID ARGUMENT PATTERN:

THE ARGUMENT:

IF SAM HAD 12¢, THEN SAM HAD 11¢
SAM HAD 12¢

∴ SAM HAD 11¢

IS AN INSTANCE OF EACH OF THESE

(1)	(2)	(3)
$p \rightarrow q$	p	$p \rightarrow q$
p	q	r
$\therefore q$	$\therefore r$	$\therefore q$
VALID	INVALID	INVALID

THE OUTCOME OF ARGUMENT PATTERN (1) DETERMINED THE FATE OF THE ARGUMENT.

NOTE: BEING AN INSTANCE OF AN INVALID ARGUMENT PATTERN DOES NOT MAKE AN ARGUMENT INVALID.

THIS ARGUMENT IS VALID.

J. <u>NEED</u>: GIVEN AN <u>ARGUMENT</u>, FIND AN <u>ARGUMENT PATTERN</u> WHOSE OUTCOME COINCIDES WITH THE OUTCOME OF THE ARGUMENT. DETERMINE WHETHER THE ARGUMENT <u>PATTERN</u> IS VALID OR NOT; YOU THEN KNOW THE OUTCOME OF THE CORRESPONDING <u>ARGUMENT</u>. TO FIND SUCH AN ARGUMENT PATTERN WE NEED TO KNOW ABOUT <u>PRIME</u> STATEMENTS.

K. <u>PRIME</u> <u>STATEMENT</u>: A STATEMENT THAT IS NEITHER A NEGATION, DISJUNCTION, CONJUNCTION, IMPLICATION, NOR BICONDITIONAL.

$1 \nless 2$. NOT PRIME

$1<2$ OR SAM WINS. NOT PRIME

$1<2$ AND SAM WINS. NOT PRIME

$1<2$ IMPLIES SAM WINS. NOT PRIME

$1<2$ IFF SAM WINS. NOT PRIME

$1<2$ PRIME

SAM WINS. PRIME

L. ARGUMENT VALIDITY/INVALIDITY

GET AN ARGUMENT PATTERN SO THAT

1. THE ARGUMENT IS AN INSTANCE OF THE ARGUMENT PATTERN.
2. ONLY PRIME STATEMENTS WERE SUBSTITUTED FOR STATEMENT VARIABLES.
3. THE SAME PRIME STATEMENT DID NOT REPLACE 2 DIFFERENT STATEMENT VARIABLES.
4. NO STATEMENT VARIABLE HAD 2 DIFFERENT STATEMENTS REPLACE IT.
5. ARGUMENT AND ARGUMENT PATTERN HAVE COMMON FATE.

M. EXAMPLE: $1 \not< 2$ OR SUE RUNS.

$1 < 2$

$\therefore$ SUE DOES NOT RUN

$$\begin{array}{c} \sim p \vee q \\ p \\ \hline \therefore \sim q \end{array}$$

$[(\sim p \vee q) \wedge p] \longrightarrow \sim q$

$\sim$	p	$\vee$	q	$\wedge$	p	$\to$	$\sim$	q
F	T	T	T	T	T	F	F	T
F	T	F	F	F	T	T	T	F
T	F	T	T	F	F	T	F	T
T	F	T	F	F	F	T	T	F

INVALID ARGUMENT PATTERN

SO INVALID ARGUMENT!

N. <u>ARGUMENT</u>: VALID OR INVALID?

SAM DID NOT RUN OR JOE WALKED.
SAM RAN.
IF JOE DID NOT WALK, THEN SUE GOT MAD.

∴ SUE DID NOT GET MAD

CORRESPONDING ARGUMENT PATTERN:

$$\begin{array}{l} \sim p \vee q \\ p \\ \sim q \rightarrow r \\ \hline \therefore \sim r \end{array}$$

MAKE A TRUTH TABLE FOR:

$$[(\sim p \vee q) \wedge p \wedge (\sim q \rightarrow r)] \rightarrow \sim r$$

TO FIND IT IS NOT A TAUTOLOGY

HENCE, INVALID <u>ARGUMENT</u>

⊖ <u>HOMEWORK</u>: DO 9.23 AND 9.24 IN PART I OF THIS BOOK.

P. WHEN PEOPLE REASON THEY DO NOT STOP EVERY FEW SENTENCES AND MAKE TRUTH TABLES. SOME FAMOUS VALID ARGUMENT PATTERNS ARE LEARNED ONCE AND THEN USED REPEATEDLY.

Q. FAMOUS ARGUMENT PATTERNS:

1. $$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

$[(p \rightarrow q) \wedge p] \rightarrow q$ IS A TAUTOLOGY.

VALID: MODUS PONENS ≡ MODE THAT AFFIRMS

2. $$\begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore p \end{array}$$

$[(p \rightarrow q) \wedge q] \rightarrow p$ IS <u>NOT</u> A TAUTOLOGY

INVALID: CONVERSE REASONING

DO <u>NOT</u> USE

3. $$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ IS A TAUTOLOGY

VALID: MODUS TOLLENS OR CONTRAPOSITIVE REASONING

MODUS TOLLENS ≡ MODE THAT DENIES

4. $p \to q$ $\quad [(p \to q) \wedge \sim p] \to \sim q$ IS

$\underline{\sim p}$ $\quad$ <u>NOT</u> A TAUTOLOGY

$\therefore \sim q$

INVALID: INVERSE REASONING

DO NOT USE

5. $p \to q$ $\quad [(p \to q) \wedge (q \to r)] \to (p \to r)$

$\underline{q \to r}$ $\quad$ <u>IS</u> A TAUTOLOGY

$\therefore p \to r$

VALID: SYLLOGISM

LEARN THESE STRUCTURES. USE THE VALID ONES. DO NOT USE INVALID ONES.

R. A DRILL ON RECOGNIZING THESE STRUCTURES

1. $\sim A \to B$ $\quad$ VALID

$\underline{\sim A}$ $\quad$ MODUS PONENS

$\therefore B$

2. $\sim A \to \sim B$ $\quad$ VALID

$\underline{B}$ $\quad$ MODUS TOLLENS OR CONTRAPOSITIVE REASONING

$\therefore A$

3. $\sim E \rightarrow \sim W$
$\sim W \rightarrow F$
$\therefore \sim E \rightarrow F$

VALID
SYLLOGISM

4. $\sim(A \vee B) \rightarrow \sim C$
$A \vee B$
$\therefore C$

INVALID
INVERSE REASONING

5. $\sim A \rightarrow (B \vee E)$
$\sim B \wedge \sim E$
$\therefore A$

VALID
MODUS TOLLENS

6. $(A \rightarrow B) \rightarrow \sim C$
C
$\therefore A \wedge \sim B$

VALID
MODUS TOLLENS

7. IF SAM RAN, THEN SUE DID NOT WALK.
SUE DID NOT WALK.
$\therefore$ SAM RAN.

INVALID
CONVERSE REASONING

8. $A \rightarrow \sim B$
B
$\therefore A$

NONE OF THE FAMOUS NAMED ONES. MAKE A TRUTH TABLE TO DECIDE

5. HOMEWORK: DO 9.22, 9.25, 9.26 PART I OF BOOK.

[CHAPTER 10]

DIRECT PROOF

A. DOES THIS SOUND LIKE A VALID ARGUMENT?

IF $4>3$, THEN $2+2=4$

$\therefore\ 2+2=4$

LET'S CHECK

$$\begin{array}{l} p \to q \\ \hline \therefore\ q \end{array}$$

p	$\to$	q	$) \to$	q
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	F	F

(table headed $(p \to q) \to q$)

INVALID

NOT A TAUTOLOGY

B. VALIDITY DEPENDS ON STRUCTURE, NOT CONTENT. LOOK FOR A CERTAIN FORM. DO NOT BE SWAYED BY CONTENT!

C. WHY VALID ARGUMENT WAS DEFINED THE WAY IT WAS: VALID ARGUMENTS ARE "TRUTH PRESERVING" (R. KRUSCHWITZ). THAT IS, IF THE PREMISES OF A VALID ARGUMENT ARE TRUE THEN THE CONCLUSION MUST BE TRUE. WHEN

AN ARGUMENT IS VALID, IT DOES NOT NECESSARILY MEAN THE PREMISES ARE TRUE. IT DOES NOT NECESSARILY MEAN THE CONCLUSION IS TRUE. IT IS SIMPLY "TRUTH PRESERVING". FOR EXAMPLE,

IF $1=2$, THEN $3=4$ — VALID

$1=2$ — MODUS PONENS

$\therefore\ 3=4$

VALID, YET FALSE CONCLUSION AND ONE FALSE PREMISE.

IF $3<4$, THEN $1<2$ — VALID

$3<4$ — MODUS PONENS.

$\therefore\ 1<2$

VALID, AND PREMISES <u>ALL</u> TRUE. THEREFORE, THE CONCLUSION <u>MUST</u> BE TRUE. VALID DOES NOT MEAN THE SAME AS TRUE.

D. SYMBOLISM STUDIED NOW. THEN SYMBOLISM WILL BE USED TO EXPLAIN WHY <u>VALID</u> WAS DEFINED SO THAT IT WOULD BE TRUTH PRESERVING.

E. DIFFERENT SYMBOLIZATIONS FOR THE SAME STATEMENT:

$$1 < 2$$

ONE IS LESS THAN TWO.

UNE EST MOINS QUE DEUX. (FRENCH)

EINS IST WENIGER ALS ZWEI. (GERMAN)

MATHEMATICIANS HAVE THE POWER TO CHOOSE SYMBOLS TO DENOTE THINGS, SO...

LET p' (READ "p PRIME") DENOTE THE STATEMENT SYMBOLIZED BY "$1<2$".

SO, p' IS A STATEMENT, NOT A STATEMENT VARIABLE, HENCE, p' COULD REPLACE STATEMENT VARIABLE p.

$p \vee \sim p$ ← STATEMENT PATTERN NOT A STATEMENT

$p' \vee \sim p'$ ← STATEMENT, A TRUE ONE. AN INSTANCE OF $p \vee \sim p$

F. ARGUMENT PATTERNS AND ARGUMENTS

LET p' DENOTE "SAM RAN".
LET q' DENOTE "SUE FLEW".

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

ARGUMENT PATTERN. MADE UP OF STATEMENT PATTERNS.

$$\begin{array}{l} \text{SAM RAN IMPLIES SUE FLEW.} \\ \text{SAM RAN} \\ \hline \therefore \text{SUE FLEW} \end{array}$$

ARGUMENT MADE UP OF STATEMENTS

$$\begin{array}{l} p' \to q' \\ p' \\ \hline \therefore q' \end{array}$$

ARGUMENT. MADE UP OF STATEMENTS.

G. WHY THE DEFINITION OF VALIDITY FORCES A VALID ARGUMENT WITH ALL PREMISES TRUE TO HAVE A TRUE CONCLUSION: AN ILLUSTRATIVE EXPLANATION FOLLOWS.

ASSUME THE VALID MODUS TOLLENS ARGUMENT

$$\begin{array}{l} p' \to q' \\ \sim q' \\ \hline \therefore \sim p' \end{array}$$

HAS ALL TRUE PREMISES.

SHOW $\sim p'$ MUST BE TRUE.

$$\begin{array}{l} p' \to q' \\ \sim q' \\ \hline \therefore \sim p' \end{array}$$ IS AN INSTANCE OF $$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$ A VALID ARGUMENT PATTERN.

SO, $[(p \to q) \wedge \sim q] \to \sim p$ IS A TAUTOLOGY.

EVERY INSTANCE IS TRUE, SO

$[(p' \to q') \wedge \sim q'] \to \sim p'$ IS TRUE.

$[(p' \to q') \wedge \sim q']$ IS TRUE SINCE BOTH

$p' \to q'$ AND $\sim q'$ ARE TRUE. SINCE

THE TRUE IMPLICATION

$[(p' \to q') \wedge \sim q'] \to \sim p'$ HAS A TRUE ANTECEDENT, THE CONSEQUENT $\sim p'$ MUST BE TRUE. THIS IS WHAT WE WANTED TO SHOW.

H. DIRECT PROOF IS COMING. A DRILL TO HELP IN DIRECT PROOF:

SUPPOSE p', q', r' ARE STATEMENTS AND WE ARE IN THE MIDDLE OF A PROOF

⋮

SUPPOSE LINES 7 AND 8 ARE TRUE FOR SOME MYSTERIOUS REASON.

7.	$p' \to q'$	MYSTERIOUS REASON
8.	p'	MYSTERIOUS REASON

NOW 7 AND 8 ARE THE <u>TRUE</u> PREMISES OF A <u>VALID</u> ARGUMENT SO THE CONCLUSION <u>MUST</u> BE TRUE

9.	q'	7, 8, MODUS PONENS

⋮

TRY SOME MORE

20.	$\sim r' \to \sim p'$	MYSTERIOUS REASON
21.	p'	MYSTERIOUS REASON
22.	r'	20, 21, MODUS TOLLENS

⋮

76	$q' \to r'$	MYSTERIOUS REASON
77.	p'	MYSTERIOUS REASON
78	$p' \to q'$	MYSTERIOUS REASON
79.	$p' \to r'$	78, 76, SYLLOGISM

SUPPOSE A', B', E' ARE STATEMENTS, ALSO.

94. $\sim A' \rightarrow (B' \vee \sim E')$ MYSTERIOUS REASON
95. $p' \rightarrow r'$ MYSTERIOUS REASON
96. $\sim B' \wedge E'$ MYSTERIOUS REASON
97. A' 94, 96, MODUS TOLLENS

I. DIRECT PROOF TO PROVE AN ARGUMENT PATTERN VALID. (PREVIOUSLY STUDIED WAS THE TRUTH TABLE METHOD FOR PROVING AN ARGUMENT PATTERN VALID OR INVALID).

1. ASSUME YOU HAVE AN ARBITRARY INSTANCE OF THE ARGUMENT PATTERN WITH TRUE PREMISES.

2. SHOW THE CONLUSION MUST BE TRUE USING TAUTOLOGIES, KNOWN VALID ARGUMENTS, AND TRUTH TABLE DEFINITIONS.

(NOTE: A DIRECT PROOF IS NOT USED TO PROVE AN ARGUMENT PATTERN INVALID, JUST VALID. ALSO, LATER YOU WILL BE TOLD WHY 1 AND 2 PROVE VALIDITY. JUST DO IT ON FAITH NOW.)

J. $p \to q$

$\sim r \to \sim q$

p

$\therefore r$

PROVE VALID BY DIRECT PROOF

0. ASSUME p', q', AND r' ARE STATEMENTS THAT REPLACE p, q, AND r RESPECTIVELY.

1. $p' \to q'$	ASSUMED TRUE FOR DIRECT PROOF
2. $\sim r' \to \sim q'$	ASSUMED TRUE FOR DIRECT PROOF
3. p'	ASSUMED TRUE FOR DIRECT PROOF
	(SHOW r' IS TRUE)
4. $q' \to r'$	2, CONTRAPOSITIVE
5. $p' \to r'$	1, 4, SYLLOGISM
6. r'	5, 3, MODUS PONENS

↑ LIST OF TRUE STATEMENTS

↑ REASONS

K. $p \to r$

$q \to s$

$\sim r \wedge \sim s$

$\therefore \sim(p \vee q)$

PROVE VALID BY DIRECT PROOF

0. ASSUME p', q', r', s' ARE STATEMENTS THAT REPLACE p, q, r, s RESPECTIVELY.

1.	$p' \to r'$	ASSUMED TRUE FOR DIRECT PROOF
2.	$q' \to s'$	" " " " "
3.	$\sim r' \wedge \sim s'$	" " " " "
		(SHOW $\sim(p' \vee q')$ TRUE)
4.	$\sim r'$	3, TRUTH TABLE DEFINITION OF $\wedge$
5.	$\sim s'$	3, TRUTH TABLE DEFINITION OF $\wedge$
6.	$\sim p'$	1, 4, MODUS TOLLENS
7.	$\sim q'$	2, 5, MODUS TOLLENS
8.	$\sim p' \wedge \sim q'$	6, 7, TRUTH TABLE DEF. OF $\wedge$
9.	$\sim(p' \vee q')$	8, DEMORGAN'S LAW.

VALID BY DIRECT PROOF

L. $\sim(p \rightarrow \sim q)$
$(p \wedge q) \rightarrow (\sim r \vee s)$
$\therefore\ r \rightarrow s$

PROVE VALID BY DIRECT PROOF

0. ASSUME p', q', r', s' ARE STATEMENTS THAT REPLACE p, q, r, s RESPECTIVELY.
1. $\sim(p' \rightarrow \sim q')$ ASSUMED TRUE FOR DIRECT PROOF
2. $(p' \wedge q') \rightarrow (\sim r' \vee s')$ " " " " "
   (SHOW $r' \rightarrow s'$ TRUE)
3. $p' \wedge q'$ 1, NEGATION OF IMPLICATION
4. $\sim r' \vee s'$ 2, 3, MODUS PONENS
5. $r' \rightarrow s'$ 4, IMPLICATION EQUIVALENCE

M. HOW DO YOU KNOW WHAT TO PUT NEXT IN A DIRECT PROOF?

1. HAVE 5 BIG EQUIVALENCES, MODUS PONENS, MODUS TOLLENS, SYLLOGISM, AND TRUTH TABLE DEFINITIONS LEARNED UNTIL NON-NERVOUSLY INSTINCTIVE.
2. USE SCRATCH WORK FIRST. ASSUME PREMISES TRUE. BRAINSTORM. HAVE A LOGICAL FEEDING FRENZY.
3. CREATIVELY PICK OUT THE PROOF. TRY HARD.

N. PHILOSOPHIZING: AT THE MOMENT WE KNOW 2 WAYS OF PROVING AN ARGUMENT PATTERN VALID

1. TRUTH TABLE
2. DIRECT PROOF

WE KNOW ONLY ONE WAY TO PROVE INVALID: TRUTH TABLE

O. QUESTION: WHAT IF I ASSUME THE PREMISES TRUE AND CANNOT SHOW THE CONCLUSION TRUE, DOES THAT PROVE IT INVALID?

NO. SOMEONE ELSE MIGHT BE ABLE TO DO IT. IF YOU HAD TRIED HARDER YOU MIGHT BE ABLE TO DO IT. HOWEVER, IF YOU HAVE TRIED HARD, IT COULD BE AN INDICATION OF INVALIDITY. THEN TRY THE TRUTH TABLE METHOD.

P. HOMEWORK: DO 10.12 AND 10.13 IN PART I OF THIS BOOK.

CHALLENGE: FEW PEOPLE ARE ABLE TO GET A DIRECT PROOF OF 10.12(j) USING OUR CURRENT TAUTOLOGY LIST. MANY THINK THEY GET A DIRECT PROOF BUT IT CONTAINS CIRCULAR REASONING OF USING 10.12(j) TO PROVE 10.12(j).

Q. OPPORTUNITY FOR INNER SATISFACTION (OIS): YOU HAVE EXPERIENCED THE JOY OF GETTING PROOFS. DOING DIRECT PROOF ENCAPSULATES MUCH OF THE NATURE OF ALL MATH. DISCOVERY: HOMEWORK HAS BEEN MISNAMED. IT SHOULD BE: OPPORTUNITY FOR INNER SATISFACTION HENCEFORTH DENOTED BY OIS!

R. WHY THE DIRECT PROOF PROCESS PROVES AN ARGUMENT PATTERN VALID.

1. NOTE: A WAY TO PROVE $L \to R$ TRUE IS TO ASSUME L TRUE AND SHOW R MUST BE TRUE. THIS SHOWS THE IMPOSSIBILITY OF TRUE $\to$ FALSE, THE ONLY WAY AN IMPLICATION COULD BE FALSE.

2. IN PART J OF THIS CHAPTER'S NOTES, A DIRECT PROOF WAS GIVEN FOR THE ARGUMENT PATTERN AT THE RIGHT. IT WILL BE USED TO SHOW WHY DIRECT PROOF PROVED IT VALID.

$$\begin{array}{l} p \to q \\ \sim r \to \sim q \\ \underline{p} \\ \therefore r \end{array}$$

3. NEED TO SHOW

$$[(p \to q) \wedge (\sim r \to \sim q) \wedge p] \to r$$

A TAUTOLOGY.

4. NEED TO SHOW: EVERY INSTANCE OF $[(p \to q) \wedge (\sim r \to \sim q) \wedge p] \to r$ TRUE

5. SO, ASSUME $[(p' \to q') \wedge (\sim r' \to \sim q') \wedge p'] \to r'$ IS AN <u>ARBITRARY</u> INSTANCE OF $[(p \to q) \wedge (\sim r \to \sim q) \wedge p] \to r$ AND SHOW THAT IMPLICATION IS TRUE.

6. BY STEP 1 OF R, ASSUME $[(p' \to q') \wedge (\sim r' \to \sim q') \wedge p']$ TRUE AND SHOW r' TRUE. NOW THIS IS WHAT THE DIRECT PROOF DID SINCE IT ASSUMED EACH OF $p' \to q'$, $\sim r' \to \sim q'$, AND p' TRUE AND SHOWED r' TRUE. THEREFORE, THE GIVEN ARGUMENT PATTERN IS VALID SINCE THE ASSOCIATED IMPLICATION IS A TAUTOLOGY.

S. <u>LOGICAL MISTAKES</u>! SOME ROOKIE MISTAKES WILL BE SHOWN. <u>AVOID THESE</u>!!!

1. ROOKIE MISTAKE 1:

⋮

76. $p' \to q'$ TRUE FOR SOME GOOD REASON

77. p' 76

BAD: JUST BECAUSE AN IMPLICATION IS TRUE THAT DOES NOT FORCE THE ANTECEDENT TO BE TRUE.

2. ROOKIE MISTAKE 2:

⋮

76 $p' \to q'$ TRUE FOR SOME GOOD REASON

77. q' 76

BAD: JUST BECAUSE AN IMPLICATION IS TRUE THAT DOES NOT FORCE THE CONSEQUENT TO BE TRUE.

3. ROOKIE MISTAKE 3: MAJOR MISTAKE

⋮

76. $p' \vee q'$ TRUE FOR SOME GOOD REASON

77. $\sim p' \wedge \sim q'$ 76, DEMORGAN'S LAW

VERY BAD: 77 IS THE NEGATION OF 76. A PROOF IS A LIST OF TRUE STATEMENTS. IF 76 IS TRUE, 77 MUST BE FALSE.

T. RECALL: TWO WAYS TO PROVE VALID

1. TRUTH TABLE 2. DIRECT PROOF

ONE WAY TO PROVE INVALID: TRUTH TABLE. NEXT COMES A THIRD WAY TO PROVE VALID.

INDIRECT PROOF

A. SUPPOSE YOU KNOW $E \wedge P$ IS <u>TRUE</u> AND YOU KNOW ALSO E IS <u>TRUE</u>. IS P TRUE OR FALSE? OF COURSE, P IS TRUE SINCE

<u>IF</u> P WERE FALSE,

WE WOULD HAVE TRUE $\wedge$ FALSE WHICH IS <u>FALSE</u>, BUT THE STATEMENT $E \wedge P$ IS TRUE.

IF YOU BELIEVE THE ABOVE REASONING LIKE THE AUTHORS DO, YOU BELIEVE IN...

<u>INDIRECT PROOF</u>!

B. TO PROVE A STATEMENT P TRUE BY INDIRECT PROOF:

1. <u>ASSUME</u> $\sim P$ TRUE (I.E. ASSUME P FALSE)
2. USE VALID REASONING.
3. GET <u>ANY</u> CONTRADICTION

NOTE: THESE WERE THE 3 STEPS USED IN PART A ABOVE TO CONVINCE YOU P WAS TRUE.

C. INDIRECT PROOF OF AN ARGUMENT PATTERN.

$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ q \rightarrow r \\ \hline \therefore r \end{array}$$

ASSUME THERE IS AN INSTANCE WITH TRUE PREMISES AND NEGATION OF THE CONCLUSION TRUE. GET ANY CONTRADICTION.

0. ASSUME p', q', r' ARE STATEMENTS THAT REPLACE p, q, r RESPECTIVELY.

1.	$p' \vee q'$	ASSUMED FOR INDIRECT PROOF
2.	$p' \rightarrow r'$	ASSUMED FOR INDIRECT PROOF
3.	$q' \rightarrow r'$	ASSUMED FOR INDIRECT PROOF
4.	$\sim r'$	ASSUMED FOR INDIRECT PROOF (GET ANY CONTRADICTION)
5.	$\sim p'$	2, 4, MODUS TOLLENS
6.	$\sim q'$	3, 4, MODUS TOLLENS
7.	$\sim p' \wedge \sim q'$	5, 6, TRUTH TABLE DEF. OF $\wedge$
8.	$\sim (p' \vee q')$	7, DEMORGAN'S LAW
9.	$p' \vee q'$	1

\# CONTRADICTION LINES 8, 9

WE USE # TO DENOTE CONTRADICTION

D. WHAT WE MEAN BY A CONTRADICTION IS A STATEMENT AND ITS <u>NEGATION</u> BOTH SHOWN <u>TRUE</u>. YOUR PROOF IS NOT CONSIDERED COMPLETE UNTIL SOME STATEMENT AND ITS NEGATION ARE LISTED ONE RIGHT AFTER THE OTHER (AS IN STEPS 8, 9 OF PRECEDING PROOF).

E. NOTE: THE PRECEDING PROBLEM WAS GIVEN FOR DIRECT PROOF HOMEWORK. IT IS VERY HARD TO DO BY DIRECT PROOF. AN INDIRECT PROOF IS MUCH EASIER.

F. WHY DID THE PRECEDING PROVE THE ARGUMENT PATTERN VALID?

1 $p' \vee q'$
2 $p' \to r'$
3. $q' \to r'$

WHEN THESE WERE ASSUMED TRUE, ALL THAT WOULD HAVE BEEN NEEDED TO HAVE A DIRECT PROOF WOULD HAVE BEEN TO PROVE r' TRUE. WE IN ESSENCE PROVED r' TRUE USING INDIRECT PROOF: ASSUME $\sim r'$ TRUE. WE GOT A CONTRADICTION.

G. GIVE INDIRECT PROOF: ILLUSTRATE IT CAN BE ANY CONTRADICTION

$$\begin{array}{l} p \to \sim q \\ r \to q \\ \hline \therefore \sim(p \wedge r) \end{array}$$

0. ASSUME p', q', r' ARE STATEMENTS THAT REPLACE p, q, r RESPECTIVELY
1. $p' \to \sim q'$ ASSUMED FOR INDIRECT PROOF.
2. $r' \to q'$ ASSUMED FOR INDIRECT PROOF.
3. $p' \wedge r'$ ASSUMED FOR INDIRECT PROOF. (GET ANY CONTRADICTION)
4. p' 3, TRUTH TABLE DEF. OF $\wedge$
5. r' 3, TRUTH TABLE DEF. OF $\wedge$
6. $\sim q'$ 1, 4, MODUS PONENS
7. q' 2, 5, MODUS PONENS

\# CONTRADICTION: LINES 6, 7

IN THE FIRST INDIRECT PROOF THE CONTRADICTION INVOLVED CONTRADICTING ONE OF THE PREMISES. THIS EXAMPLE ILLUSTRATES THAT IT DOES NOT HAVE TO BE A PREMISE THAT IS CONTRADICTED. NEITHER $\sim q'$ NOR q' IS A PREMISE. IT CAN BE ANY CONTRADICTION.

H. 3 WAYS TO PROVE AN ARGUMENT PATTERN VALID:
1. TRUTH TABLE
2. DIRECT PROOF
3. INDIRECT PROOF

1 WAY TO PROVE INVALID: TRUTH TABLE.

AN <u>ARGUMENT</u> WILL BE PROVED <u>VALID</u> 3 WAYS

JOE WINS IMPLIES SUE SITS.
NOT (BOB RUNS OR SUE SITS).

∴ JOE DID NOT WIN

$$\begin{array}{l} p \rightarrow q \\ \sim(r \vee q) \\ \hline \therefore \sim p \end{array}$$

1. VALID BY TRUTH TABLE

$$[(p \rightarrow q) \wedge \sim(r \vee q)] \rightarrow \sim p$$

p	$\rightarrow$	q	$\wedge$	$\sim$	r	$\vee$	q	$\rightarrow$	$\sim$	p
T	T	T	F	F	T	T	T	T	F	T
T	T	T	F	F	F	T	T	T	F	T
T	F	F	F	F	T	T	F	T	F	T
T	F	F	F	T	F	F	F	T	F	T
F	T	T	F	F	T	T	T	T	T	F
F	T	T	F	F	F	T	T	T	T	F
F	T	F	F	F	T	T	F	T	T	F
F	T	F	T	T	F	F	F	T	T	F

TAUTOLOGY: THEREFORE, VALID.

2. VALID BY <u>DIRECT</u> <u>PROOF</u>:

0. ASSUME p', q', r' ARE STATEMENTS THAT REPLACE p, q, r RESPECTIVELY

1.	$p' \to q'$	ASSUMED FOR DIRECT PROOF
2.	$\sim(r' \vee q')$	ASSUMED FOR DIRECT PROOF. (SHOW $\sim p'$ TRUE)
3.	$\sim r' \wedge \sim q'$	2, DEMORGAN'S LAW
4.	$\sim q'$	3, TRUTH TABLE DEF. OF $\wedge$
5.	$\sim p'$	1, 4, MODUS TOLLENS

3. VALID BY <u>INDIRECT</u> <u>PROOF</u>

0. ASSUME p', q', r' ARE STATEMENTS THAT REPLACE p, q, r RESPECTIVELY

1.	$p' \to q'$	ASSUMED FOR <u>INDIRECT</u> PROOF
2	$\sim(r' \vee q')$	ASSUMED FOR <u>INDIRECT</u> PROOF
2A.	p'	ASSUMED FOR <u>INDIRECT</u> PROOF (GET ANY CONTRADICTION)
3.	$\sim r' \wedge \sim q'$	2, DEMORGAN'S LAW
4.	$\sim q'$	3, TRUTH TABLE DEF. OF $\wedge$
5.	$\sim p'$	1, 4, MODUS TOLLENS
6	p'	2A

CONTRADICTION: LINES 5, 6

I. DIRECT PROOF ⟶ INDIRECT PROOF
NOT VICE VERSA.
NOTE: IN THE LAST DIRECT AND INDIRECT PROOF LINES 1, 2, 3, 4, 5 WERE ESSENTIALLY THE SAME. IF YOU CAN DO A DIRECT PROOF YOU CAN PROVE THE CONCLUSION TRUE. IN ANY INDIRECT PROOF YOU ASSUME THE <u>NEGATION</u> OF THE CONCLUSION, HENCE, AN AUTOMATIC CONTRADICTION WITH THE PROVEN TRUE CONCLUSION.

J. DO NOT USE AN INDIRECT PROOF ATTEMPT TO PROVE <u>INVALID</u>. IF YOU CANNOT DEDUCE A CONTRADICTION, POSSIBLY SOMEONE ELSE COULD.

K. HOMEWORK (OIS = OPPORTUNITY FOR INNER SATISFACTION) DO 11.7 AND 11.8 IN PART I OF THIS BOOK.

[CHAPTER 12] 12-196

OPEN SENTENCES

A. THIS STUDIES: PREDICATE CALCULUS
JUST FINISHED: PROPOSITIONAL CALCULUS
"FOR EVERY" & "THERE EXISTS" WILL BE STUDIED.

B. VARIABLES IN GENERAL

TYPE	REPLACEMENTS
STATEMENT VARIABLES	STATEMENTS
REAL VARIABLES	REAL NUMBERS

DEFINITION:

VARIABLE: PLACE-HOLDER FOR ANY ELEMENT OF A GIVEN CLASS.

1. CLASS = SET
2. STATEMENT VARIABLE: PLACE-HOLDER FOR ANY ELEMENT FROM THE CLASS OF STATEMENTS.
3. REAL VARIABLE: PLACE-HOLDER FOR ANY ELEMENT FROM THE CLASS OF REAL NUMBERS

4. PERSON VARIABLE: PLACE-HOLDER FOR ANY ELEMENT FROM THE CLASS OF PEOPLE

x DELIGHTS IN MATHEMATICS.

x IS A PERSON VARIABLE

REPLACE x WITH SAM.

SAM DELIGHTS IN MATHEMATICS

C. REPLACEMENT = CONSTANT

SO CONSTANTS REPLACE VARIABLES

ALGEBRAIC EXPRESSION: $x < 5$

VARIABLE: x CONSTANT: 2

REPLACE x WITH CONSTANT 2

INSTANCE: $2 < 5$

D. OPEN SENTENCE:

1. AN EXPRESSION WITH VARIABLES
2. NOT A STATEMENT
3. BECOMES A STATEMENT WHEN CONSTANTS REPLACE VARIABLES

E. INSTANCE OF OPEN SENTENCE:

THE STATEMENT FORMED BY REPLACING VARIABLES IN AN OPEN SENTENCE.

F. EXAMPLES:

1. OPEN SENTENCE: $x+1<10$

   INSTANCES: $7+1<10$ TRUE

   $20+1<10$ FALSE

   x IS A <u>REAL</u> VARIABLE

2. OPEN SENTENCE:

   y HAD A 90 AVERAGE.

   INSTANCE:

   SAM HAD A 90 AVERAGE.

   y IS A PERSON VARIABLE

3. OPEN SENTENCE: $p \longrightarrow q$

   INSTANCE: JOE RAN IMPLIES SUE SITS

   p, q ARE STATEMENT VARIABLES

4. OPEN SENTENCE: $x+y=10$

   INSTANCE

   $2+3=10$ FALSE

   x AND y ARE REAL VARIABLES

G. RECALL:

OPEN SENTENCE: <u>NOT</u> A STATEMENT

INSTANCE OF AN OPEN SENTENCE: STATEMENT

H. "EVERY" AND "THERE IS" ASSOCIATED WITH OPEN SENTENCES: LET $J = \{5, 6, 9\}$

1. TRUE OR FALSE: EVERY REPLACEMENT FOR x FROM J MAKES A TRUE INSTANCE OF "$x + 1 < 10$".

   FALSE: 9 IS A REPLACEMENT FOR x FROM J THAT MAKES A FALSE INSTANCE OF "$x+1<10$". ($9+1<10$ F)

2. TRUE OR FALSE: THERE IS A REPLACEMENT FOR x FROM J THAT MAKES A TRUE INSTANCE OF "$x+1<10$".

   TRUE 6 IS A REPLACEMENT FOR x FROM J THAT MAKES A TRUE INSTANCE OF "$x + 1 < 10$". ($6+1<10$ T)

3. TRUE OR FALSE: EVERY REPLACEMENT FOR x FROM J MAKES A TRUE INSTANCE OF "$x < 20 \rightarrow x$ IS ODD".

   FALSE 6 IS A REPLACEMENT FOR x FROM J THAT MAKES A FALSE INSTANCE OF "$x<20 \rightarrow x$ IS ODD". ($6<20 \rightarrow 6$ IS ODD F).

4. TRUE OR FALSE: THERE IS A REPLACEMENT FOR x FROM J THAT MAKES A TRUE INSTANCE OF "$x<20$ AND x IS NOT ODD". (RECALL $J=\{5,6,9\}$)

TRUE: 6 IS A REPLACEMENT FOR x FROM J THAT MAKES A TRUE INSTANCE OF "$x<20$ AND x IS NOT ODD". ($6<20$ AND 6 IS NOT ODD T)

I. HOMEWORK (OIS) DO 12.13 THROUGH 12.16 IN PART I OF THIS BOOK.

[CHAPTER 13]
QUANTIFIERS

A. $\forall$ IS READ "FOR EVERY", "FOR ALL", "FOR EACH", OR "FOR ANY"

B. DIFFERENCE BETWEEN THE WAY SOMETHING IS READ AND HOW IT IS DEFINED.

1. $L \subseteq R$ IS READ "L IS A SUBSET OF R"
$L \subseteq R$ CAN BE DEFINED "EVERY ELEMENT IN L IS AN ELEMENT OF R."

2\. FOR THOSE WHO HAVE HAD CALCULUS: $\lim_{x \to 3} 2x+1 = 7$ IS <u>READ</u> "THE LIMIT AS x APPROACHES 3 OF $2x+1$ IS 7".

$\lim_{x \to 3} 2x+1 = 7$ IS <u>DEFINED</u> "FOR EVERY $\varepsilon > 0$, THERE IS A $\delta > 0$ SUCH THAT FOR ALL REALS x, IF $0 < |x-3| < \delta$, THEN $|(2x+1) - 7| < \varepsilon$".

C. THE WAY SOMETHING IS <u>READ</u> CAN BE INTUITIVELY LIKE THE <u>DEFINITION</u>, BUT IT IS <u>NOT</u> THE <u>DEFINITION</u>.

D. BUILDUP TO A DEFINITION INVOLVING $\forall$.

GENERAL NOTATION FOR AN OPEN SENTENCE, $P(x)$.

LET $P(x)$ DENOTE "$x < 10$"

$P(3)$ WOULD BE "$3 < 10$"

LET $P(x)$ DENOTE "x RAN"

P(JOE) WOULD BE "JOE RAN".

P(SUE) WOULD BE "SUE RAN".

E. <u>READ</u>: $\forall x \in I, P(x)$ IS <u>READ</u> "FOR ALL x IN I, P OF x" OR "FOR EVERY x IN I, P OF x"

1. $\forall x \in I, x < 7$ IS <u>READ</u> "FOR ALL x IN I, $x < 7$".
2. $\forall y \in J, y+1 = 10$ IS <u>READ</u> "FOR ALL y IN J, $y+1 = 10$"

F. <u>DEFINED</u>: $\forall x \in I, P(x)$ IS <u>DEFINED</u> "EVERY REPLACEMENT FOR x FROM I MAKES A TRUE INSTANCE OF $P(x)$".

1. $\forall x \in I, x < 7$ IS <u>DEFINED</u> "EVERY REPLACEMENT FOR x FROM I MAKES A TRUE INSTANCE $x < 7$".
2. $\forall y \in J, y+1 = 10$ IS <u>DEFINED</u> "EVERY REPLACEMENT FOR y FROM J MAKES A TRUE INSTANCE $y+1 = 10$"

G. <u>ACADEMIC TROUBLE</u>: IF YOU CHOOSE NOT TO LEARN HOW $\forall x \in I, P(x)$ IS <u>READ</u> AND <u>DEFINED</u>, ACADEMIC TROUBLE AWAITS.

H. DRILL: READ, DEFINE, TELL TRUTH VALUE
LET $J = \{5, 6, 9\}$

1. GIVEN: $\forall x \varepsilon J, x+1 < 10$.
   READ: FOR EACH x IN J, $x+1 < 10$
   DEFINITION: EVERY REPLACEMENT FOR x FROM J MAKES A TRUE INSTANCE OF $x+1<10$.
   TRUTH VALUE: FALSE (REPLACE x WITH 9)
2. GIVEN: $\forall x \varepsilon J, (x<20 \rightarrow x$ IS ODD$)$
   READ: FOR EACH x IN J, $x<20$ IMPLIES x IS ODD.
   DEFINITION: EVERY REPLACEMENT FOR x FROM J MAKES A TRUE INSTANCE OF
   $x<20 \rightarrow x$ IS ODD.
   TRUTH VALUE: FALSE (REPLACE x WITH 6)
3. GIVEN: $\forall y \varepsilon J, y<20$.
   READ: FOR ALL y IN J, $y<20$.
   DEFINITION: EVERY REPLACEMENT FOR y FROM J MAKES A TRUE INSTANCE OF $y<20$.
   TRUTH VALUE: TRUE

I. UNIVERSAL QUANTIFIER = $\forall$

J. $\exists$ IS READ "THERE IS", "THERE EXISTS", OR "FOR SOME".

K. $\exists$ = EXISTENTIAL QUANTIFIER

L. PREDICATE CALCULUS STUDIES THE QUANTIFIERS $\forall$, $\exists$.

M. READ: $\exists x \in I \ni P(x)$ IS READ "THERE IS AN x IN I SUCH THAT P OF x" OR "FOR SOME x IN I, P OF x"

1. $\ni$ IS READ "SUCH THAT". IT IS ONLY THERE TO MAKE THE ENGLISH FLOW SMOOTHLY. IT CAN BE LEFT OUT
2. $\exists x \in I \ni x < 7$ IS READ "THERE IS AN x IN I SUCH THAT $x < 7$."
3. $\exists x \in J \ni x + 1 = 8$ IS READ "FOR SOME x IN J, $x + 1 = 8$"

N. DEFINED: $\exists x \in I \ni P(x)$ IS DEFINED "THERE IS A REPLACEMENT FOR x FROM I THAT MAKES A TRUE INSTANCE OF $P(x)$."

1. $\exists x \in I \ni x < 7$ IS DEFINED "THERE IS A REPLACEMENT FOR x FROM I THAT MAKES A TRUE INSTANCE OF $x < 7$".

2. $\exists x \in J \ni x+1=8$ IS DEFINED "THERE IS A REPLACEMENT FOR x FROM J THAT MAKES A TRUE INSTANCE OF $x+1=8$".

O. ACADEMIC TROUBLE PREVENTION: LEARN THOROUGHLY HOW $\exists x \in I \ni P(x)$ IS READ AND DEFINED.

P. THESE TWO MEAN THE SAME:

1. $\exists x \in I \ni P(x)$
2. $\exists x \in I, P(x)$

R. $\boxed{\text{HOMEWORK (OIS) DO 13.6 THROUGH 13.17 IN PART I OF THIS BOOK.}}$

S. BUILDUP TO UNDERSTOOD UNIVERSE

1. IN THE STATEMENT "$\forall x \in I, x<7$", I IS THE UNIVERSE OF REPLACEMENTS FOR THE VARIABLE x.
2. IN THE STATEMENT "$\exists y \in P \ni y+1=4$", P IS THE UNIVERSE OF REPLACEMENTS FOR THE VARIABLE y.
3. IN THE STATEMENT "$\forall x, x^2 \geq 0$" THE UNIVERSE OF REPLACEMENTS FOR THE VARIABLE x IS AN UNDERSTOOD UNIVERSE.

4. NOTATION: I_x IS THE UNDERSTOOD UNIVERSE FOR THE VARIABLE x. SIMILARLY I_y IS THE UNDERSTOOD UNIVERSE FOR THE VARIABLE y.

5 DEFINITION: $\forall x, P(x)$ MEANS $\forall x \varepsilon I_x, P(x)$. $\exists x \ni P(x)$ MEANS $\exists x \varepsilon I_x \ni P(x)$.

T. CONTEXT HELPS CLARIFY I_x. (LET R DENOTE THE SET OF REAL NUMBERS.)

1. THE SCENE: YOU ARE IN A CLASS WHERE THE ONLY NUMBERS THAT ARE TALKED ABOUT ARE REAL NUMBERS.

2. A THEOREM IS GIVEN: $\forall x, x^2 \geq 0$ THE MEANING $\forall x \varepsilon R, x^2 \geq 0$. THUS $I_x = R$.

3. EXAMPLE: $\forall x, (x \varepsilon B \rightarrow x \varepsilon E)$ MEANS $\forall x \varepsilon I_x, (x \varepsilon B \rightarrow x \varepsilon E)$ WHERE $I_x = \{y \mid y \varepsilon B \text{ OR } y \varepsilon E\}$.

4. EXAMPLE: $\forall z, (z \varepsilon W \leftrightarrow z < 7)$ MEANS $\forall z \varepsilon I_z, (z \varepsilon W \leftrightarrow z < 7)$ WHERE $I_z = \{x \mid x \varepsilon W \text{ OR } x \varepsilon R\}$

5. IN GENERAL, I_x, THE UNDERSTOOD UNIVERSE FOR x IN $\forall x, P(x)$ (OR IN $\exists x \ni P(x)$) IS THE SET OF ALL ELEMENTS THAT ARE IN ANY SET REFERRED TO IN $P(x)$.

6. GENERALLY, IN REAL LIFE, THE CONTEXT WILL MAKE I_x CLEAR; ALSO UNDERSTOOD UNIVERSE IS MOSTLY USED WITH $\forall$ QUANTIFIER WITH AN IMPLICATION OR BICONDITIONAL FOLLOWING.

U. <u>NOTATION</u>: UNLESS CONTEXT DEMANDS OTHERWISE, R DENOTES THE SET OF REAL NUMBERS AND $N = \{1, 2, 3, \ldots\}$ DENOTES THE SET OF <u>POSITIVE</u> INTEGERS (I.E. THE SET OF NATURAL NUMBERS.)

V. EXAMPLE: LET $B = \{1,2\}$ AND $E = \{2,3,4\}$

1. GIVEN: $\forall x, (x \in B \to x \in E)$.
2. EQUIVALENT TO: $\forall x \in I_x, (x \in B \to x \in E)$ WHERE $I_x = \{1,2,3,4\}$.
3. DEFINITION: EVERY REPLACEMENT FOR x FROM I_x MAKES A TRUE INSTANCE OF $x \in B \to x \in E$.
4. TRUTH VALUE: FALSE (SINCE $1 \in B \to 1 \in E$ IS A FALSE INSTANCE.)

W. TWO WAYS TO CHANGE AN OPEN SENTENCE INTO A STATEMENT

LET $B = \{1,2\}$ AND $E = \{2,3,4\}$

OPEN SENTENCE: $x \in B \to x \in E$ (NEITHER TRUE NOR FALSE. NOT A STATEMENT).

1. MAKE AN INSTANCE
   $2 \in B \to 2 \in E$ STATEMENT: TRUE
2. PREFIX WITH QUANTIFIER(S)
   $\forall x, (x \in B \to x \in E)$ STATEMENT: FALSE

X HOMEWORK (OIS). DO 13.18 THROUGH 13.20 IN PART I OF THE BOOK.

[CHAPTER 14]

OPEN SENTENCES INVOLVING TWO OR MORE VARIABLES

A. FREE AND COVERED VARIABLES

1. OPEN SENTENCE, 2 FREE VARIABLES

$$x+y=10$$

2. OPEN SENTENCE, 1 FREE VARIABLE x, 1 COVERED VARIABLE y (COVERED BY "$\exists y \in R$").

$\exists y \in R \ni x+y=10$ NOT A STATEMENT

3. INSTANCES BY REPLACING x

REPLACE x WITH	INSTANCE	TRUTH VALUE
5	$\exists y \in R \ni 5+y=10$	T
-1	$\exists y \in R \ni -1+y=10$	T
7	$\exists y \in R \ni 7+y=10$	T

4. EVERY REPLACEMENT FOR x FROM R WILL MAKE A TRUE INSTANCE OF $\exists y \in R \ni x+y=10$. HENCE, $\forall x \in R(\exists y \in R \ni x+y=10)$ IS TRUE BY <u>DEF.</u>

B. NOTE: 1. $x+y=10$ OPEN SENTENCE, TWO FREE VARIABLES, NOT A STATEMENT

2. $\exists y \in R \ni x+y=10$ OPEN SENTENCE IN ONE FREE VARIABLE, NOT A STATEMENT

3. $\forall x \in R (\exists y \in R \ni x+y=10)$ STATEMENT, TRUE, NOT AN OPEN SENTENCE

C. ORDER OF QUANTIFIERS CAN MAKE A DIFFERENCE.

CONSIDER $\exists y \in R \ni (\forall x \in R, x+y=10)$

1. $\forall x \in R, x+y=10$ OPEN SENTENCE 1 FREE VARIABLE, y.

2. INSTANCES

REPLACE y WITH	INSTANCE	TRUTH VALUE
7	$\forall x \in R, x+7=10$	F
-10	$\forall x \in R, x+(-10)=10$	F
3	$\forall x \in R, x+3=10$	F

3. NOT TRUE: THERE IS A REPLACEMENT FOR y FROM R THAT MAKES A TRUE INSTANCE OF $\forall x \in R, x+y=10$.

4. NOT TRUE: $\exists y \in R \ni (\forall x \in R, x+y=10)$ BY DEF.

5. TRUE: $\forall x \in R, (\exists y \in R \ni x+y=10)$ SEE B.3.

6. ORDER OF QUANTIFIERS IMPORTANT.

D. ANOTHER LOOK: ORDER OF QUANTIFIERS <u>CAN</u> MAKE A DIFFERENCE.

LET M = SET OF MARRIED PEOPLE

$\exists x \in M \ni [\forall y \in M,$ y IS MARRIED TO $x]$ F

$\forall y \in M, [\exists x \in M \ni y$ IS MARRIED TO $x]$ T

E. ORDER OF QUANTIFIERS <u>MIGHT NOT</u> MAKE A DIFFERENCE.

$\forall x \in R, [\exists y \in R \ni xy = 0]$ T

$\exists y \in R \ni [\forall x \in R, xy = 0]$ T

F. REMOVING GROUPING SYMBOLS

1. $\forall x \in I, [\exists y \in J \ni P(x,y)]$ MEANS $\forall x \in I, \exists y \in J \ni P(x,y)$. GROUPING SYMBOLS CAN BE UNDERSTOOD.
2. $\forall x \in N [\exists y \in R, (\forall z \in N, xy < x+z)]$ MEANS $\forall x \in N, \exists y \in R, \forall z \in N, xy < x+z$
3. $\forall x \in R, [\forall y \in R, xy = yx]$ MEANS $\forall x \in R, \forall y \in R, xy = yx$.

G. INTUITION AND RIGOR:

1. SYMBOLS THAT ARE READ SO THAT SO THAT THEY INTUITIVELY SUPPORT THE CONCEPT THEY DEFINE, CAN BE HELPFUL. A CALCULUS EXAMPLE FOLLOWS:

2. $\lim_{x \to 3} 2x = 6$ IS READ "THE LIMIT AS x APPROACHES 3 OF $2x$ EQUALS 6"

INTUITIVELY YOU CAN THINK OF x GETTING CLOSER TO 3 MAKING $2x$ GET CLOSER TO 6. FELLOW HUMANS CAN USE <u>INTUITION</u> TO GET THE ANSWER 6. TO <u>PROVE</u> YOUR INTUITION WAS CORRECT YOU WOULD HAVE TO SHOW <u>RIGOROUSLY</u> THE DEFINTION TRUE: FOR EVERY $\varepsilon > 0$, THERE IS A $\delta > 0$ SUCH THAT FOR ALL $x \in R$, IF $0 < |x-3| < \delta$, THEN $|2x-6| < \varepsilon$. IT IS OK NOT TO HAVE HAD CALCULUS. IT IS OK NOT TO UNDERSTAND THE DEFINITION. THIS POINTS OUT THAT SYMBOLISM READ SO THAT IT SUPPORTS THE DEFINITION CAN BE HELPFUL TO GET THE ANSWER. TO PROVE IT CORRECT IS ANOTHER MATTER.

H. INTUITION AND RIGOR IN DETERMINING THE TRUTH VALUE OF STATEMENTS LIKE $\forall x \in I, \exists y \in J \ni P(x,y)$

1. WE HAVE SEEN $\forall x \in R, \exists y \in R \ni x+y=10$ IS TRUE. AN INTUITIVE WAY TO GET THIS ANSWER CAN BE ARRIVED AT BY: HAVE 2 DIFFERENT DECKS OF CARDS. EACH DECK HAS 1 CARD FOR EACH REAL NUMBER. FROM LEFT TO RIGHT THE $\forall$ QUANTIFIER COMES FIRST, SO <u>RANDOMLY</u> PICK <u>ANY</u> CARD FROM THE <u>FIRST</u> DECK, TURN IT FACE UP (SAY 7 SHOWS). THE ONE WITH THE SECOND DECK THEN GETS TO LOOK AT ALL CARDS IN THE SECOND DECK TO FIND ONE THAT CAN ADD TO 7 TO GET 10. OBVIOUSLY PULL OUT CARD 3 FROM THE SECOND DECK. NOW PUT 3 BACK IN THE SECOND DECK. DO <u>NOT</u> PUT 7 BACK IN THE FIRST DECK. CONTINUE THIS PROCESS UNTIL ALL CARDS FROM THE FIRST DECK ARE PLAYED. IF THE ONE WITH THE SECOND DECK CAN ALWAYS FIND A CARD TO MAKE THE SUM 10, $\forall x \in R, \exists y \in R \ni x+y=10$ IS TRUE.

2. WE HAVE SEEN $\exists y \in R, \forall x \in R, x+y=10$ IS FALSE. INTUITIVELY, LET THE 1ST DECK OF CARDS, 1 FOR EACH REAL, BE ASSOCIATED WITH THE VARIABLE y, THE 2ND DECK WITH x. IF THE ONE WITH THE 1ST DECK CLAIMS "$\exists y \in R, \forall x \in R, x+y=10$" IS TRUE, THEY ARE CLAIMING THERE IS A CARD SO POWERFUL, THEY CAN LAY IT FACE UP, THEN EVERY CARD PLACED FACE UP BESIDE IT FROM THE SECOND DECK WILL CAUSE THE SUM TO BE 10. OBVIOUSLY NO SUCH 'SILVER BULLET' CARD EXISTS. FOR EXAMPLE IF THE PERSON WITH THE 1ST DECK THOUGHT 7 WAS THE 'SILVER BULLET' CARD AND PLAYED IT, THEN THE ONE WITH THE 2ND DECK COULD SUCCESSFULLY PLAY 3, BUT THAT IS IT. IF THE 3 WERE PICKED UP AND 5 PLAYED, $7+5=12$ NOT 10. SO EVERY CARD IN THE SECOND DECK WOULD NOT CAUSE THE SUM TO BE 10, ANY CARD FROM THE 1ST DECK BESIDES 7 WOULD ALSO FAIL. $\exists y \in R, \forall x \in R, x+y=10$ IS FALSE!

I. USE INTUITION TO FIND THE TRUTH VALUES OF THE FOLLOWING

1. $\forall x \in R, \exists y \in R \ni xy \neq 1$

   TRUE

2. $\exists x \in R, \forall y \in R \quad xy = 1$

   FALSE

3. $\exists x \in R, \forall y \in R \quad xy = 0$

   TRUE

4. $\exists x \in R, \exists y \in R, \quad xy = 2$

   TRUE

5. $\forall x \in R, \exists y \in R, \quad xy = 1$

   FALSE

J. WE LEARN LATER IN THE COURSE HOW TO <u>PROVE</u> THESE <u>INTUITIVE</u> ANSWERS CORRECT

K. HOMEWORK (OIS) DO 14.2 THROUGH 14.10 IN PART I OF THIS BOOK

[CHAPTER 15] 15-216

TRANSLATIONS TO QUANTIFIERS

A. TRANSLATION FROM EVERYDAY LANGUAGE TO PROPERLY QUANTIFIED OPEN SENTENCES WITH NO HIDDEN QUANTIFIERS

B. EXAMPLES: LET P = THE SET OF ALL PEOPLE. LET M BE THE SET OF ALL MATH COURSES.

1. ANYONE CAN LEARN LOGIC.

   $\forall x \in P$, x CAN LEARN LOGIC.

2. SOME PEOPLE CAN LEARN LOGIC.

   $\exists x \in P \ni x$ CAN LEARN LOGIC.

3. EVERY PERSON SHOULD TAKE SOME MATH COURSE.

   BAD: $\forall x \in P$, x SHOULD TAKE SOME MATH COURSE. REASON BAD $\to$ HIDDEN QUANTIFIER.

   GOOD: $\forall x \in P$, $\exists y \in M \ni x$ SHOULD TAKE y.

4. THERE IS A MATH COURSE EVERYONE SHOULD TAKE.

$\exists x \in M \ni \forall y \in P$, y SHOULD TAKE x

5. NOBODY LIKES LOGIC.

$\forall x \in P$, x DOES NOT LIKE LOGIC

C. HOMEWORK (OIS) DO 15.5 THROUGH 15.33 IN PART I OF THE BOOK, ALSO, TRANSLATE THE FOLLOWING TO PROPERLY QUANTIFIED OPEN SENTENCES WITH NO HIDDEN QUANTIFIERS.

1. EVERY RUBBER BAND HAS ASSOCIATED WITH IT A FORCE SO THAT IF IT IS STRETCHED WITH MORE THAN THAT FORCE, IT WILL SNAP. (LAURA SELL) (USE 3 QUANTIFIERS)
2. HE HAS FAILED TO SHAKE HANDS WITH EVERYONE IN THE ROOM. (NOTE: IN EVERYDAY LANGUAGE THIS COULD HAVE TWO ENTIRELY DIFFERENT INTERPRETATIONS. GIVE THEM IN PROPERY QUANTIFIED OPEN SENTENCES.)

[CHAPTER 16]

NEGATIONS

A. SUPPOSE YOU WERE ASKED TO NEGATE:
$\forall \epsilon > 0, \exists \delta > 0 \ni \forall x \in R$, if $0 < |x-3| < \delta$, then $|x^2 - 9| < \epsilon$. DOES THIS CHALLENGE YOU?
YOU WILL LEARN SOON!

B. WHAT IS THE NEGATION OF:

EVERYONE IN THE ROOM IS MALE?

1. EVERYONE IN THE ROOM IS FEMALE

OR

2. SOMEONE IN THE ROOM IS FEMALE

THE SECOND CHOICE IS CORRECT.
OBSERVE THE FORM.
LET $\mathcal{M}$ BE THE SET OF PEOPLE IN THE ROOM.

$\forall x \in \mathcal{M}$, x IS MALE

AND

$\exists x \in \mathcal{M} \ni x$ IS NOT MALE

ARE NEGATIONS OF EACH OTHER.

C. ABSTRACT FORM: $\forall x \in M, P(x)$ AND $\exists x \in M \ni \sim P(x)$ ARE NEGATIONS OF EACH OTHER.

D. LIKEWISE: $\exists x \in M, Q(x)$ AND $\forall x \in M, \sim Q(x)$ ARE NEGATIONS OF EACH OTHER.

E. NEGATION RULES SAID EQUIVALENTLY.

$\sim(\forall x \in M, P(x))$ IS EQUIVALENT TO $\exists x \in M \ni \sim P(x)$

$\sim(\exists x \in M, Q(x))$ IS EQUIVALENT TO $\forall x \in M, \sim Q(x)$.

[IN BOTH CASES, SWITCH THE QUANTIFIER AND NEGATE WHAT FOLLOWS.]

F. <u>NEGATE</u>: $\forall x \in A, x < 7$

$\exists x \in A \ni x \not< 7$, EQUIVALENTLY

$\exists x \in A \ni x \geq 7$

G. <u>NEGATE</u>: $\exists x \varepsilon P \ni x > 5$

$\forall x \varepsilon P,\ x \leq 5$

H. <u>NEGATE</u>: $\forall x \varepsilon P, (x \varepsilon A \text{ OR } x \notin B)$

$\exists x \varepsilon P, \sim(x \varepsilon A \text{ OR } x \notin B)$

$\exists x \varepsilon P, (x \notin A \text{ AND } x \varepsilon B)$

I. <u>NEGATE</u>: $\forall x \varepsilon A, \exists y \varepsilon B \ni x < y$.

$\exists x \varepsilon A, \sim(\exists y \varepsilon B, x < y)$

$\exists x \varepsilon A, \forall y \varepsilon B, x \geq y$.

THE INTERMEDIATE STEPS WILL NOW BE LEFT OUT

J. <u>NEGATE</u>: $\forall x \varepsilon R, \exists y \varepsilon B, (x > 7 \rightarrow y > 3)$.

$\exists x \varepsilon R, \forall y \varepsilon B, (x > 7 \wedge y \leq 3)$

K. NOW LET'S REVISIT THE CHALLENGING ONE AT THE START OF THIS CHAPTER

NEGATE: $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in R$, if $0 < |x-3| < \delta$, then $|x^2-9| < \varepsilon$.

ANSWER: $\exists \varepsilon > 0, \forall \delta > 0, \exists x \in R \ni$ $0 < |x-3| < \delta$ and $|x^2-9| \geq \varepsilon$

L. NEGATE: EVERY PERSON SHOULD TAKE SOME MATH COURSE.

1. FIRST TRANSLATE: P = SET OF PEOPLE
   M = SET OF MATH COURSES
   $\forall x \in P, \exists y \in M$, x SHOULD TAKE y
2. NEGATION: $\exists x \in P, \forall y \in M$, x SHOULD NOT TAKE y.
3. TRANSLATE BACK TO EVERYDAY ENGLISH IF DESIRED: THERE IS A PERSON THAT SHOULD TAKE NO MATH COURSE.

M. TRUTH VALUE OF: $\forall x \in \phi, P(x)$
WHAT IS IT?

1. NEGATION OF $\forall x \in \phi, P(x)$ IS $\exists x \in \phi \ni \sim P(x)$ THIS IS OBVIOUSLY FALSE SINCE IT CLAIMS, AMONG OTHER THINGS THAT THE EMPTY SET HAS AN ELEMENT! SINCE IT IS FALSE, ITS NEGATION $\forall x \in \phi, P(x)$ IS TRUE!
2. $\forall x \in \phi, P(x)$ IS TRUE (SOMETIMES CALLED VACUOUSLY TRUE.)
3. <u>EXAMPLES</u>:
   a. $\forall x \in \phi, x > 20$ IS TRUE.
   b. $\forall x \in \phi, x < 3$ IS TRUE.
   c. $\forall x \in \phi, x \geq 3$ IS TRUE.

N. HOMEWORK (OIS) DO 16.16 THROUGH 16.28 IN PART I OF THIS BOOK.

THIS ENDS PART II OF THIS BOOK. FROM NOW ON, THE NOTES ARE ALL THERE IS. THERE IS NO REGULAR BOOK TO REFER TO.

PART III

OVERHEADS FOR THE TEACHER AND NOTES FOR THE STUDENT OVER MATERIAL <u>NOT</u> IN PART I. THIS IS ALL THERE IS. IT SHOULD BE ENOUGH FOR COMPETENT STUDENTS THAT HAVE A COMPETENT TEACHER.

[CHAPTER 17]

DIRECT PROOF OF IF..., THEN... STATEMENTS.

A. AS SEEN IN CHAPTER 10: A WAY TO PROVE $L \to R$ TRUE IS TO ASSUME L TRUE (SHOW R TRUE)

B. DRILL ON HOW TO <u>START</u> "IF.., THEN.." PROOFS

1. PROVE: IF $p \varepsilon E$, THEN $p < 7$
   ASSUME $p \varepsilon E$ (SHOW $p < 7$).
2. PROVE: $p \varepsilon W \longrightarrow p \varepsilon Q$.
   ASSUME $p \varepsilon W$ (SHOW $p \varepsilon Q$)
3. PROVE: IF $p \varepsilon M$, THEN $(q \varepsilon A \longrightarrow q \varepsilon B)$
   a. ASSUME $p \varepsilon M$ (SHOW $q \varepsilon A \to q \varepsilon B$)
   b. ASSUME $q \varepsilon A$ (SHOW $q \varepsilon B$).

C. <u>NOTE</u>: YOU NOW KNOW HOW TO AT LEAST START A PROOF STATED IN $L \to R$ FORM. YOU CAN USE WHAT YOU ASSUME. YOU DO NOT USE WHAT YOU ARE TRYING TO SHOW... THAT IS YOUR GOAL, WHERE YOU ARE GOING.

D. A COMPLETE IF..., THEN... PROOF.

PROVE: IF $h < 5$, THEN $2h - 7 < 8$

A_1 1. ASSUME $h < 5$. (SHOW $2h - 7 < 8$)

2. $2h < 10$ 1, MULT. BY 2

3. $2h - 7 < 3$ 2, SUBTRACT 7

4. $3 < 8$ ARITHMETIC KNOWLEDGE

5. $2h - 7 < 8$ 3,4, TRANSITIVITY

A_1^* 6. IF $h < 5$, THEN $2h - 7 < 8$ 1-5

<u>NOTE</u>: A_1 DENOTES THE START IN THE PROOF OF THE <u>FIRST</u> ASSUMPTION (HENCE, THE SUBSCRIPT <u>1</u>). A_1^* MEANS STARTING NOW WE ARE NO LONGER CONSIDERING ASSUMPTION 1 TRUE. WE WILL NOW PHILOSOPHIZE WHAT HAS BEEN PROVEN BY LOOKING AT THIS ASSUMPTION AND ITS CONSEQUENCES

<u>NOTE</u>: IN STEP 5 WE SHOWED WHAT WE WERE TRYING TO SHOW. OUR GOAL OF SHOWING $2h-7<8$ WAS SATISFIED.

E. SET NOTATION

1. $\{1, 2\}$ IS READ "THE SET WHOSE ELEMENTS ARE 1 AND 2".
2. $\{x \mid x < 10\}$ IS READ "THE SET OF ALL x SUCH THAT $x < 10$."
3. $\{4\}$ IS READ "THE SET WHOSE ONLY ELEMENT IS 4".

F. THE DUMMY VARIABLE IN SET BUILDER NOTATION

1. $\{x \mid x < 10\}$ IS CALLED SET BUILDER NOTATION. x IS A DUMMY VARIABLE
2. LET $E = \{x \mid x < 10\}$. $E = \{y \mid y < 10\}$. $E = \{z \mid z < 10\}$.
3. THINGS YOU CAN DO IN A PROOF:

   a. 27. $p \in E$ GIVEN

      WHAT CAN YOU SAY NEXT?

      28. $p < 10$ 27, DEFINITION OF E

   b. 32. $5 < 10$ ARITHMETIC KNOWLEDGE

      33. $5 \in E$ 32, DEFINITION OF E.

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RECALL $E = \{x \mid x < 10\}$

c. 47 $W = \{x \mid x \in E \text{ AND } x > 4\}$ GIVEN

48. $q \in W$ GIVEN

WHAT CAN YOU SAY NEXT?

49. $q \in E$ AND $q > 4$ 48, 47, DEF. OF W

50. $q \in E$ 49, TT* DEF OF $\wedge$

51. $q < 10$ 50, DEF. OF E

---

d. 72. $4.2 < 10$ ARITHMETIC KNOWLEDGE

73. $4.2 \in E$ 72, DEF. OF E.

74. $4.2 > 4$ ARITHMETIC KNOWLEDGE

75. $4.2 \in E$ AND $4.2 > 4$ 73, 74, TT DEF. OF $\wedge$

76 $4.2 \in W$ 75, 47, DEF OF W.

G. THE LISTING METHOD FOR SETS

1. $\{1,2\} = \{2,1\} = \{1,2,2,1,1\}$

2. $\{1,2\}$ SET DEFINED BY LISTING METHOD

3. THINGS YOU CAN DO IN A PROOF

93. $M = \{1,2\}$ GIVEN

94 $d \in M$ GIVEN

WHAT CAN YOU SAY NEXT

95. $d = 1$ OR $d = 2$ 93, 94, DEF. OF M

YOU COULD ALSO FROM LOOKING AT 93 SAY $2 \in M$.

*TT ABBREVIATES "TRUTH TABLE"

H. AN EXAMPLE OF AN IF.., THEN.. PROOF INVOLVING SETS.

$P = \{x \mid x$ IS A POSITIVE INTEGER AND $x < 10\}$

$L = \{x \mid x \in R$ AND $x > 7\}$

(RECALL R = Reals N = POSITIVE INTEGERS)

PROVE: IF $d \in P$ AND $d > 8$, THEN $d \in L$

A_1 1. ASSUME $d \in P$ AND $d > 8$. (Show $d \in L$)
(SHOW $d \in R$ AND $d > 7$)

2. $d \in P$ — 1, TT DEF. OF $\wedge$

3. d IS A POSITIVE INTEGER AND $d < 10$ — 2, DEF P

4. d IS A POSITIVE INTEGER — 3, TT DEF. OF $\wedge$

5. $d \in R$ — 4, EVERY POS. INT. IS REAL

6. $d > 8$ — 1, TT DEF $\wedge$

7. $8 > 7$ — ARITHMETIC KNOWLEDGE

8. $d > 7$ — 6, 7, TRANSITIVITY

9. $d \in R$ AND $d > 7$ — 5, 8, TT DEF OF $\wedge$

10. $d \in L$ — 9, DEF OF L

A_1^* 11. IF $d \in P$ AND $d > 8$, THEN $d \in L$ — 1-10

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I. HOMEWORK (OIS)

1. PROVE: IF $e > 4$, THEN $3e - 5 > 2$

2. PROVE: IF $g < 7$, THEN (IF $g < 5$, THEN $3g - 10 < 8$)

3. LET $B = \{x \mid x \in R \wedge x > 10\}$

   $W = \{x \mid x \in R \wedge x > 9 \wedge x \in M\}$

   $M = \{x \mid x \in R \wedge x < 21\}$

   PROVE: IF ($K \in B$ AND $K < 15$), THEN $K \in W$.

[CHAPTER 18]

DIRECT PROOF OF "EVERY" STATEMENTS

A. "EVERY" STATEMENTS ARE AS FOLLOWS:

1. EVERY ELEMENT OF A IS IN B.
2. EVERY NUMBER LESS THAN 20 IS IN T
3. EVERY ELEMENT OF SET I HAS PROPERTY P

B. A WAY TO DIRECTLY PROVE "EVERY ELEMENT OF SET E HAS A CERTAIN PROPERTY":

ASSUME $p \in E$ (SHOW p HAS THE CERTAIN PROPERTY)

↑

CHOOSE ANY ARBITRARY SYMBOL THAT IS NON-AMBIGUOUS.

C. DRILL ON STARTING "EVERY" PROOFS

1.) PROVE: EVERY ELEMENT OF **E** IS IN B.

A_1 1. ASSUME $p \in E$ (SHOW $p \in B$)

OR

A_1 1. ASSUME $q \in E$ (SHOW $q \in B$)

STARTING "EVERY" PROOFS (CONTINUED)

2). <u>PROVE</u>: EVERY REAL NUMBER LESS THAN 20 IS IN T.

A, 1. ASSUME w IS A REAL NUMBER LESS THAN 20 (SHOW $w \in T$)

D. EXAMPLE "EVERY" PROOF.

LET $H = \{x \mid x < 10\}$ AND $K = \{x \mid x < 20\}$

<u>PROVE</u>: EVERY ELEMENT OF H IS IN K.

A_1 1.	ASSUME $p \in H$ (SHOW $p \in K$)	
2.	$p < 10$	1, DEF. OF H
3.	$10 < 20$	ARITHMETIC KNOWLEDGE
4.	$p < 20$	2, 3, TRANSITIVITY.
5.	$p \in K$	4, DEF. OF K
A_1^* 6.	EVERY ELEMENT OF H IS IN K	1-5

E. WHY DOES PROVING IT FOR p (IN THE JUST FINISHED PROOF) PROVE IT FOR <u>EVERY</u> ELEMENT OF H?

ANSWER: THE SYMBOL p WAS

ARBITRARILY SELECTED. THE ONLY CLAIMS MADE ABOUT p WERE TRUE FOR ALL ELEMENTS OF H. HENCE, ANY SPECIFIC ELEMENT FROM H COULD BE SUBSTITUTED FOR p AND ALL STATEMENTS REMAIN TRUE INCLUDING 5 THAT SAYS IT IS IN K. (BESIDE EXAMINING EACH ELEMENT OF H ONE AT A TIME AND PROVING IT IN K WOULD TAKE AN INFINITE AMOUNT OF TIME!)

F. <u>PROVE</u>: IF EVERY ELEMENT OF B IS IN E AND EVERY ELEMENT OF E IS IN G, THEN EVERY ELEMENT OF B IS IN G.

A_1 1. ASSUME EVERY ELEMENT OF B IS IN E AND EVERY ELEMENT OF E IS IN G (SHOW EVERY ELEMENT OF B IS IN G)

A_2 2. ASSUME $\omega \in B$ (SHOW $\omega \in G$)

3. EVERY ELEMENT OF B IS IN E (1, TT DEF $\wedge$)

4. $\omega \in E$ 2, 3, INSTANCE OF 3

5. EVERY ELEMENT OF E IS IN G (1, TT DEF OF $\wedge$)

6. $\omega \in G$ 4, 5, INSTANCE OF 5

A_2^* 7. EVERY ELEMENT OF B IS IN G 2–6

A_1^* 8. IF EVERY ELEMENT OF B IS IN E AND EVERY ELEMENT OF E IS IN G, THEN EVERY ELEMENT OF B IS IN G 1, 7

G. <u>NOTE:</u> INSTANCES OF "EVERY" STATEMENTS.

FORM THAT PRODUCES AN INSTANCE:

YOU HAVE A SPECIFIC ELEMENT OF B

YOU KNOW EVERY ELEMENT OF B HAS A CERTAIN PROPERTY.

SO AS AN INSTANCE, YOU KNOW THE SPECIFIC ELEMENT OF B HAS THE CERTAIN PROPERTY.

IN THE LAST PROOF WE SAW IT AS

2. $\omega \in B$

3. EVERY ELEMENT OF B IS IN E

THEN CAME THE INSTANCE

4. $\omega \in E$ 2, 3 INSTANCE OF 3

H. HOMEWORK (OIS) LET

$T = \{x \mid x \in R \wedge x < 20 \wedge x \in B\}$

$B = \{x \mid x \in R \wedge x > 10\}$

$W = \{x \mid x \in R \wedge x > 9 \wedge x \in M\}$

$M = \{x \mid x \in R \wedge x < 21\}$

1. PROVE: EVERY REAL NUMBER LESS THAN 20 IS IN M.

2. PROVE: EVERY REAL NUMBER GREATER THAN 5, WHEN DOUBLED, IS IN B.

3. PROVE: EVERY ELEMENT OF T IS IN W.

4. PROVE: IF EVERY REAL NUMBER LESS THAN 30 IS IN E, THEN EVERY ELEMENT OF M IS IN E.

5. PROVE: IF $g \in B$, THEN EVERY REAL NUMBER LESS THAN 7 IS LESS THAN g.

[CHAPTER 19] 19-235

DIRECT PROOF OF $\forall x \in I, P(x)$ FORM

A. PROVE THE FOLLOWING "EVERY" STATEMENT
LET $H = \{y \mid y < 10\}$ <u>PROVE</u>: EVERY REPLACEMENT FOR x FROM H MAKES A TRUE INSTANCE OF $2x - 1 < 40$.

A, 1. ASSUME q IS A REPLACEMENT FOR x FROM H. (SHOW q MAKES A TRUE INSTANCE OF $2x - 1 < 40$, I.E. SHOW $2q - 1 < 40$)

2.	$q \in H$	1
3.	$q < 10$	2, DEF. OF H
4.	$2q < 20$	3, MULT. BY 2
5.	$2q - 1 < 19$	4, SUBT. 1
6.	$19 < 40$	ARITHMETIC KNOWLEDGE
7.	$2q - 1 < 40$	5, 6, TRANSITIVITY

A,* 8. EVERY REPLACEMENT FOR x FROM H MAKES A TRUE INSTANCE OF $2x - 1 < 40$. 1-7

NOTE: WE PROVED BY <u>DEFINITION</u> $\forall x \in H, 2x - 1 < 40$.

B. NOTE q WAS SYMBOL CHOSEN TO REPLACE x IN THE PREVIOUS PROOF. THE SYMBOL USED TO REPLACE x NEEDS TO BE DIFFERENT FROM x. A SIMILAR PROOF WILL NOW BE GIVEN USING THE SYMBOL x' TO REPLACE x INSTEAD OF q.

A, 1. ASSUME x' IS A REPLACEMENT FOR x FROM H. (SHOW x' MAKES A TRUE INSTANCE OF $2x-1<40$, I.E. SHOW $2x'-1<40$)

2.	$x' \in H$	1
3.	$x' < 10$	2, DEF. OF H
4.	$2x' < 20$	3, MULT. BY 2
5.	$2x'-1 < 19$	4, SUBTRACT 1
6.	$19 < 40$	ARITHEMETIC KNOWLEDGE
7.	$2x'-1 < 40$	5,6, TRANSITIVITY

A_1^* 8. EVERY REPLACEMENT FOR x FROM H MAKES A TRUE INSTANCE OF $2x-1<40$. 1-7

<u>NOTE:</u> WE PROVED BY <u>DEFINITION</u> $\forall x \in H, 2x-1<40$ AGAIN! SUMMARY OF WHAT WE DID:

ASSUME $x' \in H$ (SHOW $2x'-1<40$)

C. TO PROVE DIRECTLY: $\forall x \in I,\ P(x)$

1. ASSUME $x' \in I$.
2. SHOW $P(x')$ TRUE.

D. DRILL ON HOW TO START $\forall x \in I,\ P(x)$ PROOFS:

a. PROOF START FOR: $\forall x \in R,\ x^2 \geq 0$

A_1 1. ASSUME $x' \in R$ (SHOW $(x')^2 \geq 0$)

b. PROOF START FOR: $\forall x \in A,\ x \in B$

A_1 1. ASSUME $x' \in A$ (SHOW $x' \in B$)

c. PROOF START FOR: $\forall x \in E,\ (x \in B \rightarrow x \in T)$.

A_1 1. ASSUME $x' \in E$ (SHOW $x' \in B \rightarrow x' \in T$).

A_2 1. ASSUME $x' \in B$ (SHOW $x' \in T$)

d. PROOF START FOR:

IF $7 \in T$, THEN $\forall x \in E,\ x \in G$

A_1 1. ASSUME $7 \in T$. (SHOW $\forall x \in E,\ x \in G$)

A_2 2. ASSUME $x' \in E$. (SHOW $x' \in G$)

e. PROOF START FOR:

IF $(\forall x \in A,\ x \in B)$, THEN $(\forall x \in E,\ x \in G)$.

A_1 1. ASSUME $\forall x \in A,\ x \in B$. (SHOW $\forall x \in E,\ x \in G$)

A_2 2. ASSUME $x' \in E$ (SHOW $x' \in G$)

f. PROOF START FOR: $\forall y \in E, y \in T$

A_1 1. ASSUME $y' \in E$ (SHOW $y' \in T$)

g. PROOF START FOR: $\forall x \in E, \forall y \in T, x+y<10$.

A_1 1. ASSUME $x' \in E$. (SHOW $\forall y \in T, x'+y<10$)

A_2 2. ASSUME $y' \in T$. (SHOW $x'+y'<10$)

h. PROOF START FOR:

$\forall x \in E$, (IF $x<3$, THEN $\forall y \in B, x+y>5$).

A_1 1. ASSUME $x' \in E$ (SHOW IF $x'<3$, THEN $\forall y \in B, x'+y>5$)

A_2 2. ASSUME $x'<3$ (SHOW $\forall y \in B, x'+y>5$)

A_3 3. ASSUME $y' \in B$ (SHOW $x'+y'>5$)

E. OUR PLAN FOR VARIABLE REPLACEMENT:

TO PROVE $\forall x \in I, P(x)$, BY <u>DEFINITION</u> YOU HAVE TO PROVE: EVERY REPLACEMENT FOR x FROM I MAKES A TRUE INSTANCE OF $P(x)$. SO ASSUME $x' \in I$ (SHOW $P(x')$).

FOR PROOF OF $\forall z \in I, P(z)$:

ASSUME $z' \in I$ (SHOW $P(z')$)

FOR PROOF OF $\forall q \in I, P(q)$:

ASSUME $q' \in I$ (SHOW $P(q')$)

NOTE: q' IS CONSIDERED A DIFFERENT SYMBOL FROM q

F. LET $H = \{y \mid y < 10\}$. PROVE $\forall y \in H$, $2y - 1 < 40$.

A_1 1. ASSUME $y' \in H$ (SHOW $2y' - 1 < 40$)

2. $y' < 10$ 1, DEF. OF H

3. $2y' < 20$ 2, MULT. BY 2

4. $2y' - 1 < 19$ 3, SUBTRACT 1

5. $19 < 40$ ARITHMETIC KNOWLEDGE

6. $2y' - 1 < 40$ 4,5, TRANSITIVITY

A_1^* 7. $\forall y \in H$, $2y - 1 < 40$ 1–6

G. PROVE: $\forall y \in R$, (IF $y < 3$, THEN $2y - 1 < 5$).

A_1 1. ASSUME $y' \in R$ (SHOW IF $y' < 3$, THEN $2y' - 1 < 5$)

A_2 2. ASSUME $y' < 3$ (SHOW $2y' - 1 < 5$)

3. $2y' < 6$ 2, MULT. BY 2

4. $2y' - 1 < 5$ 3, SUBTRACT 1

A_2^* 5. IF $y' < 3$, THEN $2y' - 1 < 5$ 2–4

A_1^* 6. $\forall y \in R$, (IF $y < 3$, THEN $2y - 1 < 5$)   1,5

H. INSTANCES OF $\forall x \in I, P(x)$

a. SUPPOSE YOU ARE GIVEN:

1. $4 \in A$ GIVEN
2. $\forall x \in A, x \in B$ GIVEN

WHAT CAN YOU SAY?

3. $4 \in B$ 1, 2, INSTANCE OF 2

b. SUPPOSE YOU ARE GIVEN:

1. $2 \in T$ GIVEN
2. $\forall y \in T$, (IF $y \in B$, THEN $y \in E$) GIVEN

WHAT CAN YOU SAY?

3. IF $2 \in B$, THEN $2 \in E$ 1, 2, INSTANCE OF 2

c. SUPPOSE YOU ARE GIVEN:

1. $x' \in T$ GIVEN
2. $\forall x \in T, x < 7$ GIVEN

WHAT CAN YOU SAY?

3. $x' < 7$ 1, 2, INSTANCE OF 2

D. SUPPOSE YOU ARE GIVEN:

1. $x' \in T$ GIVEN
2. $\forall x \in T, \forall y \in E, x + y < 10$ GIVEN

WHAT CAN YOU SAY?

3. $\forall y \in E, x' + y < 10$ 1, 2, INSTANCE OF 2

I. USING $\forall x \in I, P(x)$ IN A PROOF:

PROVE: IF $4 \in E$ AND ($\forall x \in E, 2x \in T$) AND ($\forall x \in T, x+3 \in G$), THEN $11 \in G$.

A, 1. ASSUME $4 \in E$ AND ($\forall x \in E, 2x \in T$) AND ($\forall x \in T, x+3 \in G$). (SHOW $11 \in G$)

2.	$4 \in E$	1, TT DEF. OF AND
3.	$\forall x \in E, 2x \in T$	1, TT DEF. OF AND
4.	$2(4) \in T$	2, 3, INSTANCE OF 3
5.	$8 \in T$	4
6.	$\forall x \in T, x+3 \in G$	1, TT DEF. OF AND.
7.	$8+3 \in G$	5, 6, INSTANCE OF 6
8.	$11 \in G$	7

A,* 9. IF $4 \in E$ AND ($\forall x \in E, 2x \in T$) AND ($\forall x \in T, x+3 \in G$), THEN $11 \in G$ 1–8

J. PROVE: IF $[(\forall x \varepsilon A, x \varepsilon B)$ AND $(\forall x \varepsilon A, x \varepsilon E)]$, THEN $[\forall x \varepsilon A, (x \varepsilon B \wedge x \varepsilon E)]$.

A_1 1. ASSUME $(\forall x \varepsilon A, x \varepsilon B)$ AND $(\forall x \varepsilon A, x \varepsilon E)$ (SHOW $\forall x \varepsilon A, (x \varepsilon B \wedge x \varepsilon E)$)

A_2 2. ASSUME $x' \varepsilon A$ (SHOW $x' \varepsilon B \wedge x' \varepsilon E$)

3. $\forall x \varepsilon A, x \varepsilon B$ — 1, TT DEF. OF AND

4. $x' \varepsilon B$ — 2, 3, INSTANCE OF 3

5. $\forall x \varepsilon A, x \varepsilon E$ — 1, TT DEF. OF AND

6. $x' \varepsilon E$ — 2, 5, INSTANCE OF 5

7. $x' \varepsilon B \wedge x' \varepsilon E$ — 4, 6, TT DEF OF $\wedge$

A_2^* 8. $\forall x \varepsilon A, (x \varepsilon B \wedge x \varepsilon E)$ — 2–7

A_1^* 9. IF $[(\forall x \varepsilon A, x \varepsilon B)$ AND $(\forall x \varepsilon A, x \varepsilon E)]$, THEN $[\forall x \varepsilon A, (x \varepsilon B \wedge x \varepsilon E)]$ — 1, 8

K. CAN <u>YOU</u> DEDUCE? SOME MIGHT SAY THE ABOVE IS OBVIOUS AND DOES NOT NEED TO BE PROVEN. HOWEVER, MANY CANNOT DO A PROOF LIKE ABOVE. A PURPOSE OF THE BOOK SO FAR HAS BEEN TO TRAIN YOU WHERE YOU CAN DO CLEAR DEDUCTIVE THOUGHT.

L. HOMEWORK (OIS)

1. LET $T = \{y \mid y < 8\}$. PROVE: $\forall x \in T,\ 3x - 2 < 25$.

2. PROVE: $\forall x \in R$, IF $x < 7$, THEN $4x - 1 < 32$.

3. INSTANCE: GIVEN $x' \in M$
   GIVEN: $\forall x \in M$, (IF $x \in T$, THEN $x \in W$.)
   NAME AN INSTANCE.

4. PROVE: IF $(\forall x \in H, [x \in K \wedge x \in L])$, THEN
   $(\forall x \in H, [x \in L \vee x \in F])$

5. PROVE: IF $[(\forall x \in A, x > 5)$ AND $(\forall x \in B, x > 6)]$,
   THEN $\forall g \in B, \forall z \in A,\ 2z + 3g > 28$.

6. INSTANCE: GIVEN $y' \in E$
   GIVEN $\forall y \in E, \forall z \in B, (x + y < 10 \rightarrow y \in T)$
   NAME AN INSTANCE

[CHAPTER 20]

PROOF OF $L \leftrightarrow R$

A. A WAY TO PROVE $L \leftrightarrow R$ IS TO PROVE $(L \rightarrow R)$ AND $(R \rightarrow L)$

1. NOTE: THIS IS A CONJUCTION TO BE PROVEN TRUE.
2. PROVE $L \rightarrow R$ TRUE. THEN PROVE $R \rightarrow L$ TRUE. THEN JOIN TOGETHER WITH AND.

B. AN $\longleftrightarrow$ PROOF

PROVE: $p > 3 \Longleftrightarrow 2p-6 > 0$

(SHOW $p > 3 \longrightarrow 2p-6 > 0$)

A_1 1. ASSUME $p > 3$ (Show $2p-6 > 0$)
2. $2p > 6$ — 1, MULT. BY 2
3. $2p-6 > 0$ — 2, SUBTRACT 6

A_1^* 4. $p > 3 \longrightarrow 2p-6 > 0$ — 1–3

(SHOW $2p-6 > 0 \longrightarrow p > 3$)

A_2 5. ASSUME $2p-6 > 0$ (SHOW $p > 3$)
6. $2p > 6$ — 5, ADD 6
7. $p > 3$ — 6, DIVIDE BY 2

A_2^* 8 $2p-6 > 0 \longrightarrow p > 3$ — 5–7

9. $(p > 3 \longrightarrow 2p-6 > 0)$ AND $(2p-6 > 0 \longrightarrow p > 3)$ — 4, 8, TT DEF. OF $\wedge$

10. $p > 3 \longleftrightarrow 2p-6 > 0$ — 9, DEF OF $\longleftrightarrow$

C. PROVE: $\forall x \in R, (2x<6 \leftrightarrow 3x-9<0)$

A_1 1. ASSUME $x' \in R$ (SHOW $2x'<6 \leftrightarrow 3x'-9<0$)

(SHOW $2x'<6 \longrightarrow 3x'-9<0$)

A_2 2. ASSUME $2x'<6$ (SHOW $3x'-9<0$)

3. $x'<3$ — 2, DIVIDE BY 2
4. $3x'<9$ — 3, MULTIPLY BY 3
5. $3x'-9<0$ — 4, SUBTRACT 9

A_2^* 6 $2x'<6 \longrightarrow 3x'-9<0$ — 2-5

(SHOW $3x'-9<0 \longrightarrow 2x'<6$)

A_3 7. ASSUME $3x'-9<0$ (SHOW $2x'<6$)

8 $3x'<9$ — 7, ADD 9
9. $x'<3$ — 8, DIVIDE BY 3
10. $2x'<6$ — 9, MULTIPLY BY 2

A_3^* 11. $3x'-9<0 \rightarrow 2x'<6$ — 7-10

12 $(2x'<6 \rightarrow 3x'-9<0)$ AND $(3x'-9<0 \rightarrow 2x'<6)$ — 6, 11, TT DEF OF $\wedge$

13. $2x'<6 \longleftrightarrow 3x'-9<0$ — 12, DEF OF $\leftrightarrow$

A_1^* 14 $\forall x \in R, (2x<6 \longleftrightarrow 3x-9<0)$ — 1, 13

D. HOMEWORK (OIS)

1. PROVE: $4b > 12 \leftrightarrow 15 < 5b$
2. LET $B = \{x \mid x < 2\}$ PROVE:
   $\forall x \varepsilon R, (0 > 2x - 4 \leftrightarrow x \varepsilon B)$
3. SUPPOSE YOU ARE GIVEN $p \varepsilon T \leftrightarrow p < 5$. SUPPOSE YOU ARE ALSO GIVEN $p \varepsilon T$. THEN YOU KNOW $p < 5$ BECAUSE THE ONLY WAY TO HAVE A TRUE BICONDITIONAL WITH THE LEFT PART TRUE IS TO HAVE THE RIGHT PART TRUE ALSO. YOU MAY NEED TO USE AN IDEA LIKE THIS IN THE NEXT PROOF. PROVE:

   IF $[\forall x \varepsilon A, (x \varepsilon B \leftrightarrow x < 5)]$, THEN
   $[\forall z \varepsilon B, (z \varepsilon A \rightarrow 2z < 10)]$

[CHAPTER 21] 21-247

SUBSET

A. DEFINITION: A IS A SUBSET OF B, DENOTED $A \subseteq B$, IFF $\forall x \in A, x \in B$.

B. OBSERVATIONS: $A \subseteq B$ MEANS EVERY ELEMENT IN A IS IN B (BUT NOT NECESSARILY VICE VERSA).

1. LET $E = \{1,2\}$ AND $F = \{1,2,3,4\}$
   $E \subseteq F$
2. NOTE $E \subseteq E$. ANY SET IS A SUBSET OF ITSELF. SO IF $M \subseteq Q$ IT IS POSSIBLE THAT $M = Q$, BUT ALSO POSSIBLE $M \neq Q$.

C. LET $B = \{x \mid x < 9\}$ PROVE:
IF$(\forall x \in A, x < 7)$, THEN $A \subseteq B$

A_1 1. ASSUME $\forall x \in A, x < 7$ (SHOW $A \subseteq B$)
(SHOW $\forall x \in A, x \in B$)

A_2 2. ASSUME $x' \in A$ (SHOW $x' \in B$)

3. $x' < 7$ 2, 1, INSTANCE OF 1

4. $7 < 9$ ARITHMETIC KNOWLEDGE

5. $x' < 9$ 3, 4, TRANSITIVITY.

6. $x' \in B$ 5, DEFINITION OF B

A_2^* 7. $\forall x \in A, x \in B$ 2-6

8. $A \subseteq B$ 7, DEFINITION OF $\subseteq$

A_1^* 9. IF $(\forall x \in A, x < 7)$, THEN $A \subseteq B$.

D. LET $B = \{x \mid x < 9\}$ AND $T = \{x \mid 2x - 3 < 20\}$

PROVE: $B \subseteq T$ (SHOW $\forall x \in B, x \in T$)

A_1 1. ASSUME $x' \in B$ (SHOW $x' \in T$)

(SHOW $2x' - 3 < 20$)

2. $x' < 9$ 1, DEFINITION OF B

3. $2x' < 18$ 2, MULTIPLY BY 2

4. $2x' - 3 < 15$ 3, SUBTRACT 3

5. $15 < 20$ ARITHMETIC KNOWLEDGE

6. $2x' - 3 < 20$ 4, 5, TRANSITIVITY

7. $x' \in T$ 6, DEFINITION OF T

A_1^* 8. $\forall x \in B, x \in T$ 1—7

9 $B \subseteq T$ 8, DEFINITION OF $\subseteq$

21-249

E. HOMEWORK (OIS)

1. LET $T = \{x \mid x < 5\}$ PROVE:
IF $(\forall x \in E, 2x < 9)$, THEN $E \subseteq T$

2. PROVE: IF $B \subseteq E$ AND $E \subseteq G$,
THEN $B \subseteq G$.

3. LET $I = \{x \mid x \in E \wedge x \in T\}$ PROVE:
IF $[\forall x \in E, (x \notin T \vee x \in M)]$, THEN $I \subseteq M$.

[CHAPTER 22] 22-250

UNION AND INTERSECTION

A. $L \cup R$ IS READ "L UNION R".

B. $L \cap R$ IS READ "L INTERSECTION R".

C. DEFINITION

$$L \cup R = \{x \mid x \in L \text{ OR } x \in R\}$$
$$= \{x \mid x \text{ IS IN AT LEAST ONE OF THE SETS } L, R\}.$$

$$L \cap R = \{x \mid x \in L \text{ AND } x \in R\}$$
$$= \{x \mid x \text{ IS IN BOTH THE SETS } L, R\}$$

NOTE: IN THE ABOVE YOU CAN THINK OF L AS LEFT AND R AS RIGHT (NOT THE SET OF REAL NUMBERS).

D. EXAMPLES: LET $A = \{1, 2, 3, 4\}$ AND $B = \{2, 3, 4, 5, 6\}$.

$$A \cup B = \{1, 2, 3, 4, 5, 6\}$$

$$A \cap B = \{2, 3, 4\}$$

E. PROOF STARTS AND THINGS YOU CAN DO IN A PROOF

a. PROOF START: $P \cap Q \subseteq Z$
(SHOW $\forall x \in P \cap Q$, $x \in Z$)

A, 1. ASSUME $x' \in P \cap Q$ (SHOW $x' \in Z$)
2. $x' \in P$ AND $x' \in Q$ 1, DEF. OF $\cap$

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b. PROOF START: $H \cap K \subseteq A \cup P$
(SHOW $\forall x \in H \cap K$, $x \in A \cup P$)

A, 1. ASSUME $x' \in H \cap K$ (SHOW $x' \in A \cup P$)
(SHOW $x' \in A$ OR $x' \in P$)

---

c. YOU CAN DO THIS IN A PROOF

:

23. $x' \in T$ GIVEN
24. $x' \in T$ OR $x' \in E$ 23, TT DEF. OF OR
25. $x' \in T \cup E$ 24, DEF. OF $\cup$

---

d. YOU CAN DO THIS IN A PROOF

:

76. $y' \in M$ GIVEN
77. $y' \in E$ GIVEN
78. $y' \in M$ AND $y' \in E$ 76, 77, TT DEF OF $\wedge$
79. $y' \in M \cap E$ 78, DEF OF $\cap$

F. PROVE $E \subseteq E \cup T$

($\text{SHOW } \forall x \in E,\ x \in E \cup T$)

A, 1. ASSUME $x' \in E$ (SHOW $x' \in E \cup T$)

2. $x' \in E$ OR $x' \in T$ — 1, TT DEF. OF OR

3. $x' \in E \cup T$ — 2, DEF. OF $\cup$

A,* 4. $\forall x \in E,\ x \in E \cup T$ — 1–3

5. $E \subseteq E \cup T$ — 4, DEF. $\subseteq$

G. HOMEWORK (OIS)

1. PROVE: $E \cap T \subseteq E$

2. PROVE: $E \cap T \subseteq E \cup T$

3. PROVE: $A \cap (B \cup C) \subseteq (E \cap T) \cup A$

[CHAPTER 23]

DIFFERENCE OF SETS AND INDIRECT PROOF REVISITED

A. DEF: $L-R=\{x \mid x \in L \text{ AND } x \notin R\}$

B. EXAMPLE: LET $L=\{1,2,3,4\}$ AND $R=\{3,4,5,6\}$ $L-R=\{1,2\}$

C. THINGS YOU CAN DO IN A PROOF

a. 23. $x' \in A-(B \cap C)$ GIVEN

24. $x' \in A$ AND $x' \notin B \cap C$ 23, DEF. $-$

b. 28. $x' \in M$ GIVEN

29 $x' \notin T$ GIVEN

30. $x' \in M$ AND $x' \notin T$ 28, 29, TT DEF $\wedge$

31. $x' \in M-T$ 30, DEF. OF $-$

c. 38. $x' \in A \cup B$ GIVEN

39. $x' \notin E$ GIVEN

40. $x' \in A \cup B$ AND $x' \notin E$ 38, 39, TT DEF. $\wedge$

41. $x' \in (A \cup B)-E$ 40, DEF. OF $-$

D. PROVE $B-(E \cap F) \subseteq B \cup D$

(SHOW $\forall x \varepsilon B-(E \cap F)$, $x \varepsilon B \cup D$)

A_1 1. ASSUME $x' \varepsilon B-(E \cap F)$. (SHOW $x' \varepsilon B \cup D$)

2. $x' \varepsilon B$ AND $x' \notin E \cap F$ — 1, DEF. OF $-$

3. $x' \varepsilon B$ — 2, TT DEF. OF $\wedge$

4. $x' \varepsilon B$ OR $x' \varepsilon D$ — 3, TT DEF. OF OR

5. $x' \varepsilon B \cup D$ — 4, DEF. OF $\cup$

A_1^* 6. $\forall x \varepsilon B-(E \cap F)$, $x \varepsilon B \cup D$ — 1–5

7. $B-(E \cap F) \subseteq B \cup D$ — 6, DEF. $\subseteq$

E. INDIRECT PROOF REVISITED:

1. RECALL: TO PROVE STATEMENT P TRUE BY INDIRECT PROOF, ASSUME "NOT P" TRUE. GET <u>ANY</u> CONTRADICTION.

2. INDIRECT PROOF CAN BE DONE ANYTIME, ANYPLACE, ANYWHERE.

⋮ (SHOW $x' \notin B$)

ASSUME $x' \varepsilon B$ (INDIRECT PROOF GET ANY CONTRADICTION.)

F. PROVE: IF $A \cap B = \phi$, THEN $A \subseteq A - B$.

A_1 1. ASSUME $A \cap B = \phi$ (SHOW $A \subseteq A - B$)
(SHOW $\forall x \varepsilon A$, $x \varepsilon A - B$)

A_2 2. ASSUME $x' \varepsilon A$ (SHOW $x' \varepsilon A - B$)
(SHOW $x' \varepsilon A$ AND $x' \notin B$)
(FIRST SHOW $x' \notin B$ BY INDIRECT PROOF)

A_3 3. ASSUME $x' \varepsilon B$. (INDIRECT PROOF. GET #)

4.	$x' \varepsilon A$ AND $x' \varepsilon B$	2, 3, TT DEF. OF AND
5.	$x' \varepsilon A \cap B$	4, DEF. OF $\cap$
6.	$x' \notin A \cap B$	1, DEF. OF ϕ
	CONTRADICTION	
A_3^* 7.	$x' \notin B$	3–6, INDIRECT PROOF
8.	$x' \varepsilon A$ AND $x' \notin B$	2, 7, TT DEF. OF AND
9.	$x' \varepsilon A - B$	8, DEF. OF $-$
A_2^* 10.	$\forall x \varepsilon A$, $x \varepsilon A - B$	2, 7–9
11.	$A \subseteq A - B$	10, DEF. $\subseteq$
A_1^* 12.	IF $A \cap B = \phi$, THEN $A \subseteq A - B$	1, 11

G. THE EMPTY SET IS A SUBSET OF ANY SET, H, SINCE $\forall x \in \emptyset,\ x \in H$ IS TRUE. $\emptyset \subseteq H$.

H. HOMEWORK (OIS)

1. PROVE: $H - K \subseteq H$
2. PROVE: $H - (H - K) \subseteq K$
3. PROVE BY USING INDIRECT PROOF SOMEWHERE:
   IF $A \cap B = \emptyset$ AND $E \subseteq A$, THEN $E \subseteq A - B$

[CHAPTER 24]

AXIOM OF EXTENT

A. AXIOM OF EXTENT: A WAY TO PROVE SETS EQUAL

$L = R$ IFF $L \subseteq R$ AND $R \subseteq L$

B. DRILL ON USING THE AXIOM OF EXTENT IN PROOFS

a. PROVE $A \cap B = E \cup B$

(SHOW $A \cap B \subseteq E \cup B$ AND $E \cup B \subseteq A \cap B$)

______

b. PROVE $H - (H - K) = K$

(SHOW $H-(H-K) \subseteq K$ AND $K \subseteq H-(H-K)$)

______

c. 23. $A - B \subseteq H$ AND $H \subseteq A - B$ GIVEN

24. $A - B = H$ 23, AXIOM OF EXTENT.

______

d. 73. $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ AND $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ GIVEN

74. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ 73, AXIOM OF EXTENT

C. PROVE: IF $A \subseteq B$, THEN $A \cap B = A$

A_1 1. ASSUME $A \subseteq B$ (SHOW $A \cap B = A$)

(SHOW $A \cap B \subseteq A$)(SHOW $\forall x \varepsilon A \cap B, x \varepsilon A$)

A_2 2. ASSUME $x' \varepsilon A \cap B$ (SHOW $x' \varepsilon A$)

3. $x' \varepsilon A$ AND $x' \varepsilon B$ — 2, DEF. OF $\cap$

4. $x' \varepsilon A$ — 3, TT DEF OF AND

A_2^* 5. $\forall x \varepsilon A \cap B, x \varepsilon A$ — 2–4

6. $A \cap B \subseteq A$ — 5, DEF. OF $\subseteq$

(SHOW $A \subseteq A \cap B$) (SHOW $\forall x \varepsilon A, x \varepsilon A \cap B$)

A_3 7. ASSUME $x' \varepsilon A$ (SHOW $x' \varepsilon A \cap B$)

8. $\forall x \varepsilon A, x \varepsilon B$ — 1, DEF. $\subseteq$

9. $x' \varepsilon B$ — 7, 8, INSTANCE OF 8

10. $x' \varepsilon A$ AND $x' \varepsilon B$ — 7, 9, TT DEF OF AND.

11. $x' \varepsilon A \cap B$ — 10, DEF OF $\cap$

A_3^* 12. $\forall x \varepsilon A, x \varepsilon A \cap B$ — 7–11

13. $A \subseteq A \cap B$ — 12, DEF OF $\subseteq$

14. $A \cap B \subseteq A$ AND $A \subseteq A \cap B$ — 6, 13, TT DEF $\wedge$

15. $A \cap B = A$ — 14, AXIOM OF EXTENT

A_1^* 16 IF $A \subseteq B$, THEN $A \cap B = A$. — 1, 15

D. IT IS OK TO USE PREVIOUSLY PROVEN THEOREMS IN A PROOF. THIS WILL NOW BE SHOWN.

E. PROVE: IF $A \cap B = \phi$, THEN $A - B = A$

A_1 1. ASSUME $A \cap B = \phi$ (SHOW $A - B = A$)
(SHOW $A - B \subseteq A$ AND $A \subseteq A - B$)

2. $A - B \subseteq A$ HOMEWORK, CHAPTER 23, H, 1

3. $A \subseteq A - B$ 1, CHAPTER 23, F

4. $A - B \subseteq A$ AND $A \subseteq A - B$ 2, 3, TT DEF $\wedge$

5. $A - B = A$ 4, AXIOM OF EXTENT

A_1^* 6. IF $A \cap B = \phi$, THEN $A - B = A$.

F. AXIOM OF EXTENT IS NOT THE ONLY WAY TO PROVE SETS EQUAL. YOU COULD USE PREVIOUS THEOREMS AND STRING OUT EQUALITIES. FOR EXAMPLE, SUPPOSE WE HAD THEOREMS:

THEOREM A: $H \cup K = K \cup H$

THEOREM B: $H \cap (K \cup J) = (H \cap K) \cup (H \cap J)$

PROVE: $(A \cap B) \cup (A \cap C) = A \cap (C \cup B)$

PROOF:

$$(A \cap B) \cup (A \cap C) \stackrel{B}{=} A \cap (B \cup C) \stackrel{A}{=} A \cap (C \cup B)$$

FIRST EQUALITY, THM B; SECOND EQ., THM. A

NO AXIOM OF EXTENT USED HERE.

G. HOMEWORK (OIS) **PROVE:**

1. $H \cap K = K \cap H$
2. $J \cap (H \cap K) = (J \cap H) \cap K$
3. $H \cup \emptyset = H$
4. $H \cap \emptyset = \emptyset$
5. IF $A \subseteq B$, THEN $B - (B - A) = A$
6. MAKE UP AN EXAMPLE OF SETS A AND B SUCH THAT $B - (B - A) \neq A$

[CHAPTER 25]

PROOF BY CASES

A. FORM OF PROOF BY CASES:

$$\begin{array}{l} p \vee q \\ p \to r \\ q \to r \\ \hline \therefore r \end{array}$$

BE SURE TO HAVE LINES OF THIS FORM IN THE PROOF.

B. PROVE: IF (1) $7 \in B$ OR $7 \in D$, (2) $B \subseteq E$, AND (3) $D \subseteq W$, THEN $\underline{7 \in E \cup W}$

SCRATCH WORK: NOT PART OF PROOF

CASE 1: $7 \in B$	CASE 2: $7 \in D$
$B \subseteq E$	$D \subseteq W$
$7 \in E$	$7 \in W$
$7 \in E \cup W$	$7 \in E \cup W$

THERE NEED TO BE THESE LINES IN THE PROOF:

$$\begin{array}{l} 7 \in B \text{ OR } 7 \in D \\ 7 \in B \longrightarrow 7 \in E \cup W \\ 7 \in D \longrightarrow 7 \in E \cup W \\ \hline \therefore 7 \in E \cup W \end{array}$$

A "PROOF BY CASES PROOF" NOW FOLLOWS:

A_1 1. ASSUME (1) $7 \in B$ OR $7 \in D$, (2) $B \subseteq E$, AND (3) $D \subseteq W$ (SHOW $7 \in E \cup W$)

2. $7 \in B$ OR $7 \in D$ — 1, TT DEF OF AND

(SHOW $7 \in B \rightarrow 7 \in E \cup W$)

A_2 3. <u>CASE 1</u>: ASSUME $7 \in B$ (SHOW $7 \in E \cup W$)

4. $B \subseteq E$ — 1, TT DEF. OF $\wedge$

5. $\forall x \in B, x \in E$ — 4, DEF OF $\subseteq$

6. $7 \in E$ — 3, 5, INSTANCE OF 5

7. $7 \in E$ OR $7 \in W$ — 6, TT DEF. OF OR

8. $7 \in E \cup W$ — 7, DEF. OF $\cup$

A_2^* 9. $7 \in B \longrightarrow 7 \in E \cup W$ — 3–8

(SHOW $7 \in D \longrightarrow 7 \in E \cup W$)

A_3 10. <u>CASE 2</u>: ASSUME $7 \in D$. (SHOW $7 \in E \cup W$)

11. $D \subseteq W$ — 1, TT DEF. OF AND

12. $\forall x \in D, x \in W$ — 11, DEF. OF $\subseteq$

13. $7 \in W$ — 10, 12, INSTANCE OF 12

14. $7 \in E$ OR $7 \in W$ — 13, DEF. OF OR

15. $7 \in E \cup W$ — 14, DEF. OF $\cup$

A_3^* 16. $7 \in D \longrightarrow 7 \in E \cup W$ — 10–15

17. $7 \in E \cup W$ — 2, 9, 16, <u>PROOF</u> <u>BY</u> <u>CASES</u>!

A_1^* 18. IF (1) $7 \in B$ OR $7 \in D$, (2) $B \subseteq E$, AND (3) $D \subseteq W$, THEN $7 \in E \cup W$ — 1, 17

C. PROVE: IF $(\forall x \in A, x \in B)$ OR $(\forall x \in A, x < 7)$, THEN $(\forall x \in A, x \geq 7 \rightarrow x \in B)$.

A_1 1. ASSUME $(\forall x \in A, x \in B)$ OR $(\forall x \in A, x < 7)$
(SHOW $\forall x \in A, x \geq 7 \rightarrow x \in B$)

A_2 2. ASSUME $x' \in A$. (SHOW $x' \geq 7 \rightarrow x' \in B$)
(SHOW $x' < 7$ OR $x' \in B$)

(SHOW $[\forall x \in A, x \in B] \longrightarrow [x' < 7$ OR $x' \in B]$)

A_3 3. CASE 1: ASSUME $\forall x \in A, x \in B$
(SHOW $x' < 7$ OR $x' \in B$)

4. $x' \in B$ 2, 3, INSTANCE OF 3

5. $x' < 7$ OR $x' \in B$ 4, TT DEF. OF OR

A_3^* 6. $[\forall x \in A, x \in B] \longrightarrow [x' < 7$ OR $x' \in B]$ 3–5

(SHOW $[\forall x \in A, x < 7] \rightarrow [x' < 7$ OR $x' \in B]$)

A_4 7. CASE 2: ASSUME $\forall x \in A, x < 7$
(SHOW $x' < 7$ OR $x' \in B$)

8. $x' < 7$ 2, 7, INSTANCE OF 7

9. $x' < 7$ OR $x' \in B$ 8, TT DEF. OF OR

A_4^* 10 $[\forall x \in A, x < 7] \longrightarrow [x' < 7$ OR $x' \in B]$

11. $x' < 7$ OR $x' \in B$ 1, 6, 10, PROOF BY CASES

12. $x' \geq 7 \rightarrow x' \varepsilon B$ 11, IMPLICATION EQUIV.

A_2^*13. $\forall x \varepsilon A,\ x \geq 7 \rightarrow x \varepsilon B$ 2, 12

A_1^*14. IF $(\forall x \varepsilon A,\ x \varepsilon B)$ OR $(\forall x \varepsilon A,\ x < 7)$,
THEN $(\forall x \varepsilon A,\ x \geq 7 \rightarrow x \varepsilon B)$ 1, 13

NOTE THE PROOF BY CASES FORM IN LINES 1, 6, 10, 11

$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ q \rightarrow r \\ \hline \therefore r \end{array}$$

1. $(\forall x \varepsilon A,\ x \varepsilon B)$ OR $(\forall x \varepsilon A,\ x < 7)$
6. $(\forall x \varepsilon A,\ x \varepsilon B) \rightarrow (x' < 7$ OR $x' \varepsilon B)$
10. $(\forall x \varepsilon A,\ x < 7) \rightarrow (x' < 7$ OR $x' \varepsilon B)$

$\therefore$ 11 $x' < 7$ OR $x' \varepsilon B$

D. HOMEWORK (OIS)

1. PROVE: IF $A \subseteq E$ AND $B \subseteq F$, THEN $A \cup B \subseteq E \cup F$.
2. PROVE: IF $(A \subseteq B$ OR $A \subseteq T)$, THEN $A \subseteq T \cup B$

E. PROOF BY CASES WITHIN PROOF BY CASES

A FORM FOLLOWS

	3.	$p \vee q$	
A_2	4	CASE 1 ASSUME p (SHOW r)	
	⋮		
	8	r	
A_2^*	9.	$p \to r$	
A_3	10.	CASE 2 ASSUME q (SHOW r)	
		PROOF BY CASES WITHIN CASE 2	
	11.	$d \vee h$	
A_4	12	CASE 2A ASSUME d (SHOW r)	
	⋮		
	17	r	
A_4^*	18	$d \to r$	12-17
A_5	19	CASE 2B ASSUME h (SHOW r)	
	⋮		
	24	r	
A_5^*	25	$h \to r$	19-24
	26.	r	11, 18, 25, PROOF BY CASES
A_3^*	27	$q \to r$	10, 26
	28.	r	3, 9, 27, PROOF BY CASES

NOTE THE PROOF BY CASES WITHIN PROOF BY CASES IS OF THE FORM (11, 18, 25, 26):

$$\begin{array}{l} d \vee h \\ d \to r \\ h \to r \\ \hline \therefore\ r \end{array}$$

F. PROVE $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ USING PROOF BY CASES WITHIN PROOF BY CASES.
(SHOW $\forall x \in A \cup (B \cup C)$, $x \in (A \cup B) \cup C$)

A_1 1. ASSUME $x' \in A \cup (B \cup C)$. (SHOW $x' \in (A \cup B) \cup C$)
2. $x' \in A$ OR $x' \in B \cup C$ — 1, DEF. OF $\cup$

A_2 3. <u>CASE 1</u>: ASSUME $x' \in A$ (SHOW $x' \in (A \cup B) \cup C$)
4. $x' \in A$ OR $x' \in B$ — 3, TT DEF. OF OR
5. $x' \in A \cup B$ — 4, DEF. OF $\cup$
6. ($x' \in A \cup B$) OR $x' \in C$ — 5, TT DEF. OF OR
7. $x' \in (A \cup B) \cup C$ — 6, DEF. OF $\cup$

A_2^* 8. $x' \in A \longrightarrow x' \in (A \cup B) \cup C$ — 3-7

A_3 9. <u>CASE 2</u>: ASSUME $x' \in B \cup C$
(SHOW $x' \in (A \cup B) \cup C$)
10. $x' \in B$ OR $x' \in C$ — 9, DEF. OF $\cup$

A_4 11. <u>CASE 2A</u>: ASSUME $x' \in B$
(SHOW $x' \in (A \cup B) \cup C$)
12. $x' \in A$ OR $x' \in B$ — 11, TT DEF. OF OR
13. $x' \in A \cup B$ — 12, DEF. OF $\cup$
14. $x' \in A \cup B$ OR $x' \in C$ — 13, TT DEF. OF OR
15. $x' \in (A \cup B) \cup C$ — 14, DEF. OF $\cup$

A_4^*16. $x' \varepsilon B \rightarrow x' \varepsilon (A \cup B) \cup C$ 11-15

A_5 17. <u>CASE 2B</u>: ASSUME $x' \varepsilon C$

(SHOW $x' \varepsilon (A \cup B) \cup C$)

18. $x' \varepsilon A \cup B$ OR $x' \varepsilon C$ 17, TT DEF. OF OR

19. $x' \varepsilon (A \cup B) \cup C$ 18, DEF. OF $\cup$

A_5^* 20. $x' \varepsilon C \rightarrow x' \varepsilon (A \cup B) \cup C$ 17-19

21. $x' \varepsilon (A \cup B) \cup C$ 10, 16, 20 PROOF BY CASES (WITHIN PROOF BY CASES)

A_3^* 22. $x' \varepsilon B \cup C \longrightarrow x' \varepsilon (A \cup B) \cup C$ 9, 21

23. $x' \varepsilon (A \cup B) \cup C$ 2, 8, 22, PROOF BY CASES

A_1^* 24. $\forall x \varepsilon A \cup (B \cup C)$, $x \varepsilon (A \cup B) \cup C$ 1, 23

25. $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ 24, DEF. OF $\subseteq$

G. HOMEWORK (OIS) PROVE

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ USING PROOF BY CASES WITHIN PROOF BY CASES. (DO NOT USE TAUTOLOGIES $T_{16}, T_{17}, T_{18}, T_{19}$ IN CHAPTER 8)

[CHAPTER 26]

DIRECT PROOF OF $\exists$

A. A WAY TO PROVE $\exists$.. "EXHIBIT IT"

B. EXAMPLE: LET $A = \{1,3,5\}$ AND $B = \{0,1,2,3\}$. PROVE:

$\exists x \in A \cap B \ni x < 4$

1.	$3 \in A$	DEF. OF A
2.	$3 \in B$	DEF. OF B
3.	$3 \in A$ AND $3 \in B$	1, 2, TT DEF. OF $\wedge$
4.	$3 \in A \cap B$	3, DEF. OF $\cap$
5.	$3 < 4$	ARITHMETIC KNOWLEDGE
6.	$\exists x \in A \cap B \ni x < 4$	4, 5, NAMELY REPLACE x WITH 3.

NOTE: ACCORDING TO THE DEFINITION, WE NEEDED A REPLACEMENT FOR x FROM $A \cap B$ THAT MAKES A TRUE INSTANCE OF $x < 4$. WE EXHIBITED SUCH A REPLACEMENT, 3. (NOW 1 COULD HAVE ALSO BEEN A REPLACEMENT. $\exists$ MEANS 'AT LEAST 1' NOT 'ONLY 1'.)

C. HOMEWORK (OIS) LET $A = \{1, 3, 5\}$ AND $B = \{2, 3, 4\}$

1. PROVE $\exists x \in A \cup B \ni x > 4$.
2. PROVE $\exists x \in A \cup B \ni x < 3$.

D. NOTE FOR A DIRECT PROOF OF A $\exists$ STATEMENT, WE DID NOT HAVE THE WORD "ASSUME" NOR DISCHARGING OF ASSUMPTIONS

E. DRILL ON WHAT CAN BE SAID IN A PROOF:

SUPPOSE YOU HAVE GIVEN

23. $5 \in P \cap Q$ GIVEN
24. $5 \in Q - (Z \cap W)$ GIVEN
25. $7 \in A \cup Q$ GIVEN
26. $5 \in H \cup K$ GIVEN

WHAT COULD BE SAID NEXT?

27. $\exists x \in P \cap Q \ni x \in H \cup K$ 23, 26, REPLACE x WITH 5.

WHAT ELSE COULD HAVE BEEN SAID?

28. $\exists x \in H \cup K \ni x \in Q - (Z \cap W)$ 26, 24, REPLACE x WITH 5

(DRILL CONTINUED)

SUPPOSE YOU HAVE GIVEN

76. $3x'+y' \varepsilon B$ GIVEN

77. $5x'+y' \varepsilon E$ GIVEN

78. $3x'+y' \varepsilon H-E$ GIVEN

LIST THINGS THAT CAN BE SAID

79. $\exists x \varepsilon B \ni x \varepsilon H-E$ 76, 78, REPLACE x WITH $3x'+y'$

80. $\exists z \varepsilon H-E \ni z \varepsilon B$ 78, 76, REPLACE z WITH $3x'+y'$

SUPPOSE YOU HAVE GIVEN

96. $x'' \varepsilon R$ GIVEN

97. $y'' \varepsilon T$ GIVEN

98. $4x''+3y'' < 7$ GIVEN

LIST THINGS THAT CAN BE SAID

99. $\exists x \varepsilon R \ni 4x+3y'' < 7$ 96, 98 REPLACE x WITH x''

100. $\exists y \varepsilon T, \exists x \varepsilon R \ni 4x+3y < 7$ 97, 99, REPLACE y WITH y''.

101. $\exists y \varepsilon T \ni 4x''+3y < 7$ REASON ______

102. $\exists x \varepsilon R, \exists y \varepsilon T \ni 4x+3y < 7$ REASON ______

F. PROVE: IF $8 \in E$ AND $E \subseteq F$, THEN $\exists x \in F \ni x > 7$.

A_1 1. ASSUME $8 \in E$ AND $E \subseteq F$.
(SHOW $\exists x \in F \ni x > 7$)

2. $E \subseteq F$ — 1, TT DEF. OF AND
3. $\forall x \in E, x \in F$ — 2, DEF. $\subseteq$
4. $8 \in E$ — 1, TT DEF. OF AND
5. $8 \in F$ — 3,4, INSTANCE OF 3
6. $8 > 7$ — ARITHMETIC KNOWLEDGE
7. $\exists x \in F \ni x > 7$ — 5,6, REPLACE x WITH 8.

A_1^* 8. IF $8 \in E$ AND $E \subseteq F$, THEN $\exists x \in F \ni x > 7$ — 1-7

G. HOMEWORK (OIS)

1. PROVE: IF $7 \in A$ AND $A \cup B \subseteq E$, THEN $\exists x \in E \cup F \ni x > 5$
2. $T = \{x \mid x \in R \text{ AND } x = x^2\}$ PROVE: $\exists x \in R \ni (x \in T \text{ AND } x > 0)$.

[CHAPTER 27]

THE "LET" RULE AND USING $\exists$

A. LET RULE: IF YOU KNOW B IS A NON-EMPTY SET, THEN YOU CAN SAY "LET $x' \in B$"

B. DRILL ON USING THE LET RULE

SUPPOSE YOU ARE GIVEN

27. $A \neq \phi$ GIVEN

THEN YOU CAN SAY:

28 LET $x' \in A$ 27, LET RULE

SUPPOSE YOU ARE GIVEN

46. $E \cap F \neq \phi$ GIVEN

THEN YOU CAN SAY

47. LET $x' \in E \cap F$ 46, LET RULE
48. $x' \in E$ AND $x' \in F$ 47, DEF. OF $\wedge$

SUPPOSE YOU ARE GIVEN

83. $H - K \neq \phi$ GIVEN

THEN YOU CAN SAY:

84. LET $y' \in H - K$ 83, LET RULE
85. $y' \in H$ AND $y' \notin K$ 84, DEF. OF $-$

C. IN THIS BOOK "LET" DOES NOT MEAN THE SAME AS "ASSUME". LET MEANS WE KNOW AT LEAST ONE ELEMENT OF THIS TYPE EXISTS. WE ARE THEN CHOOSING AN ELEMENT AND A NAME AND LETTING THAT BE THE NAME OF THE ELEMENT.

D. PROVE: IF $A \neq \emptyset$ AND $A \subseteq B$ AND $(\forall x \in B, x < 10)$ AND $B \subseteq C$, THEN $\exists x \in C, x < 10$.

A_1 1. ASSUME $A \neq \emptyset$ AND $A \subseteq B$ AND $(\forall x \in B, x < 10)$ AND $B \subseteq C$ (SHOW $\exists x \in C \ni x < 10$)

2. $A \neq \emptyset$ — 1, TT DEF. OF AND

3. LET $x' \in A$ — 2, LET RULE

4. $A \subseteq B$ — 1, TT DEF. OF AND

5. $\forall x \in A, x \in B$ — 4, DEF. OF $\subseteq$

6. $x' \in B$ — 3, 5, INSTANCE OF 5

7. $\forall x \in B, x < 10$ — 1, TT DEF. OF AND

8. $x' < 10$ — 6, 7, INSTANCE OF 7

9. $B \subseteq C$ — 1, TT DEF. OF AND

10. $\forall x \in B, x \in C$ — 9, DEF. OF $\subseteq$

11. $x' \in C$ — 6, 10, INSTANCE OF 10

12. $\exists x \in C, x < 10$ — 11, 8, REPLACE x WITH x'

A_1^* 13. IF $A \neq \emptyset$ AND $A \subseteq B$ AND $(\forall x \in B, x < 10)$ AND $B \subseteq C$, THEN $(\exists x \in C \ni x < 10)$ — 1-12

E. THE LET RULE IS ALSO CALLED THE AXIOM OF CHOICE. WE CAN CHOOSE AN ELEMENT FROM A NONEMPTY SET.

F. "ASSUME" VS. "LET" IN THIS BOOK

1. WHEN YOU SAY "ASSUME $x' \varepsilon A$", YOU ARE NOT CLAIMING A HAS ANY ELEMENTS. YOU ARE JUST PRETENDING IT DOES AND SEEING WHAT THAT PRETENSE IMPLIES.
2. WHEN YOU SAY "LET $x' \varepsilon A$", YOU ARE CLAIMING A HAS AT LEAST ONE ELEMENT, YOU ARE RANDOMLY PICKING OUT ONE OF THOSE ELEMENTS, AND GIVING IT A NAME x', AND THEN USING x'.

G. HOMEWORK (OIS)

1. PROVE: IF $E \cap F \neq \phi$ AND $F \subseteq P \cap Q$, THEN $\exists\, x \varepsilon P \ni x \varepsilon Q$
2. PROVE: IF $H - K \neq \phi$ AND $F \subseteq K$, THEN $\exists\, x \varepsilon H, x \notin F$.

H. USING $\exists$: SUPPOSE YOU HAVE $\exists\, x \varepsilon A,\ x \varepsilon B$. YOU CAN SAY "LET $x' \varepsilon A \ni x' \varepsilon B$". REASON: WE KNOW, BY DEFINITION, THERE IS A REPLACEMENT FOR x FROM A THAT MAKES A TRUE INSTANCE OF $x \varepsilon B$. SINCE WE KNOW AT LEAST ONE SUCH REPLACEMENT EXISTS, CHOOSE ONE, AND CALL IT x' BY THE LET RULE.

I DRILL ON USING $\exists$

SUPPOSE YOU ARE GIVEN

27. $\exists x \in P \ni x \in Q$ GIVEN

YOU CAN SAY

28. LET $x' \in P \ni x' \in Q$ 27, LET RULE
29. $x' \in P$ 28
30. $x' \in Q$ 28

SUPPOSE YOU ARE GIVEN

46. $\exists x \in A \cap B \ni x \in E - D$ GIVEN

YOU CAN SAY

47. LET $x' \in A \cap B \ni x' \in E - D$ 46, LET RULE
48. $x' \in A \cap B$ 47
49 $x' \in E - D$ 47

J. DO <u>NOT</u> DO THIS:

27 $\exists x \in P \ni x \in Q$ GIVEN

28. LET $x' \in P$ 27, LET RULE

29. $x' \in Q$ ← BAD. GRANTED 27 LETS US KNOW P IS NONEMPTY, BUT IF YOU SIMPLY PICK ANY ELEMENT FROM P YOU ARE NOT ASSURED IT WILL BE IN Q.

K. PROVE: IF ($\exists x \in A, x \in E \cap F$) AND $F \subseteq G$, THEN $\exists x \in G, x \in E$.

A_1 1. ASSUME ($\exists x \in A, x \in E \cap F$) AND $F \subseteq G$.
(SHOW $\exists x \in G, x \in E$)

2.	$\exists x \in A, x \in E \cap F$	1, TT DEF OF AND
3.	LET $x' \in A \ni x' \in E \cap F$	2, LET RULE
4.	$x' \in E \cap F$	3
5.	$x' \in E$ AND $x' \in F$	4, DEF OF $\cap$
6.	$x' \in F$	5, TT DEF OF AND
7.	$F \subseteq G$	1, TT DEF OF AND
8.	$\forall x \in F, x \in G$	7, DEF. OF $\subseteq$
9.	$x' \in G$	6, 8, INSTANCE OF 8
10.	$x' \in E$	5, TT DEF. OF AND
11.	$\exists x \in G, x \in E$	9, 10, REPLACE x WITH x'

A_1^* 12. IF ($\exists x \in A, x \in E \cap F$) AND $F \subseteq G$, THEN $\exists x \in G, x \in E$. 1–11

L. HOMEWORK (OIS)

1. PROVE: IF $(\exists y \in B,\ y \in H \cap K)$ AND $H \subseteq E$, THEN $\exists x \in K \cap E,\ x \in B$.

2. PROVE: IF $(\exists x \in A,\ x \in E \cup F)$, $E \subseteq B$, AND $F \subseteq K$, THEN $\exists x \in A,\ x \in B \cup K$.

3. PROVE: IF (1) $A \cap B \neq \phi$,
   (2) $B \subseteq H$, AND
   (3) IF $(\exists x \in A,\ x \in H)$, THEN $K \neq \phi$,

   THEN

   IF $K \subseteq J$ AND $J \cap M = \phi$, THEN $\exists x \in J \ni x \notin M$.

[CHAPTER 28]

PROPER SUBSET

A. $\subset$ IS READ "IS A PROPER SUBSET OF".

B. DEFINITION: $L \subset R$ IFF $L \subseteq R$ AND $\exists x \varepsilon R \ni x \notin L$.

C. EXAMPLE: LET $A = \{2,3\}$ AND $B = \{1,2,3,4\}$. $A \subset B$ SINCE $A \subseteq B$ AND $\exists x \varepsilon B \ni x \notin A$ (NAMELY REPLACE x WITH 1 OR 4 }).

D. PROVE: IF $A \neq \phi$, THEN $\phi \subset A$.

A, 1. ASSUME $A \neq \phi$. (SHOW $\phi \subset A$)
(SHOW $\phi \subseteq A$ AND $\exists x \varepsilon A \ni x \notin \phi$)

2. $\forall x \varepsilon \phi, x \varepsilon A$ — VACUOUSLY TRUE
3. $\phi \subseteq A$ — 2, DEF. OF $\subseteq$
4. LET $x' \varepsilon A$ — 1, LET RULE
5. $x' \notin \phi$ — DEF. OF ϕ
6. $\exists x \varepsilon A \ni x \notin \phi$ — 4,5, REPLACE x WITH x'
7. $\phi \subseteq A$ AND $\exists x \varepsilon A \ni x \notin \phi$ — 3,6, TT DEF $\wedge$
8. $\phi \subset A$ — 7, DEF. OF $\subset$

A_1^*9. IF $A \neq \phi$, THEN $\phi \subset A$. — 1–8

E. PROVE: IF $A \subset B$ AND $B \subseteq H$ AND $\forall x \in H, (x \in R \wedge x < 3)$, THEN $\exists x \in H \ni 2x < 6$.

A, 1. ASSUME $A \subset B$ AND $B \subseteq H$ AND $\forall x \in H, (x \in R \wedge x < 3)$. (SHOW $\exists x \in H \ni 2x < 6$)

2. $A \subset B$ — 1, TT DEF. OF AND

3. $A \subseteq B$ AND $\exists x \in B \ni x \notin A$ — 2, DEF. OF $\subset$

4. $\exists x \in B \ni x \notin A$ — 3, TT DEF OF AND

5. LET $x' \in B \ni x' \notin A$ — 4, LET RULE

6. $x' \in B$ — 5

7. $B \subseteq H$ — 1, TT DEF OF AND

8. $\forall x \in B, x \in H$ — 7, DEF. OF $\subseteq$

9. $x' \in H$ — 6, 8, INSTANCE OF 8

10. $\forall x \in H, (x \in R \wedge x < 3)$ — 1, TT DEF OF AND

11. $x' \in R \wedge x' < 3$ — 9, 10, INSTANCE OF 10

12. $x' < 3$ — 11, TT DEF OF AND

13. $2x' < 6$ — 12, MULTIPLY BY 2

14. $\exists x \in H \ni 2x < 6$ — 13, 9, REPLACE x WITH x'

A, *15. IF $A \subset B$ AND $B \subseteq H$ AND $\forall x \in H, (x \in R \wedge x < 3)$, THEN $\exists x \in H \ni 2x < 6$ — 1-14

F. HOMEWORK (OIS)

1. PROVE: IF $A \subset B$ AND $B \subseteq H$, THEN $A \subset H$.
2. PROVE: IF $A \cap B \neq \phi$, THEN $A - B \subset A$
3. PROVE: IF $A \neq \phi$ AND $A \subset B$, THEN $B - A \subset B$
4. PROVE: $A \subset B$ IFF $[A \subseteq B$ AND $A \neq B]$

ABBREVIATIONS OF PROOFS

A. <u>RULE FOR ABBREVIATION</u>: YOU CAN UNHESITATINGLY PUT IN THE DETAILED INFORMATION IF CALLED UPON TO DO SO.

B. <u>ANOTHER RULE</u>: YOU DO NOT HAVE TO ABBREVIATE UNLESS YOU WANT TO AND IT MAKES THE FLOW OF LOGIC EASIER TO FOLLOW.

C. IT IS OK TO DROP THE PRIMES IF YOUR UNDERSTANDING REMAINS CLEAR.

LET $A=\{y \mid y>3\}$ PROVE $\forall x \in A, 2x>6$

PROOF 1 OF $\forall x \in A, 2x>6$

A_1 1. ASSUME $x' \in A$ (SHOW $2x'>6$)

2. $x'>3$ — 1, DEF. OF A

3. $2x'>6$ — 2, MULT. BY 2

A_1^* 4. $\forall x \in A, 2x>6$ — 1–3

PROOF 2 OF $\forall x \in A, 2x>6$

A_1 1. ASSUME $r \in A$ (SHOW $2r>6$)

2. $r>3$ — 1, DEF. OF A

3. $2r>6$ — 2, MULT. BY 2

A_1^* 4 $\forall x \in A, 2x>6$ — 1–3

NOTE: IN PROOF 1 WE PROVED "EVERY REPLACEMENT FOR x FROM A MAKES A TRUE INSTANCE OF $2x>6$" BY REPLACING x WITH x'. IN PROOF 2 WE REPLACED x WITH r. IT WOULD HAVE STILL BEEN A PROOF IF WE HAD REPLACED x WITH s, t, u, v, or w! THE ESSENCE IS: PICK A SYMBOL. ASSUME SYMBOL IS IN A. SHOW 2(SYMBOL) > 6. LET'S USE SYMBOL x.

0.	PROVE $\forall x \in A,\ 2x>6$	
A,1.	ASSUME $x \in A$ (SHOW $2x>6$)	
2.	$x>3$	1, DEF. OF A
3.	$2x>6$	2, MULT. BY 2
A,*4	$\forall x \in A,\ 2x>6$	1–3

THIS IS PERMISSIBLE PROVIDED YOU KNOW THE x IN LINES 0 AND 4 IS NOT THE x IN LINES 1, 2, 3. THE x IN LINES 0, 4 IS A DUMMY VARIABLE. THE x IN LINES 1, 2, 3 IS A REPLACEMENT FOR THE DUMMY VARIABLE.

IF YOU LIKE THIS ABBREVIATION, YOU CAN USE IT, BUT YOU DON'T HAVE TO.

D. YOU CAN STOP DISCHARGING ASSUMPTIONS AND STOP WHEN YOU HAVE DONE THE LAST "SHOW"

PROVE: $\forall x \in R$, (IF $x<5$, THEN (IF $x<3$, THEN $2x<7$))

1. ASSUME $x \in R$ (SHOW IF $x<5$, THEN (IF $x<3$, THEN $2x<7$))
2. ASSUME $x<5$ (SHOW IF $x<3$, THEN $2x<7$)
3. ASSUME $x<3$ (SHOW $2x<7$)
4. $2x<6<7$ 3, MULT. BY 2, ARIT. KNOW.

E. YOU CAN COMBINE ASSUMPTIONS AND YOU DO NOT HAVE TO BRING DOWN EACH PART OF AN "AND" STATEMENT INDIVIDUALLY.

REDO THE PROOF ABOVE:

1. ASSUME $x \in R$, $x<5$, AND $x<3$ (SHOW $2x<7$)
2. $2x<6<7$ 1, MULT. BY 2, ARIT. KNOW.

F. IF YOU HAVE

28. $A \subseteq B$ GIVEN
29. $p \in A$ GIVEN

YOU DO NOT HAVE TO SAY $\forall x \in A, x \in B$. YOU CAN SIMPLY SAY

30. $p \in B$ 28, 29, DEF. OF $\subseteq$

G. FOR <u>PROOF</u> <u>BY</u> <u>CASES</u>, YOU CAN LEAVE OFF THE WORD ASSUME IN EACH CASE (IT IS UNDERSTOOD). YOU DO NOT HAVE TO PUT DOWN THE $p \to r$, $q \to r$ LINES. YOU DO NOT HAVE TO WRITE DOWN r AGAIN WHEN YOU HAVE IT IN BOTH CASES.

PROVE: IF ($p \in A$ OR $p \in B$) AND $B \subseteq F$, THEN $p \in A \cup F$.

1. ASSUME ($p \in A$ OR $p \in B$) AND $B \subseteq F$ (SHOW $p \in A \cup F$)
2. <u>CASE 1</u> $p \in A$ (SHOW $p \in A \cup F$)
3. $p \in A$ OR $p \in F$ — 2, TT DEF OF OR
4. $p \in A \cup F$ — 3, DEF. OF $\cup$
5. <u>CASE 2</u> $p \in B$ (SHOW $p \in A \cup F$)
6. $p \in F$ — 5, 1, SINCE $B \subseteq F$
7. $p \in A$ OR $p \in F$ — 6, TT DEF OF OR.
8. $p \in A \cup F$ — 7, DEF OF $\cup$

H. COMBINING $\forall$

$\forall x \in R, \forall y \in R, \forall z \in R$, IF $x < y$, THEN $x + z < y + z + 3$

MEANS

$\forall x, y, z \in R$, IF $x < y$, THEN $x + z < y + z + 3$

PROOF START OF:

$\forall x, y, z \in R$, IF $x<y$, THEN $x+z<y+z+3$

1. ASSUME $x, y, z \in R$ AND $x<y$
   (SHOW $x+z<y+z+3$)

G. HOMEWORK (OIS): PROVE EACH OF THE FOLLOWING IN ABBREVIATED STYLE

1. PROVE: $\forall x, y \in R$, IF $x<y<10$, THEN $x+y<20$

2. PROVE: IF $A \subseteq B$ AND $E \subseteq F$, THEN $A \cup E \subseteq B \cup F$

3. PROVE: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

4. BELOW IS A TYPE OF PROOF SEEN IN MOST BOOKS. PUT IN OUR ABBREVIATED PROOF STYLE.

   PROVE: $|x|<a$ IFF $-a<x<a$

   PART I. ASSUME $|x|<a$. HENCE $0<a$; SO $-a<0$. WHEN $0 \leq x$ WE WOULD HAVE $|x|=x$, SO $-a<0 \leq x=|x|$. SO $-a<x<a$. WHEN $x<0$ WE WOULD HAVE $|x|=-x$, SO $-a<0<-x=|x|<a$. HENCE, $-a<-x<a$ OR $-a<x<a$

(CONTINUED NEXT PAGE)

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PART II ASSUME $-a < x < a$. WHEN $0 \leq x$ WE WOULD HAVE $|x| = x$ AND HENCE $|x| < a$. WHEN $x < 0$, WE WOULD HAVE $|x| = -x$. RECALL $-a < x$, SO $-x < a$. THAT GIVES $|x| < a$.

RECALL THE DEFINITION OF $|x|$

$$|x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x < 0 \end{cases}$$

H ABBREVIATIONS INVOLVING $\exists$

1. PREVIOUS WAY

27. $\exists x \in A \ni x < 7$ GIVEN
28. LET $x' \in A \ni x' < 7$ 27, LET RULE
29. $x' < 7$ 28
30. $2x' < 14$ 29, MULT. BY 2

2. NEW ABBREVIATED WAY

27. $\exists x \in A \ni x < 7$ GIVEN
28 $2x < 14$ 27, MULT. BY 2

3. ANOTHER ABBREVIATED WAY

27. $\exists x_1 \in A \ni x_1 < 7$ GIVEN
28. $2x_1 < 14$ 27, MULT. BY 2

(NOTE: 27 COULD BE THOUGHT OF AS SAYING LOOSELY, "THERE IS AN ELEMENT IN A THAT WE WILL CALL X_1 SUCH THAT $X_1 < 7$.")

4. ABBREVIATION INVOLVING $\exists, \exists$

$\exists x \in R, \exists y \in R \ni x + y = 10$

CAN BE ABBREVIATED

$\exists x, y \in R \ni x + y = 10$

I. PROVE: IF $H \subset K$ AND $\forall x \in K, x < 5$ THEN $\exists x \in K \ni 2x < 14$

1. ASSUME $H \subset K$ AND $\forall x \in K, x < 5$
   (SHOW $\exists x \in K \ni 2x < 14$)
2. $H \subseteq K$ AND $\exists x_1 \in K \ni x_1 \notin H$ 1, DEF $\subset$
3. $x_1 < 5$ 2, 1, INSTANCE OF 1
4. $2x_1 < 10 < 14$ 3, MULT. BY 2
5. $\exists x_1 \in K \ni 2x_1 < 14$ 2, 4

J. HOMEWORK (OIS) PROVE BY ABBREVIATED PROOF

1. IF $A \subset B$, THEN $A - B \subset A \cup B$

K SLOPPY TO PURE SET DEFINING

1. $\{n^2 \mid n \in R \text{ AND } n > 10\} =$

$\{x \mid \exists n \in R \ni n > 10 \text{ AND } x = n^2\}$

2. $\{3x+5 \mid x \in [2,3]\} =$

$\{y \mid \exists x \in [2,3] \ni y = 3x+5\}$

3. $\{4x+3y \mid x \in [0,1] \text{ AND } y \in [2,4]\}$

$\{z \mid \exists x \in [0,1], \exists y \in [2,4] \ni z = 4x+3y\}$

I. LET $P = \{2x+3 \mid x \in [0,1]\}$

LET $Q = \{5x+2y \mid x \in [0,1] \wedge y \in [1,3]\}$

PROVE $P \subseteq Q$ (SHOW $\forall z \in P$, $z \in Q$)

1.	ASSUME $z \in P$ (SHOW $z \in Q$)	
2.	$\exists x \in [0,1] \ni z = 2x+3$	1, DEF. OF P
3.	$0 \le x \le 1$	2
4.	$0 \le 2x/5 \le 2/5 \le 1$	3, MULT. BY $2/5$
5.	$2x/5 \in [0,1]$	4
6.	$1 \le 3/2 \le 3$	ARITHMETIC KNOWLEDGE
7.	$5(2x/5) + 2(3/2) = 2x+3 = z$	ALGEBRA, 2
8.	$\exists p \in [0,1], \exists q \in [1,3] \ni z = 5p+2q$ REPLACE p WITH $2x/5$, q WITH $3/2$	5, 6, 7
9.	$z \in Q$	8, DEFINITION OF Q

J. HOMEWORK (OIS)

1. LET $A = \{2x+8 \mid x \in [0,1]\}$

   LET $B = \{4x+3y \mid x \in [0,3] \wedge y \in [2,4]\}$

   PROVE $A \subseteq B$

[CHAPTER 30]

MATHEMATICAL INDUCTION

A. ATTEMPTS TO PROVE "$\forall$ POSITIVE INTEGER n, $P(n)$".

1. ATTEMPT 1: DIRECT PROOF

   ASSUME n IS A POSITIVE INTEGER.

   (SHOW $P(n)$)

   SUPPOSE THIS ATTEMPT FAILS

2. ATTEMPT 2: INDIRECT PROOF

   ASSUME $\exists$ A POSITIVE INTEGER $n \ni \sim P(n)$

   (TRY TO GET ANY CONTRADICTION)

   SUPPOSE THIS ATTEMPT ALSO FAILS.

   THERE IS STILL HOPE USING THE PRINCIPLE OF MATHEMATICAL INDUCTION!

B. AN ILLUSTRATION OF SOMETHING WHERE DIRECT AND INDIRECT PROOF WILL FAIL, BUT MATHEMATICAL INDUCTION WILL PREVAIL.

FOR EVERY POSITIVE INTEGER n,

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

TO BE PROVEN LATER

BUILD-UP TO MATH INDUCTION

C. SUPPOSE YOU ARE GIVEN:

0: $P(1)$ IS TRUE.

1: IF $P(1)$ IS TRUE, THEN $P(2)$ IS TRUE.

2: IF $P(2)$ IS TRUE, THEN $P(3)$ IS TRUE.

3: IF $P(3)$ IS TRUE, THEN $P(4)$ IS TRUE

⋮ KEEP GOING. ONE STATEMENT FOR EACH POSITIVE INTEGER n

n: IF $P(n)$ IS TRUE, THEN $P(n+1)$ IS TRUE.

⋮

D. NOTE: C MEANS THE SAME AS "FOR EVERY POSITIVE INTEGER n, $P(n)$ IS TRUE"

1. $P(1)$ IS TRUE BY (C, 0:)
2. $P(2)$ IS TRUE BY (D,1), (C,1:), MODUS PONENS
3. $P(3)$ IS TRUE BY (D,2), (C,2:), MODUS PONENS

⋮

ONE CAN CONTINUE FOREVER WITH THIS REASONING, SO

$\forall$ POSITIVE INTEGER n, $P(n)$ IS TRUE

IS EQUIVALENT TO

C ABOVE

E. NOW C, ON THE PRECEDING PAGE CAN BE REWRITTEN IN COMPACT FORM:

P(1) IS TRUE

$\forall$ POSITIVE INTEGER n, IF P(n) IS TRUE, THEN P(n+1) IS TRUE

SO TO PROVE THIS, YOU

1. PROVE P(1) IS TRUE
2. ASSUME n IS A POSITIVE INTEGER AND P(n) IS TRUE* (SHOW P(n+1) IS TRUE)

THIS IS THE <u>PRINCIPLE OF MATHEMATICAL INDUCTION</u>. IT PROVES "$\forall$ POSITIVE INTEGER n, P(n) IS TRUE"

(SINCE IT PROVES C ON THE PREVIOUS PAGE TRUE AND C IS EQUIVALENT TO "$\forall$ POSITIVE INTEGER n, P(n) IS TRUE".)

F. PMI = PRINCIPLE OF MATHEMATICAL INDUCTION.

* INDUCTION HYPOTHESIS

G. DRILL ON RECOGNIZING P(n)

1. $P(n): 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$P(5): 1+2+3+4+5 = \frac{5(5+1)}{2}$

$P(2): 1+2 = \frac{2(2+1)}{2}$

$P(1): 1 = \frac{1(1+1)}{2}$

$P(K): 1+2+3+\cdots+K = \frac{K(K+1)}{2}$

$P(K+1): 1+2+3+\cdots+(K+1) = \frac{(K+1)((K+1)+1)}{2}$

$= 1+2+3+\cdots+K+(K+1)$

2. $P(n): 2+8+14+\cdots+(6n-4) = n(3n-1)$

$P(4): 2+8+14+20 = 4(3(4)-1)$

$P(2): 2+8 = 2(3(2)-1)$

$P(1): 6(1)-4 = 1(3(1)-1)$

$P(K): 2+8+14+\cdots+(6K-4) = K(3K-1)$

$P(K+1): 2+8+14+\cdots+(6(K+1)-4) =$

$(K+1)(3(K+1)-1) =$

$2+8+14+\cdots+(6K-4)+(6(K+1)-4)$

H. A WAY TO PROVE $L = R$ IS TO START WITH L, STRING EQUALITIES OUT, REACH R.

$$L = M_1 = M_2 = M_3 = R$$

IT CAN BE WRITTEN VERTICALLY AS: $\longrightarrow$

$L =$
$M_1 =$
$M_2 =$
$M_3 =$
R

I. PROVE BY MATHEMATICAL INDUCTION (PMI)

FOR EVERY POSITIVE INTEGER n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

1. PROVE $P(1)$ TRUE (SHOW $1 = \frac{1(1+1)}{2}$ BY STRINGING OUT EQUALITIES)

$$\frac{1(1+1)}{2} = \frac{1(2)}{2} = \frac{2}{2} = 1$$

2. ASSUME n IS A POSITIVE INTEGER AND $P(n)$ IS TRUE* (SHOW $P(n+1)$ TRUE, THAT IS ASSUME n IS A POSITIVE INTEGER AND $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ (SHOW $1 + 2 + 3 + \cdots + (n+1) = \frac{(n+1)((n+1)+1)}{2}$ BY STRINGING OUT EQUALITIES.)

* INDUCTION HYPOTHESIS

3. $1+2+3+\cdots+(n+1)$ = ASSOCIATIVITY

4. $[1+2+3+\cdots+n]+(n+1)$ = 2, INDUCTION HYPOTHESIS

5. $\frac{n(n+1)}{2} + (n+1)$ = FACTOR OUT $(n+1)$

6. $(n+1)\left[\frac{n}{2} + 1\right]$ = ALGEBRA

7. $(n+1)\left[\frac{(n+2)}{2}\right]$ = ALGEBRA

8. $(n+1)\left[\frac{((n+1)+1)}{2}\right]$

NOTE: THE REASON FOR EACH EQUALITY IS TO THE RIGHT OF THE EQUALITY.

NOTE: IN LINES 1 AND 2 WE MADE OUR PROVE AND ASSUME STATEMENTS FIRST, GENERALLY, THEN SPECIFICALLY TO THE PROBLEM. FROM NOW ON WE WILL ONLY BE SPECIFIC.

NOTE: DO NOT ATTEMPT THE PROOF IN LINE 1 BY

$$1 = \frac{1(1+1)}{2}$$
$$1 = \frac{1(2)}{2}$$
$$1 = \frac{2}{2}$$
$$1 = 1$$

THIS IS STARTING WITH WHAT YOU WANT TO PROVE!

J. PROVE BY MATHEMATICAL INDUCTION FOR EVERY POSITIVE INTEGER n,

$2+8+14+\cdots+(6n-4)=n(3n-1)$

(SHOW $6(1)-4=1(3(1)-1)$)

1. $6(1)-4=6-4=2=1(2)=1(3-1)=1(3(1)-1)$

2. ASSUME n IS A POSITIVE INTEGER AND $2+8+14+\cdots+(6n-4)=n(3n-1)$ INDUCTION HYPOTHESIS; (NOW SHOW $2+8+\cdots+(6(n+1)-4)=(n+1)(3(n+1)-1)$)

3. $2+8+\cdots+(6(n+1)-4)$ = ASSOCIATIVITY

4. $[2+8+\cdots+(6n-4)]+(6(n+1)-4)$ = 2, IND. HYP.

5. $n(3n-1)+(6(n+1)-4)$ = ALGEBRA

6. $n(3n-1)+(6n+6-4)$ = ALGEBRA

7. $3n^2-n+(6n+2)$ = ALGEBRA

8. $3n^2+5n+2$ = ALGEBRA

9. $(n+1)(3n+2)$ = ALGEBRA

10 $(n+1)(3n+3-1)$ = ALGEBRA

11. $(n+1)(3(n+1)-1)$

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K HOMEWORK (COIS) USING MATHEMATICAL INDUCTION, PROVE EACH OF THE FOLLOWING TRUE FOR EVERY POSITIVE INTEGER n.

1. $1+3+5+\cdots+(2n-1)=n^2$

2. $1+5+9+\cdots+(4n-3)=n(2n-1)$

3. $7+10+13+\cdots+(3n+4)=\dfrac{n(3n+11)}{2}$

4. $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

[CHAPTER 31]

NUMERATION SYSTEMS

A. DECIMAL (BASE TEN)

1. DIGITS: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

2. EXPANDED FORM (ILLUSTRATES POSITIONAL NOTATION)

$$234 = 2(10^2) + 3(10^1) + 4(10^0)$$

$$426.35 = 4(10^2) + 2(10^1) + 6(10^0) + 3(10^{-1}) + 5(10^{-2})$$

3. THE "." BETWEEN THE 6 AND THE 3 IN 426.35 WE CALL THE DECIMAL POINT; SINCE WE WILL LOOK AT OTHER BASES BESIDES BASE TEN (DECIMAL), IT WILL GENERALLY BE CALLED THE RADIX POINT.

B. BASE FIVE

1. DIGITS: 0, 1, 2, 3, 4

2. NOTATION 24_5 IS READ "TWO FOUR BASE FIVE". NO SUBSCRIPT MEANS BASE TEN

3. COUNTING: $0_5, 1_5, 2_5, 3_5, 4_5, 10_5, 11_5, 12_5, 13_5, 14_5, 20_5, 21_5, 22_5, 23_5, 24_5, 30_5, 31_5, 32_5, 33_5, 34_5, 40_5, 41_5, 42_5, 43_5, 44_5, 100_5, 101_5, \ldots$

4. EXPANDED FORM: $423.413_5 =$

$$4(5^2) + 2(5^1) + 3(5^0) + 4(5^{-1}) + 1(5^{-2}) + 3(5^{-3})$$

NOTE: THE RADIX POINT IS BETWEEN 3 AND 4 IN 423.413_5

ALSO, ON THE LEFT SIDE OF THE EQUALITY WE ARE IN BASE FIVE AND ON THE RIGHT SIDE WE ARE IN BASE TEN.

C. BASE FOUR

1. DIGITS: 0, 1, 2, 3
2. COUNTING: $0_4, 1_4, 2_4, 3_4, 10_4, 11_4, 12_4, 13_4, 20_4, 21_4, 22_4, 23_4, 30_4, 31_4, 32_4, 33_4, 100_4, 101_4, 102_4, 103_4, 110_4, 111_4, 112_4, 113_4, 120_4, \ldots$
3. EXPANDED FORM: $231.23_4 =$

$$2(4^2) + 3(4^1) + 1(4^0) + 2(4^{-1}) + 3(4^{-2})$$

D. BASE TWO (BINARY NUMBER SYSTEM)

1. DIGITS: 0, 1
2. COUNTING: $0_2, 1_2, 10_2, 11_2, 100_2, 101_2, 110_2, 111_2, 1000_2, 1001_2, 1010_2, 1011_2, 1100_2, \ldots$

3. EXPANDED FORM: $1101.11_2 =$

$$1(2^3)+1(2^2)+0(2^1)+1(2^0)+1(2^{-1})+1(2^{-2})$$

E. BASE EIGHT (OCTAL NUMBER SYSTEM)

1. DIGITS: 0, 1, 2, 3, 4, 5, 6, 7

F. BASE SIXTEEN (HEXADECIMAL NUMBER SYSTEM)

1. DIGITS: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

2. CORRESPONDENCE BETWEEN DECIMAL AND HEXADECIMAL NUMERATION

DECIMAL	HEXADECIMAL
9	9
10	A
11	B
12	C
13	D
14	E
15	F

3. EXPANDED FORM: $F07.9B_{16} =$
$15(16^2)+0(16^1)+7(16^0)+9(16^{-1})+11(16^{-2})$

4. COUNTING: $1_{16}, 2_{16}, 3_{16}, 4_{16}, 5_{16}, 6_{16}, 7_{16}, 8_{16}, 9_{16}, A_{16}, B_{16}, C_{16}, D_{16}, E_{16}, F_{16}, 10_{16}, 11_{16}, 12_{16}, 13_{16}, 14_{16}, 15_{16}, 16_{16}, 17_{16}, 18_{16}, 19_{16}, 1A_{16}, 1B_{16}, 1C_{16}, 1D_{16}, 1E_{16}, 1F_{16}, 20_{16}, 21_{16}, 22_{16}, 23_{16}, 24_{16}, 25_{16}, 26_{16}, 27_{16}, 28_{16}, 29_{16}, 2A_{16}, 2B_{16}, 2C_{16}, 2D_{16}, 2E_{16}, 2F_{16}, 30_{16}$

NEXT NUMBER AFTER 99_{16} IS $9A_{16}$.
NEXT NUMBER AFTER $9F_{16}$ IS $A0_{16}$.
NEXT NUMBER AFTER FF_{16} IS 100_{16}.

G. HOMEWORK (OIS)

1. COUNT FROM 0 TO 34_8 IN OCTAL
2. WRITE 2034.506_8 IN EXPANDED FORM.
3. WHAT IS THE NEXT NUMBER AFTER
   2033_4
   5077_8
   $B9_{16}$
   EF_{16}

H. CONVERSION TO BASE TEN

1. WRITE IN EXPANDED FORM, THEN PERFORM THE ARITHMETIC.

2. CONVERT 235.14_8 TO BASE TEN

$$235.14_8 = 2(8^2)+3(8^1)+5(8^0)+1(8^{-1})+4(8^{-2}) = 2(64)+3(8)+5(1)+1(1/8)+4(1/8^2) = 128+24+5+(1/8)+(4/64) = 157+(12/64) = 157+(3/16)$$

I. HOMEWORK (OIS) CONVERT EACH OF THE FOLLOWING TO BASE TEN

1. 101101.101_2
2. 302.14_8
3. $FA3_{16}$
4. $20B.03_{16}$

J. CONVERTING FROM BASE TEN

1. TO CONVERT 763 TO BASE FIVE KEEP DIVIDING BY FIVE; KEEP TRACK OF THE REMAINDERS, TO GET THE ANSWER WRITE THE REMAINDERS IN REVERSE ORDER.

2.

	REMAINDERS
$5\,)\,763$	
$5\,)\,152$	3
$5\,)\,30$	2
$5\,)\,6$	0
$5\,)\,1$	1
0	1

ANSWER

11023_5

3. TO MAKE SURE THE NOTATION ABOVE IS UNDERSTOOD:

$763 \div 5 = 152$ WITH REMAINDER 3

$152 \div 5 = 30$ WITH REMAINDER 2

$30 \div 5 = 6$ WITH REMAINDER 0

$6 \div 5 = 1$ WITH REMAINDER 1

$1 \div 5 = 0$ WITH REMAINDER 1

THE REMAINDERS IN REVERSE ORDER GIVES THE ANSWER 11023_5

4. THIS WAS A WHOLE NUMBER. NEXT FRACTIONS.

5. CONVERT .5216 TO BASE <u>FIVE</u>

PLAN: KEEP MULTIPLYING BY <u>FIVE</u>.
TAKE THE INTEGER PARTS IN ORDER (WITH A DECIMAL IN FRONT) AS THE ANSWER

$5(.5216) = 2.608$ TAKE 2

.608 IS LEFT

$5(.608) = 3.04$ TAKE 3

.04 IS LEFT

$5(.04) = 0.2$ TAKE 0

.2 IS LEFT

$5(.2) = 1.0$ TAKE 1

ANSWER $.5216 = .2301_5$

<u>BRIEF</u> <u>HINT</u> WHY THIS WORKS:

CLAIM $.5216 = 2(1/5) + 3(1/5^2) + 0(1/5^3) + 1(1/5^4)$

$5(.5216) = .5216 \div (1/5) = 2.608$

SO THERE ARE 2 ONE-FIFTHS IN .5216 WITH .608 ONE-FIFTHS LEFT OVER. NEXT STEP DETERMINES $(1/5^2)$

6. TERMINATING DECIMALS IN ONE BASE NEED NOT TERMINATE IN ANOTHER BASE!

CONVERT .1 TO BASE <u>TWO</u>

$2(.1) = 0.2$ TAKE 0
$2(.2) = 0.4$ TAKE 0
$2(.4) = 0.8$ TAKE 0
$2(.8) = 1.6$ TAKE 1
$2(.6) = 1.2$ TAKE 1
$2(.2) = 0.4$ TAKE 0
$2(.4) = 0.8$ TAKE 0
$2(.8) = 1.6$ TAKE 1
$2(.6) = 1.2$ TAKE 1
...

NOTE THE ENDLESS REPEATING PATTERN

$.1 = .00011001100110011001100 11\ldots_2$

NOTE: COMPUTERS STORE REAL NUMBERS IN BASE TWO (BINARY) AND THEY DO NOT HAVE AN INFINITE AMOUNT OF SPACE, SO WHEN .1 IS PUT IN A COMPUTER AN APPROXIMATION OF IT IN BINARY FORM IS PUT IN ... ERROR ENTERS!

K. HOMEWORK (OIS)

1. CONVERT .125 TO BINARY
2. CONVERT .3 TO BINARY
3. CONVERT .125 TO OCTAL
4. CONVERT .3 TO OCTAL
5. CONVERT .75 TO HEXADECIMAL
6. CONVERT 12.125 TO BINARY

L. SHORTCUT BINARY ↔ OCTAL CONVERSION

1. OCTAL TO BINARY : REPLACE EACH OCTAL DIGIT WITH ITS 3 DIGIT BINARY EQUIVALENT.

CONVERT 213.45_8 TO BINARY

2	1	3	.	4	5_8
010	001	011	.	100	101_2

ANSWER 10001011.100101_2

NOTE $2_8 = 10_2 = 010_2$
$1_8 = 1_2 = 001_2$

ADD LEADING ZEROS TO MAKE 3-DIGIT BINARY

2. <u>BINARY</u> TO <u>OCTAL</u>: REPLACE EACH 3 BINARY DIGITS (STARTING AT THE RADIX POINT) WITH ITS OCTAL EQUIVALENT. YOU CAN ADD LEADING OR TRAILING ZEROS.

CONVERT 10111101.00101_2 TO OCTAL

$$010\ 111\ 101.\ 001\ \ 010_2$$
$$2\ \ \ 7\ \ \ 5\ .\ 1\ \ \ \ 2_8$$

ANSWER 275.12_8

NOTE $101_2 = 5_8$,
$111_2 = 7_8$, ETC.

M. HOMEWORK (OIS)

1. CONVERT 4703.051_8 TO BINARY
2. CONVERT 11101011.00111_2 TO OCTAL
3. CONVERT 11101011.00111_2 TO HEXADECIMAL
   HINT: USE GROUPS OF FOUR FROM RADIX POINT AND REPLACE WITH HEXADECIMAL EQUIVALENT.
4. CONVERT $A3B.7F_{16}$ TO BINARY. NO HINT.

N. ADDITION IN OTHER BASES

USE THE SAME METHOD FOR CARRYING THAT YOU DO IN BASE TEN

$$\begin{array}{r} {\scriptstyle 1\ 1\ \leftarrow \text{CARRY}} \\ 213_5 \\ +\ 34_5 \\ \hline 302_5 \end{array}$$

$3_5 + 4_5 = 12_5$ PUT DOWN 2, CARRY THE 1.

$3_5 + 1_5 + 1_5 = 10_5$ PUT DOWN 0, CARRY THE 1

$2_5 + 1_5 = 3_5$

SO $213_5 + 34_5 = 302_5$

O. HOMEWORK (OIS)

1. $576_8 + 435_8 =$
2. $101101_2 + 1111_2 =$
3. $A8F_{16} + 97B_{16} =$

P. SUBTRACTION IN OTHER BASES

1. THINK BUNDLES OF 5 WHEN YOU SEE BASE 5 POSITIONAL NOTATION
   243_5 CAN BE THOUGHT OF AS

PENCILS, LOOSE AND IN BUNDLES. 243_5 HAS 3 PENCILS LOOSE, 4 BUNDLES OF 5 HELD TOGETHER BY A STRING, AND 2 BUNDLES OF 5^2 HELD BY A STRING. CUT THE STRING AND 5 BUNDLES OF 5 FALL OUT.

2. "BORROWING" IS STEALING, NEVER TO RETURN AGAIN! CONSIDER $43_5 - 4_5$. YOU CANNOT TAKE AWAY 4 LOOSE PENCILS FROM 3 LOOSE PENCILS. SO CUT THE STRING ON ONE OF THE 4 BUNDLES OF 5, LEAVING 3. YOU NOW HAVE 8 (I.E. 13_5) LOOSE PENCILS; TAKE AWAY 4. RESULT 34_5.

THIS IS SHOWN BY:

$$\begin{array}{r} \overset{3}{\cancel{4}}\,{}^{1}3_5 \\ -\ 4_5 \\ \hline 34_5 \end{array}$$

$13_5 - 4_5 = 4_5$

3. FIND $243_5 - 44_5$.

$$\begin{array}{r} \overset{1}{\cancel{2}}\,\overset{\overset{1}{3}}{\cancel{4}}\,{}^{1}3_5 \\ -\ 44_5 \\ \hline 144_5 \end{array}$$

$13_5 - 4_5 = 4$

4. FIND $241_5 - 43_5$

$$\begin{array}{r} \overset{1}{\cancel{2}}\overset{1}{\overset{3}{\cancel{4}}}\overset{1}{1}_5 \\ -43_5 \\ \hline 143_5 \end{array}$$

$11_5 - 3_5 = 6 - 3 = 3$

$13_5 - 4_5 = 8 - 4 = 4$

5. DOUBLE "BORROW". FIND

$3021_5 - 1434_5$

$$\begin{array}{r} \overset{2}{\cancel{3}}\overset{4}{\cancel{0}}\overset{1}{\overset{1}{\cancel{2}}}\overset{1}{1}_5 \\ -1434_5 \\ \hline 1032_5 \end{array}$$

$11_5 - 4_5 = 6 - 4 = 2$

$11_5 - 3_5 = 6 - 3 = 3$

Q. HOMEWORK (OIS)

1. $213_5 - 24_5 =$
2. $40123_5 - 2341_5 =$
3. $23B_{16} - BF_{16} =$
4. $101101_2 - 11010_2 =$

R. MULTIPLICATION IN OTHER BASES

1. MULTIPLY 34_5 BY 3_5

$$\begin{array}{r} {}^{2\,2}\\ 34_5 \\ 3_5 \\ \hline 212_5 \end{array}$$

$(3_5)(4_5) = 12 = 22_5$

PUT DOWN 2, CARRY 2

$(3_5)(3_5) + 2_5 = 11 = 21_5$

PUT DOWN 1, CARRY 2

2. MULTIPLY 34_5 BY 23_5

$$\begin{array}{r} {}^{1\,1}\\ {}^{2\,2}\\ 34_5 \\ 23_5 \\ \hline 212_5 \\ 123_5 \\ \hline 1442_5 \end{array}$$

$(3_5)(4_5) = 12 = 22_5$

$(3_5)(3_5) + 2_5 = 11 = 21_5$

$(2_5)(4_5) = 8 = 13_5$

$(2_5)(3_5) + 1_5 = 7 = 12_5$

S. HOMEWORK (OIS)

$$\begin{array}{r} 101101_2 \\ \times\ 101_2 \\ \hline \end{array} \qquad \begin{array}{r} A7_{16} \\ \times\ 3B_{16} \\ \hline \end{array} \qquad \begin{array}{r} 27_8 \\ \times\ 35_8 \\ \hline \end{array}$$

T. DIVISION IN OTHER BASES

1. FIND $4312_5 \div 23_5$. GET THE MULTIPLES OF 23_5. FIND THE LARGEST MULTIPLE THAT DOES NOT EXCEED THE CURRENT DIVIDEND.

$$
\begin{array}{r}
134 \\
23_5\,\overline{)\,4312_5} \\
\underline{23} \\
201 \\
\underline{124} \\
222 \\
\underline{202} \\
20
\end{array}
$$

$(23_5)(1_5) = 23_5$

$(23_5)(2_5) = 101_5$

$(23_5)(3_5) = 124_5$

$(23_5)(4_5) = 202_5$

$4312_5 \div 23_5 = 134_5$ WITH REMAINDER 20_5

U. HOMEWORK (OIS)

1. $34241_5 \div 42_5 =$
2. $110101101_2 \div 101_2 =$
3. $42301_5 \div 123_5 =$

RELATIONS

A. CARTESIAN PRODUCT OF <u>SETS</u>.

1. $A \times B$ IS READ "A CROSS B" OR "THE CARTESIAN PRODUCT OF A AND B".

2. THE <u>ORDERED PAIR</u> a, b IS DENOTED (a, b). ORDER MAKES A DIFFERENCE. $(3,2) \neq (2,3)$

   $(a,b) = (c,d)$ IFF $a = c$ AND $b = d$.

   A SET OF TWO ELEMENTS IS NOT AN ORDERED PAIR SINCE $\{3,2\} = \{2,3\}$

3. DEFINITION OF CARTESIAN PRODUCT

   $$A \times B = \{(a,b) \mid a \varepsilon A \text{ AND } b \varepsilon B\}$$

4. EXAMPLES: LET $A = \{2,3\}$ $B = \{1,3,6\}$

   $A \times B = \{(2,1),(2,3),(2,6),(3,1),(3,3),(3,6)\}$

   $B \times A = \{(1,2),(1,3),(3,2),(3,3),(6,2),(6,3)\}$

   $(2,1) \varepsilon A \times B$ BUT $(2,1) \notin B \times A$, SO $A \times B$ IS NOT NECESSARILY EQUAL TO $B \times A$, GENERALLY.

   $A \times A = \{(2,2),(2,3),(3,2),(3,3)\}$

B. <u>RELATION</u>: A RELATION r <u>FROM</u> SET A <u>TO</u> SET B IS A SUBSET OF $A \times B$. (SINCE ONLY 2 SETS ARE INVOLVED, r IS CALLED A BINARY RELATION.)

1. THE <u>DOMAIN</u> OF RELATION r, DENOTED $\mathrm{dom}(r)$, IS THE SET OF ALL FIRST TERMS OF r

$$\mathrm{dom}(r) = \{ a \mid (a,b) \in r \}$$

2. THE <u>RANGE</u> OF RELATION r, DENOTED $\mathrm{ran}(r)$, IS THE SET OF ALL SECOND TERMS OF r.

$$\mathrm{ran}(r) = \{ b \mid (a,b) \in r \}$$

3. EXAMPLE: $A = \{2,3\}$ $B = \{1,3,6\}$

$A \times B = \{(2,1),(2,3),(2,6),(3,1),(3,3),(3,6)\}$

LET $r = \{(2,1),(2,6),(3,1)\}$. $r \subseteq A \times B$.

r IS A RELATION FROM A TO B

$\mathrm{dom}(r) = \{2,3\}$

$\mathrm{ran}(r) = \{1,6\}$

4. NOTATION: SUPPOSE r IS A RELATION FROM A TO B, $a \in A$, $b \in B$.

a IS RELATED TO b, DENOTED arb, IFF $(a,b) \in r$. $\boxed{\text{INFIX NOTATION}}$

SUPPOSE $r = \{(2,1),(2,6),(3,1)\}$

$2r1$ SINCE $(2,1) \in r$

$2r6$ SINCE $(2,6) \in r$

$3r1$ SINCE $(3,1) \in r$

$2 \not r 3$ SINCE $(2,3) \notin r$

$3 \not r 6$ SINCE $(3,6) \notin r$

5. THE RELATION "IS LESS THAN"

$< = \{(x,y) \mid x,y \in R$ AND x IS LESS THAN $y\}$

$(3,5) \in <$ SO $3<5$

$(2,9) \in <$ SO $2<9$

$(7,5) \notin <$ SO $7 \not< 5$

6. THE RELATION "IsTheFriendOf"

LET $A = \{$Bob, Sue, Sam$\}$ $B = \{$Joe, Jan$\}$

IsTheFriendOf $= \{$(Bob, Joe), (Sue, Jan)$\}$

(Bob, Joe) $\in$ IsTheFriendOf, SO

Bob IsTheFriendOf Joe

C. A RELATION ON A IS A SUBSET OF $A \times A$, THAT IS, A RELATION FROM A TO A.

LET $A = \{2, 3\}$ $B = \{1, 3, 6\}$

$r = \{(2,2), (2,3)\}$ IS A RELATION ON A, SINCE $r \subseteq A \times A$.

$w = \{(1,6), (6,3)\}$ IS A RELATION ON B, SINCE $w \subseteq B \times B$.

THE RELATION "IS LESS THAN" IS A RELATION ON R, THE SET OF REALS.

D. INVERSE OF A RELATION r

1. r^{-1} IS READ "r INVERSE" (NOT "r TO THE MINUS 1")

2. DEFINITION: $r^{-1} = \{(b,a) \mid (a,b) \in r\}$

3. TO GET THE INVERSE OF RELATION r, SIMPLY TURN AROUND THE ORDERED PAIRS IN r.

4. SUPPOSE $r = \{(2,2), (2,3)\}$

$r^{-1} = \{(2,2), (3,2)\}$

$dom(r^{-1}) = \{2,3\} = ran(r)$

$ran(r^{-1}) = \{2\} = dom(r)$

E. HOMEWORK (OIS)

1. LET $E = \{a, b\}$, $T = \{1, 2, 3, 4\}$

   $E \times T =$

   $T \times E =$

   NAME A RELATION r FROM E TO T

   NAME A RELATION f ON T

   $r^{-1} =$

   $dom(r) =$

   $ran(r) =$

2. DEFINE r^{-1} PURELY, WHERE r IS A RELATION FROM A TO B.

   $r^{-1} = \{(b,a) \mid (a,b) \in r\}$

   $= \{z \mid \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\}$

F. LIST OF RELATIONS TO BE USED LATER.

1. $<, >, =, \leq, \geq$ ARE ALL RELATIONS ON R.

2. LET I DENOTE THE SET OF INTEGERS

$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

DEFINITION

$a|b$ IS READ "a DIVIDES b" OR "a IS A FACTOR OF b"

$\forall a, b \in I$, $a|b$ IFF $\exists k \in I \ni ak = b$

$7|14$ SINCE $7(2) = 14$

$7|21$ SINCE $7(3) = 21$

$7|0$ SINCE $7(0) = 0$

$7|-7$ SINCE $7(-1) = -7$

$7|7$ SINCE $7(1) = 7$

DEFINITION

G. RELATION r IS REFLEXIVE ON SET A IFF $\forall x \in A, x r x$

(I.E. $\forall x \in A, (x, x) \in r$)

1. $=$ IS REFLEXIVE ON R: $\forall x \in R,\ x=x$

2. $<$ IS <u>NOT</u> REFLEXIVE ON R: $\sim(\forall x \in R,\ x<x)$

3. $\leq$ IS REFLEXIVE ON R: $\forall x \in R,\ x \leq x$

4. $|$ IS REFLEXIVE ON I: $\forall x \in I,\ x|x$

$\boxed{\text{DEFINITION}}$

H. RELATION r IS <u>SYMMETRIC</u> ON SET A IFF $\forall x,y \in A$, IF $x r y$, THEN $y r x$
(I.E. $\forall x,y \in A$, IF $(x,y) \in r$, THEN $(y,x) \in r$)

1. $=$ IS SYMMETRIC ON R
   $\forall x,y \in R$, IF $x=y$, THEN $y=x$.

2. $<$ IS NOT SYMMETRIC ON R
   $\sim(\forall x,y \in R$, IF $x<y$, THEN $y<x)$
   $2<5$ BUT NOT $5<2$

3. $|$ IS NOT SYMMETRIC ON I
   $\sim(\forall x,y \in I$, IF $x|y$, THEN $y|x)$
   $7|14$ BUT $14 \nmid 7$

DEFINITION

I. RELATION r ON SET A IS <u>TRANSITIVE</u> IFF $\forall x, y, z \in A$, IF xry AND yrz, THEN xrz.
(I.E. $\forall x, y, z \in A$, IF $(x,y) \in r$ AND $(y,z) \in r$, THEN $(x,z) \in r$)

1. $=$ IS TRANSITIVE ON R
$\forall x, y, z \in R$, IF $x=y$ AND $y=z$, THEN $x=z$.

2. $<$ IS TRANSITIVE ON R
$\forall x, y, z \in R$, IF $x<y$ AND $y<z$, THEN $x<z$.

3. LET P BE THE SET OF ALL PEOPLE. THE RELATION "LIKES" ON P IS <u>NOT</u> TRANSITIVE ON P.
BOB LIKES SUE. SUE LIKES JOE.
BUT BOB DOES NOT LIKE JOE.

DEFINITION

J. RELATION r IS <u>ANTISYMMETRIC</u> ON SET A IFF $\forall x, y \in A$, IF xry AND $x \neq y$, THEN $y \not r x$ (I.E. $\forall x, y \in A$, IF $(x,y) \in r$ AND $x \neq y$, THEN $(y,x) \notin r$).

1. $<$ <u>IS</u> ANTISYMMETRIC ON R: $\forall x, y \in R$, IF $x<y$ AND $x \neq y$, THEN $y \not< x$.

2. "IsASiblingOf" IS NOT ANTISYMMETRIC ON P: SAM IsASiblingOf JOE AND SAM $\neq$ JOE AND JOE IsASiblingOf SAM.

$\sim(\forall x, y \in P$, IF x IsASiblingOf y AND $x \neq y$, THEN NOT (y IsASiblingOf x)).

$\exists x, y \in P \ni x$ IsASiblingOf y AND $x \neq y$ AND y IsASiblingOf x (REPLACE x WITH SAM AND y WITH JOE).

K. $\boxed{\text{DEFINITION}}$ RELATION r ON SET A IS AN <u>EQUIVALENCE RELATION</u> IFF r IS REFLEXIVE, SYMMETRIC, AND TRANSITIVE ON A.

1. $=$ IS AN EQUIVALENCE RELATION ON R.
2. $<$ IS <u>NOT</u> AN EQUIVALENCE RELATION ON R. IT IS NOT REFLEXIVE ON R. IT IS NOT SYMMETRIC ON R.

DEFINITION

L. RELATION r ON SET A IS A <u>PARTIAL ORDER</u> IFF r IS REFLEXIVE, ANTI-SYMMETRIC, AND TRANSITIVE **ON** A

1. "IS A FACTOR OF" (I.E. $|$) IS A PARTIAL ORDER **ON** THE SET OF POSITIVE INTEGERS (NOT ON THE SET OF INTEGERS)
2. $<$ IS NOT A PARTIAL ORDER **ON** R.
   $<$ IS NOT REFLEXIVE **ON** R.

M. LOOKING AT RELATION PROPERTIES IN ORDERED PAIR NOTATION.

LET $A = \{2,3,4\}$.

$r = \{(2,2),(3,3),(4,4),(3,4),(4,3),(2,4),(4,2)\}$

r IS REFLEXIVE AND SYMMETRIC **ON** A.

r IS <u>NOT</u> TRANSITIVE ON A SINCE $(3,4) \in r$ AND $(4,2) \in r$, BUT $(3,2) \notin r$

r IS <u>NOT</u> AN EQUIVALENCE RELATION ON A.

r IS <u>NOT</u> ANTISYMMETRIC ON A SINCE $(3,4) \in r$ AND $3 \neq 4$ AND $(4,3) \in r$

r IS <u>NOT</u> A PARTIAL ORDER **ON** A.

N. HOMEWORK (OIS)

1. TELL WHETHER EACH OF THE FOLLOWING IS REFLEXIVE, SYMMETRIC, TRANSITIVE, EQUIVALENCE RELATION, OR PARTIAL ORDER. EXPLAIN YOUR ANSWER

   a. $\leq$ ON THE SET OF REALS

   b. $|$ ON THE SET OF INTEGERS

   c. LET $N = \{1, 2, 3, \ldots\}$ DEFINE RELATION r ON N AS FOLLOWS: $\forall p, q \in N$, $p\, r\, q$ IFF p AND q HAVE THE SAME REMAINDER WHEN DIVIDED BY 5. (NOTE $12\, r\, 22$ SINCE 12 AND 22 HAVE THE SAME REMAINDER WHEN DIVIDED BY 5, NAMELY, 2.)

   d. DEFINE RELATION r ON THE SET, I, OF INTEGERS BY: $\forall p, q \in I$, $p\, r\, q$ IFF $p^2 = q^2$

e. DEFINE r ON THE SET I BY:
$\forall x, y \in I$, $x r y$ IFF 5 IS A FACTOR OF $x - y$.

f. LET $A = \{3, 5, 7\}$
LET $r = \{(5,5), (5,7), (7,5), (7,3), (3,7), (5,3), (3,5)\}$

2. MAKE UP A RELATION r ON A THAT IS REFLEXIVE AND TRANSITIVE BUT NOT SYMMETRIC (WHERE $A = \{3, 5, 7\}$).

3. SYMMETRIC AND ANTISYMMETRIC ARE <u>NOT</u> THE NEGATION OF EACH OTHER. MAKE UP A RELATION ON A SET THAT IS BOTH SYMMETRIC AND ANTISYMMETRIC.

0. <u>PARTITION</u>: A PARTITION OF A SET E IS A COLLECTION P OF DISJOINT SUBSETS OF E WHOSE UNION IS E.

EXAMPLES: LET $E = \{1,2,3,4,5,6,7,8\}$

1. $P_1 = \{\{1,2,3,4\}, \{5,6,7,8\}\}$ YES
2. $P_2 = \{\{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}\}$ YES
3. $P_3 = \{\{1,3,5\}, \{2,4\}, \{6,7,8\}\}$ YES
4. $N_1 = \{\{1,2,3\}, \{4,5,6\}\}$ NO. THE UNION OF THE ELEMENTS OF N_1 DOES NOT EQUAL E.
5. $N_2 = \{\{1,2,3\}, \{4,5,6\}, \{3,7,8\}\}$ NO. THE SUBSETS $\{1,2,3\}$ AND $\{3,7,8\}$ ARE NOT DISJOINT. (DISJOINT MEANS THEIR INTERSECTION IS THE EMPTY SET)
6. $P_4 = \{\{1,2,3,4,5,6,7,8\}\}$ YES
7. $N_3 = \{1,2,3,4,5,6,7,8\}$ NO. THE ELEMENTS OF N_3 ARE NOT <u>SUBSETS</u> OF E. THEY ARE <u>ELEMENTS</u> OF E... BIG DIFFERENCE.

P. A <u>PARTITION INDUCES AN EQUIVALENCE RELATION</u>: LET P BE A PARTITION OF A SET E. LET r BE THE RELATION ON E SUCH THAT $(x,y) \in r$ IFF $\exists\, H \in P \ni x \in H$ AND $y \in H$. r IS AN EQUIVALENCE RELATION.

EXAMPLE: $E = \{1,2,3,4,5,6,7,8\}$

$P = \{\{1,3,5\},\{2,4\},\{6,7,8\}\}$. THE EQUIVALENCE RELATION r INDUCED BY P IS:

$$r = \{(1,1),(3,3),(5,5),(1,3),(3,1),(1,5),(5,1),(3,5),(5,3),$$
$$(2,2),(4,4),(2,4),(4,2),$$
$$(6,6),(7,7),(8,8),(6,7),(7,6),(6,8),(8,6),(7,8),(8,7)\}$$

Q. <u>EQUIVALENCE CLASSES</u>: SUPPOSE r IS AN EQUIVALENCE RELATION ON E. $\forall\, e \in E$, THE EQUIVALENCE CLASS DETERMINED BY e, DENOTED $[e]$, IS DEFINED BY:

$$[e] = \{x \mid x \in E \text{ AND } x\, r\, e\}$$

EXAMPLES OF EQUIVALENCE CLASSES AND EQUIVALENCE RELATIONS FOLLOW:

1. LET $E = \{1,2,3,4,5,6,7,8\}$

LET $r = \{(x,y) \mid x, y \varepsilon E$ AND $3 \mid (x-y)\}$

$[1] = \{1,4,7\}$ NOTE $3 \mid (1-4)$, SO $(1,4) \varepsilon r$.

$(4,1) \varepsilon r$. $4 r 1$. SO $4 \varepsilon [1]$

$[2] = \{2,5,8\} = [5] = [8]$

$[3] = \{3,6\} = [6]$

NOTE: $[1] = [4] = [7]$.

LET $P = \{[1],[2],[3],[4],[5],[6],[7],[8]\}$

$= \{[e] \mid e \varepsilon E\}$

$= \{\{1,4,7\},\{2,5,8\},\{3,6\}\}$ A PARTITION

2. LET $E = \{1,2,3,4,5\}$. LET $r = \{(1,1),(2,2),$
$(3,3),(4,4),(5,5),(1,3),(3,1),(1,5),(5,1),(3,5),(5,3),$
$(2,4),(4,2)\}$

$[1] = \{1,3,5\} = [3] = [5]$

$[2] = \{2,4\} = [4]$

$\{[1],[2],[3],[4],[5]\} = \{[e] \mid e \varepsilon E\}$

$= \{\{1,3,5\},\{2,4\}\}$ A PARTITION

NOTE: EQUIVALENCE RELATIONS INDUCE PARTITIONS.

R. <u>THEOREM</u>: IF r IS AN EQUIVALENCE RELATION ON E, THEN $P=\{[e] \mid e \in E\}$ IS A PARTITION OF **E**.

SO WE HAVE NOW SEEN THAT PARTITIONS INDUCE EQUIVALENCE RELATIONS AND EQUIVALENCE RELATIONS INDUCE PARTITIONS. THEY ARE EQUIVALENT IDEAS IN DISGUISE.

S. HOMEWORK (OIS)

1. EACH OF THE FOLLOWING RELATIONS IS AN EQUIVALENCE RELATION ON $E=\{1,2,3,4,5,6\}$

   a. $r=\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),$
   $(1,2),(2,1),(1,4),(4,1),(2,4),(4,2),(3,5),(5,3),$
   $(3,6),(6,3),(5,6),(6,5)\}$

   $[3]=$ ____________________

   $P=\{[e] \mid e \in E\}=$ ____________________

   b. $r=\{(x,y) \mid x,y \in E$ AND $4 \mid (y-x)\}$

   $P=\{[e] \mid e \in E\}=$ ____________________

2. PROVE If r is an equivalence relation on set M and $a,b \in M$, then [$a r b$ <u>iff</u> $[a]=[b]$].

2. LISTED BELOW ARE PARTITIONS OF $E = \{1, 2, 3, 4, 5, 6\}$. GIVE THE EQUIVALENCE RELATION INDUCED BY EACH PARTITION.

   a. $\{\{1, 5, 6\}, \{2, 3, 4\}\}$

   b. $\{\{1, 5\}, \{2, 3\}, \{4, 6\}\}$

   c. $\{\{1, 5\}, \{2, 6\}, \{3\}, \{4\}\}$

[CHAPTER 33]

FUNCTIONS

A. <u>DEFINITION</u>: A <u>FUNCTION</u> IS A SET OF ORDERED PAIRS SUCH THAT NO TWO ORDERED PAIRS HAVE THE SAME FIRST TERM

1. <u>FUNCTION</u> $f = \{(1,7), (3,6), (5,6)\}$

   NOTE: A FUNCTION IS A SPECIAL TYPE OF A RELATION.

   LET $A = \{1, 3, 5\}$ $B = \{6, 7\}$

   f IS A RELATION FROM A TO B

   $dom(f) = \{1,3,5\}$ $ran(f) = \{6,7\}$

2. <u>NOT</u> <u>A</u> <u>FUNCTION</u> $f = \{(2,5), (3,6), (2,7)\}$

B. <u>NOTATION</u>: $f: H \to K$ MEANS f IS A FUNCTION, $dom(f) = H$, AND $ran(f) \subseteq K$

LET $f = \{(1,7), (3,6), (5,6)\}$ $A = \{1,3,5\}$ $B = \{6,7\}$ $K = \{6,7,8\}$ IT IS TRUE THAT:

$f: A \to B$, $f: A \to K$, AND $f: A \to R$

C. $f: H \to K$ IS READ "f IS A FUNCTION FROM H <u>INTO</u> K".

D. $f(x)$ NOTATION: $f(x)$ IS READ "f OF x". $f(x)$ IS THE SECOND TERM OF THE ORDERED PAIR IN f WHOSE FIRST TERM IS x.

LET $f = \{(1,7), (3,6), (5,6)\}$

1. $f(1)=7$, $f(3)=6$, $f(5)=6$
2. NOTE $f(1)=7 \leftrightarrow (1,7) \in f$
3. $f(x)=y \leftrightarrow (x,y) \in f$

E. EQUATIONS ARE NOT FUNCTIONS BUT CAN <u>DEFINE</u> FUNCTIONS

1. $f(x)=\frac{1}{x}$ DEFINES THE FUNCTION

$$f=\left\{\left(x,\tfrac{1}{x}\right) \middle| x \in R \text{ AND } x \neq 0\right\}$$

$$\text{dom}(f)=\{x \mid x \in R \text{ AND } x \neq 0\}$$

$$f: R-\{0\} \rightarrow R$$

2. THE DOMAIN OF A FUNCTION DEFINED BY AN EQUATION IS THE SET OF ALL x VALUES THAT MAKE SENSE.

LET $f(x)=\sqrt{2-x}$

$\text{dom}(f)=\{x \mid x \leq 2\}$

SCRATCH WORK

$2-x \geq 0$

$2 \geq x$

$x \leq 2$

$\boxed{\text{DEFINITION}}$

F. ONE-TO-ONE FUNCTIONS (1-1)

GIVEN $f: A \to B$. f IS ONE-TO-ONE IFF $\forall w, v \in A$, IF $f(w) = f(v)$, THEN $w = v$

1. NOT 1-1 $\quad f = \{(1,7),(3,7)\}$

   $1 \neq 3$ BUT $f(1) = f(3) = 7$

2. 1-1 $\quad f = \{(1,5),(2,6),(3,7)\}$

   NOTE: CONTRAPOSITIVE IN THE DEFINITION

   $\forall w, v \in A$, IF $w \neq v$, THEN $f(w) \neq f(v)$

3. $f: R \to R \quad f(x) = 3x - 2 \quad f$ IS 1-1

   (SHOW $\forall w, v \in R$, IF $f(w) = f(v)$, THEN $w = v$)

   a. ASSUME $w, v \in R$ AND $f(w) = f(v)$
      (SHOW $w = v$)

   b. $3w - 2 = 3v - 2$ $\qquad$ a, DEF. OF f

   c. $3w = 3v$ $\qquad$ b

   d. $w = v$ $\qquad$ c

4. $f: R \to R \quad f(x) = x^2 \quad f$ IS NOT 1-1

   (SHOW $\exists w, v \in R \ni f(w) = f(v)$ AND $w \neq v$)

   a. $3, -3 \in R$ AND $3 \neq -3$

   b. $f(3) = 3^2 = 9 = (-3)^2 = f(-3)$

5. $f:(-\infty,0)\rightarrow R$ $f(x)=x^2+4$. PROVE f 1-1

RECALL $|x| = x$ IF $x \geq 0$ $\quad \sqrt{x^2}=|x|$

$-x$ IF $x<0$

(SHOW $\forall w,v \in (-\infty,0)$, IF $f(w)=f(v)$, THEN $w=v$.)

a. ASSUME $w,v \in (-\infty,0)$ AND $f(w)=f(v)$.

(SHOW $w=v$)

b.	$w^2+4=v^2+4$	a
c.	$w^2=v^2$	b
d	$\sqrt{w^2}=\sqrt{v^2}$	c
e.	$\lvert w\rvert=\lvert v\rvert$	d
f.	$w<0$ AND $v<0$	a
g.	$-w=-v$	e, f, DEF. OF AB. V.
h.	$w=v$	

G. HOMEWORK (OIS) PROVE ONE-TO-ONE

1. $f:R\rightarrow R$ $f(x)=4x+3$
2. $f:(-\infty,-1)\rightarrow R$ $f(x)=-2x^2+3$
3. $f:(-\infty,2)\rightarrow R$ $f(x)=x^2-4x+7$

H. ONTO FUNCTIONS:

1. DEF. $f: A \to B$ IS ONTO IFF $\text{ran}(f) = B$
2. THEOREM $f: A \to B$ IS ONTO IFF $\forall y \in B, \exists x \in A \ni f(x) = y$
3. GENERALLY, THE THEOREM WILL BE USED TO PROVE ONTO
4. IF $f: A \to B$ IS ONTO WE SAY "f IS A FUNCTION FROM A ONTO B".
5. LET $f = \{(1,0), (5,4)\}$ LET $H = \{1,5\}$, $K = \{0,4\}$, $M = \{0,1,2,3,4,5\}$

   $f: H \to K$ IS ONTO

   $f: H \to M$ IS NOT ONTO
6. $f: (-\infty, 0) \to (4, \infty)$ $f(x) = x^2 + 4$

   PROVE f IS ONTO.

   (SHOW $\forall y \in (4,\infty), \exists x \in (-\infty, 0) \ni f(x) = y$)

   SCRATCH WORK - NOT THE PROOF

   $f(x) = y$ $\qquad \sqrt{x^2} = |x| = \sqrt{y-4}$

   $x^2 + 4 = y$ $\qquad -x = \sqrt{y-4}$

   $x^2 = y - 4$ $\qquad x = -\sqrt{y-4}$

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a. PROOF: ASSUME $y \in (4,\infty)$

(SHOW $\exists x \in (-\infty, 0) \ni f(x) = y$)

b.	$y > 4$	a
c.	$y - 4 > 0$	b
d.	$\sqrt{y-4} > 0$	c
e.	$-\sqrt{y-4} < 0$	d
f.	LET $x = -\sqrt{y-4}$	
g.	$x \in (-\infty, 0)$	e, f
h.	$f(x) = x^2 + 4$	definition of $f(x)$
i.	$= (-\sqrt{y-4})^2 + 4 = (y-4)+4 = y$	f
j	$\exists x \in (-\infty, 0) \ni f(x) = y$	h, i, g

I. HOMEWORK (OIS) PROVE ONTO

1. $f: (-\infty, 0) \to (-2, \infty)$ $\quad f(x) = 3x^2 - 2$
2. $f: (-\infty, 2) \to (3, \infty)$ $\quad f(x) = x^2 - 4x + 7$
3. $f: R \to R$ $\quad f(x) = 5x - 4$

DEFINITION 33-336

J. COUNTABLE SETS : RECALL $N = \{1, 2, 3, \ldots\}$.

A SET E IS COUNTABLE IFF $E = \phi$ OR $\exists$ A FUNCTION f FROM N ONTO E.

1. LET $E = \{2, 4\}$. E IS COUNTABLE. LET $f : N \to E \ni f(1) = 2$, $f(2) = 2$, $f(3) = 4$, $f(4) = 4$ AND $\forall n \in N$, IF $n > 4$, THEN $f(n) = 4$. f IS A FUNCTION FROM N ONTO E. THIS METHOD OF DEFINING A FUNCTION CAN BE EXTENDED TO ANY FINITE SET, SO...

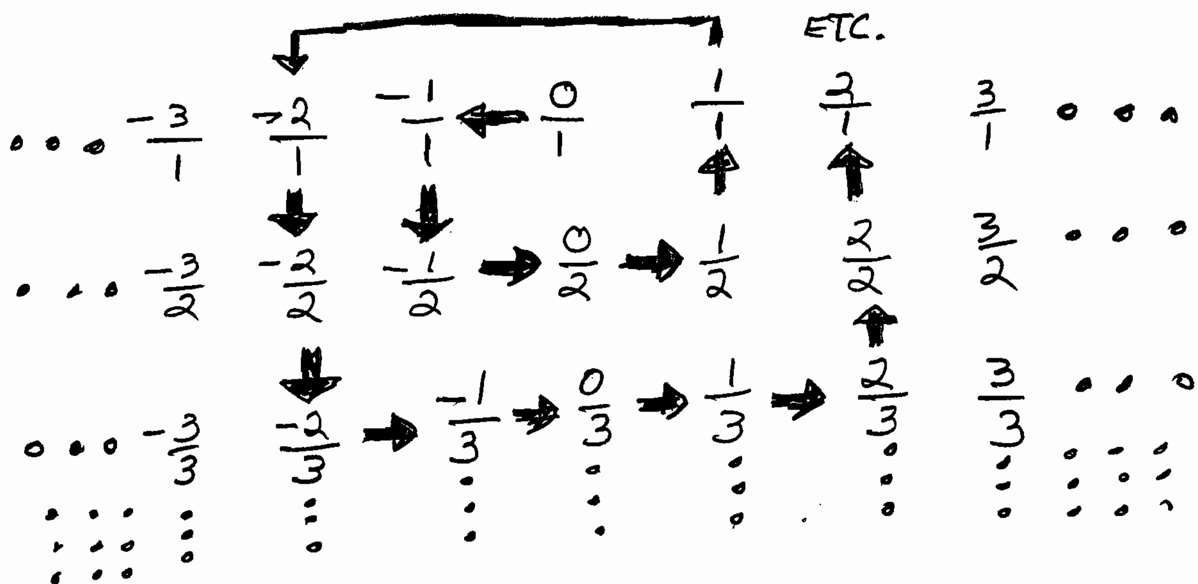
2. EVERY FINITE SET IS COUNTABLE.

3. LET $E = \{2, 4, 6, \ldots\}$. E IS COUNTABLE SINCE $f(n) = 2n$ DEFINES A FUNCTION FROM N ONTO E.

4. LET $I = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. I IS COUNTABLE. DEFINE f BY : $f(1) = 0$, $f(2) = 1$, $f(3) = -1$, $f(4) = 2$, $f(5) = -2$, $f(6) = 3$, $f(7) = -3$, $f(8) = 4$, $f(9) = -4$, .. ETC. f IS A FUNCTION FROM N ONTO I.

5. LET $Q = \left\{ \frac{p}{q} \;\middle|\; p, q \text{ ARE INTEGERS AND } q \neq 0 \right\}$. Q IS THE SET OF RATIONAL NUMBERS. (Q CAN STAND FOR "QUOTIENT"). Q IS COUNTABLE. THE FOLLOWING WILL DEFINE THE FUNCTION f FROM N ONTO Q.

THE GRID BELOW CONTAINS ALL OF THE RATIONAL NUMBERS, WITH SOME NUMBERS DUPLICATED. THE FUNCTION f DEFINED BELOW IS A FUNCTION FROM N ONTO Q. IT IS DEFINED BY AN EXPANDING CIRCULAR PATTERN THAT EVENTUALLY ENCOMPASSES ALL THE RATIONALS, STARTING AT $\frac{0}{1}$



$f(1)=\frac{0}{1}$, $f(2)=\frac{-1}{1}$, $f(3)=\frac{-1}{2}$, $f(4)=\frac{0}{2}$,
$f(5)=\frac{1}{2}$, $f(6)=\frac{1}{1}$, $f(7)=\frac{-2}{1}$, $f(8)=\frac{-2}{2}$,
$f(9)=\frac{-2}{3}$, $f(10)=\frac{-1}{3}$, $f(11)=\frac{0}{3}$, $f(12)=\frac{1}{3}$,
$f(13)=\frac{2}{3}$, $f(14)=\frac{2}{2}$, $f(15)=\frac{2}{1}$, $f(16)=\frac{-3}{1}$, ETC.

YOU MIGHT THINK ALL INFINITE SETS ARE COUNTABLE ... WRONG ... KEEP READING

K. <u>UNCOUNTABLE</u> MEANS NOT COUNTABLE.

L. THE SET OF REAL NUMBERS IS UNCOUNTABLE.

THIS CAN BE PROVEN BY AN INDIRECT PROOF THAT WILL BE INDICATED BUT NOT GIVEN. ONE CAN ASSUME THERE IS A FUNCTION f FROM N ONTO R AND GET A CONTRADICTION BY CONSTRUCTING A REAL NUMBER b NOT IN $\{f(1), f(2), f(3) \ldots\} = R$.

M. HOMEWORK (OIS)

1. PROVE $\Theta = \{1, 3, 5, 7, 9, \ldots\}$ IS COUNTABLE.
2. PROVE $T = \{1, 4, 6\}$ IS COUNTABLE.
3. THE SET OF IRRATIONAL NUMBERS IS THE SET OF ALL REALS THAT ARE NOT RATIONAL. PROVE THE SET OF IRRATIONAL NUMBERS IS UNCOUNTABLE BY <u>INDIRECT</u> PROOF. ASSUME THE IRRATIONALS ARE COUNTABLE AND GET A CONTRADICTION BY PROVING THE REALS COUNTABLE.

PROOF IDEA OF WHY THE REALS ARE UNCOUNTABLE.

SHOW $(0,1)$ IS UNCOUNTABLE (Zehna, Johnson idea)

1. ASSUME $(0,1)$ IS COUNTABLE GET A CONTRADICTION.
2. THERE IS A FUNCTION f FROM $\{1,2,3,\ldots\}$ <u>ONTO</u> $(0,1)$

SO EVERY ELEMENT OF $(0,1)$ MUST BE IN THE RANGE

FOR A CONTRADICTION WE WILL CONSTRUCT AN ELEMENT OF $(0,1)$ NOT IN THE RANGE.

CONSTRUCTION <u>IDEA</u> BY EXAMPLE

SUPPOSE

$f(1) = .314215\ldots$ NOT FIRST DIGIT

.5 SOMETHING IN $(0,1)$

$f(2) = .719345\ldots$ NOT SECOND DIGIT

$f(3) = .2500000\ldots$ NOT 3^{rd} DIGIT

$f(4) = .3145892\ldots$ NOT 4^{th} DIGIT

$\vdots$

$x =$. $\boxed{5}$ $\boxed{2}$ $\boxed{5}$ $\boxed{2}$ ... etc

CONSTRUCTED NUMBER

$x \neq f(1)$ since 1^{st} digits differ

$x \neq f(2)$ since 2^{nd} digits differ

$x \neq f(3)$ since 3^{rd} digits differ

$\vdots$

x IS NOT IN $\{f(1), f(2), f(3), \ldots\}$ but it has to be since $x \in (0,1)$ and f is ONTO $(0,1)$ CONTR.!!

N. <u>CARDINAL</u> <u>NUMBERS</u>: THE CARDINAL NUMBER FOR A SET E IS DENOTED #E. <u>INTUITIVELY</u>, IT IS THE NUMBER OF ELEMENTS IN THE SET.

1. $\#\{3,7\} = 2 = \#\{4,6\}$
2. $\#\{1,3,5\} = 3 = \#\{a,b,c\}$
3. $\#\{3,7\} < \#\{a,b,c\}$
4. <u>EQUIPOTENT</u> <u>SETS</u>: E AND F ARE EQUIPOTENT SETS IFF THERE IS A 1-1 FUNCTION FROM E ONTO F.

   THUS, E AND F ARE EQUIPOTENT SETS IFF $\#E = \#F$. $\{3,7\}$ AND $\{4,6\}$ ARE EQUIPOTENT SETS.
5. THEOREM: INFINITE SETS THAT ARE COUNTABLE ARE EQUIPOTENT, AND HENCE HAVE THE SAME CARDINAL NUMBER.

   LET Ir DENOTE THE SET OF IRRATIONALS

   $\#N = \#I = \#Q < \#R = \#Ir$
6. WE HAVE SEEN 2 INFINITE CARDINAL NUMBERS, $\#N$ AND $\#R$, WITH $\#N < \#R$. IS THERE A HIGHER DEGREE OF INFINITY AND MORE ... KEEP READING

θ THE POWER SET OF A IS THE SET OF ALL SUBSETS OF A (DENOTED $P(A)$)

1. LET $A = \{3\}$ $\quad P(A) = \{\phi, \{3\}\}$
   $\#A = 1 \qquad \#P(A) = 2$
2. LET $A = \{p, q\}$ $\quad P(A) = \{\phi, \{p,q\}, \{p\}, \{q\}\}$
   $\#A = 2 \qquad \#P(A) = 4$
3. <u>THEOREM</u>: FOR EVERY SET A, $\#A < \#P(A)$
   (THIS IS TRUE EVEN FOR INFINITE SETS A)
4. <u>COROLLARY</u> (A RESULT THAT FOLLOWS EASILY FROM A THEOREM) THERE ARE INFINITELY MANY DEGREES OF INFINITY!

   RECALL $N = \{1, 2, 3, \ldots\}$ COUNTABLE

   LET $N_1 = P(N)$ UNCOUNTABLE $\quad \#N < \#N_1$

   LET $N_2 = P(N_1)$ UNCOUNTABLE $\quad \#N_1 < \#N_2$

   LET $N_3 = P(N_2)$ UNCOUNTABLE $\quad \#N_2 < \#N_3$

   ... KEEP ON FOREVER.

   <u>MIND EXPANDING NEWS</u>: $\#N, \#N_1, \#N_2, \ldots$ REPRESENTS COUNTABLY INFINITELY MANY DEGREES OF INFINITY... BUT, THE TRUTH IS... THERE ARE UNCOUNTABLY MANY DEGREES OF INFINITY!!! THAT'S BIG.

P. HOMEWORK (OIS)

1. LET $A = \{1, 2, 3\}$. $P(A) =$
2. SUPPOSE $A = \{1, 2, 3, \ldots, n\}$. $P(A)$ HAS HOW MANY ELEMENTS?
3. LET $A = \{1\}$. $P(A) =$ ________.
   LET $A_1 = P(A)$. $P(A_1) =$ ________.
   $\# A =$ ____, $\# A_1 =$ ____, $\# P(A_1) =$ ____
4. THEOREM: IF $A \subseteq B$, THEN $\# A \leq \# B$.
   $N_1 = P(N)$, $N_2 = P(N_1)$, $N_3 = P(N_2)$, ...
   NAME A SET T SUCH THAT FOR EVERY POSITIVE INTEGER i, $\# N_i < \# T$.
   PROVE YOUR ANSWER WITH THE HELP OF THIS THEOREM.

Q CARDINAL AND ORDINAL NUMBERS (INTUITIVELY DISCUSSED).

1. THERE ARE MORE NUMBERS THAN YOU THOUGHT. THERE IS A FIRST ORDINAL NUMBER AFTER THE ORDINAL NUMBERS $0, 1, 2, 3, \ldots$ IT IS ω (LITTLE OMEGA) LET US COUNT AWHILE.

$0, 1, 2, 3, \ldots, \omega, \omega+1, \omega+2, \omega+3, \ldots, 2\omega, 2\omega+1, 2\omega+2, \ldots, 3\omega, 3\omega+1, 3\omega+2, \ldots, 4\omega, 4\omega+1, \ldots$
... KEEP GOING A LONG WHILE, ..., Ω (BIG OMEGA) $\Omega+1, \Omega+2, \ldots, \Omega+\omega, \Omega+\omega+1, \Omega+\omega+2, \ldots$,
THERE IS A SET A ASSOCIATED WITH EACH ORDINAL NUMBER, AND A CARDINAL NUMBER, **DENOTED** $\#A$, ASSOCIATED WITH THE SET A

ORDINAL NUMBER	SET A	COUNTABLE/ UNCOUNTABLE	$\#A$
0	ϕ	COUNTABLE	0
1	$\{1\}$	COUNTABLE	1
2	$\{1,2\}$	COUNTABLE	2
3	$\{1,2,3\}$	COUNTABLE	3
...	...	...	...
ω	$N=\{1,2,3,\ldots\}$	COUNTABLE	ω
$\omega+1$	$N \cup \{a\}$	COUNTABLE	ω
$\omega+2$	$N \cup \{a,b\}$	COUNTABLE	ω
...	...	...	...
Ω	$P(N)$	UNCOUNTABLE	Ω
$\Omega+1$	$P(N) \cup \{a\}$	UNCOUNTABLE	Ω
...	...	...	...

2. EVERY CARDINAL NUMBER IS AN ORDINAL NUMBER, BUT NOT VICE VERSA. EACH OF THE NONNEGATIVE INTEGERS IS BOTH A CARDINAL AND AN ORDINAL NUMBER. ω IS THE FIRST INFINITE ORDINAL NUMBER AND IS ALSO A CARDINAL NUMBER. $\omega+1$ IS AN ORDINAL NUMBER, BUT NOT A CARDINAL NUMBER. $\#N=\omega$. YOU COULD THINK OF THE ORDINAL NUMBER $\omega+1$ ASSOCIATED WITH $N\cup\{a\} = \{1,2,3,\ldots\}\cup\{a\}$. HOWEVER, $\exists$ A 1-1 FUNCTION f FROM $N\cup\{a\}$ ONTO N, THEREFORE, $N\cup\{a\}$ AND N ARE EQUIPOTENT. SO, $\#(N\cup\{a\})=\#N=\omega$. THIS WOULD ALSO BE TRUE OF $N\cup\{a,b\}$ AND THE ORDINAL NUMBER $\omega+2$. $1,2,3,\ldots$ ARE THE FINITE ORDINAL NUMBERS. $\omega, \omega+1, \omega+2, \ldots, 2\omega, 2\omega+1, \ldots, 3\omega, \ldots$ AND ALL ORDINAL NUMBERS LESS THAN Ω AS THE COUNTABLE ORDINAL NUMBERS, ALONG WITH $0,1,2,3,\ldots$. Ω IS THE FIRST UNCOUNTABLE ORDINAL AND IS ALSO A CARDINAL NUMBER. A CARDINAL NUMBER IS THE FIRST ORDINAL NUMBER OF THAT DEGREE OF FINITENESS OR INFINITENESS, LOOSELY SPEAKING.

3. THIS WAS AN ATTEMPT TO INTUITIVELY EXPLAIN A DEEP TOPIC.

R. HOMEWORK (OIS)

1. IS THERE SUCH A THING AS THE LARGEST NUMBER LESS THAN ω?

2. WHAT IS THE SMALLEST ORDINAL NUMBER GREATER THAN EACH NUMBER IN THE SEQUENCE $\omega, 2\omega, 3\omega, 4\omega, 5\omega, 6\omega, \ldots$?

3. NAME AN ORDINAL NUMBER K SUCH THAT $K > \omega$, THERE IS NO LARGEST ORDINAL NUMBER LESS THAN K, AND K IS NOT A CARDINAL NUMBER.

4. LET A BE A SET SUCH THAT $\#A = \Omega$. NAME A CARDINAL NUMBER GREATER THAN Ω.

5. WHAT IS THE SMALLEST ORDINAL NUMBER K SUCH THAT $K > \omega^2$ AND THERE IS NO LARGEST ORDINAL NUMBER LESS THAN K.

[CHAPTER 34] 34-343B

RECURSION

A. A RECURSIVE TASK IS A TASK THAT, IN ITS DEFINITION, REFERS TO ITSELF IN ALL CASES EXCEPT IN SIMPLE STOPPING CASE(S).

B. RECURSIVE TASK TO CALCULATE n FACTORIAL (n A NONNEGATIVE INTEGER)

1. NOTATION FOR n FACTORIAL : $n!$
2. EXAMPLES OF $n!$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$3! = 3 \cdot 2 \cdot 1 = 6$

$2! = 2 \cdot 1 = 2$

$1! = 1$

$0! = 1$ MATH FORMULAS REMAIN TRUE WITH THIS DEFINITION

3. RECURSIVE TASK FACTORIAL(n) WILL NOW BE GIVEN.

```
TASK FACTORIAL (n):
IF n = 0 THEN
   RETURN (1)
ELSE
   RETURN (n·FACTORIAL (n-1))
```

4. THIS WAY OF WRITING A TASK LOOKS SOMEWHAT LIKE A COMPUTER LANGUAGE AND SOMEWHAT LIKE ENGLISH. IT IS CALLED PSEUDOCODE. THE WORD PSEUDO MEANS FAKE.

5. NOTICE WHEN $n=0$, THE ANSWER OF 1 IS RETURNED. ALSO SAID, FACTORIAL (0) RETURNS 1. (I.E. $0!=1$) TO FIND FACTORIAL (1), NOTE $1 \neq 0$, SO GO TO THE ELSE STATEMENT. THE ELSE STATEMENT REFERS TO ITSELF:

FACTORIAL(1) = 1 · FACTORIAL (0)

WE HAVE SEEN FACTORIAL (0) = 1, SO

FACTORIAL(1) = 1·1 = 1 (I.E. $1! = 1$)

TO FIND FACTORIAL(2), NOTE $2 \neq 0$, SO GO TO THE ELSE STATEMENT. THE ELSE STATEMENT REFERS TO THE TASK FACTORIAL ITSELF:

FACTORIAL (2) = 2 • FACTORIAL(1)

WE HAVE SEEN FACTORIAL(1) REFERS AGAIN TO ITSELF TO GET FACTORIAL(1) = 1. HENCE, FACTORIAL(2) = 2 • 1 = 2 (I.E. 2! = 2)

C. WHAT THIS BOOK CALLS TASKS, SOME COMPUTER LANGUAGES CALL PROCEDURES, SUBPROGRAMS, OR FUNCTIONS.

D. RECURSIVE TASK TO CALCULATE FIBONACCI NUMBERS. THE FIBONACCI SEQUENCE IS: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... AFTER THE FIRST TWO TERMS 1 AND 1, TO GET THE NEXT TERM ADD TOGETHER THE TWO TERMS IMMEDIATELY PRECEDING IT.

$2 = 1+1$, $3 = 1+2$, $5 = 2+3$, $8 = 3+5$, ...

TASK FIB(n):

```
IF n=1 OR n=2 THEN
   RETURN (1)
ELSE
   RETURN ( FIB(n-1) + FIB(n-2))
```

1. NOTE FIB(n) HAS TWO STOPPING CASES $n=1$, $n=2$. IN EITHER OF THOSE CASES 1 IS RETURNED
2. FIND FIB(3), THE 3RD FIBONACCI NUMBER (IT SHOULD BE 2). NOW $3 \neq 1$ AND $3 \neq 2$ SO GO TO THE ELSE CLAUSE: FIB(3) = FIB(2) + FIB(1) = 1 + 1 = 2.
3. THROUGH REPEATED SELF REFERENCES FIB(4) WOULD RETURN 3.

E. HOMEWORK (OIS)

1. WRITE A RECURSIVE TASK THAT COMPUTES $1+2+3+\cdots+n$
2. WRITE A RECURSIVE TASK THAT COMPUTES 5^n.
3. WRITE A RECURSIVE TASK THAT COMPUTES $1^2+2^2+3^2+\cdots+n^2$

F. FOR FACTORIAL(n) AND FIB(n) WE FIRST SAW THE SEQUENCE OF TERMS AND THEN THE PSEUDOCODE WAS WRITTEN. NOW VICE VERSA.

```
TASK WHAT (n):
IF n < 5 THEN
  RETURN (8)
ELSE
  RETURN ( 4 + WHAT(n-3))
```

1. WHAT(4) RETURNS 8.
2. WHAT(7) RETURNS 4 + WHAT(4) $= 4+8 = 12$
3. WHAT(10) RETURNS 4 + WHAT(7) $= 4 + (4 + \text{WHAT}(4)) = 4 + (4+8) = 16$

G. <u>PARAMETERS</u>: THE n IN WHAT(n) IS CALLED A PARAMETER. THE ONLY RECURSION SEEN SO FAR HAS HAD ONLY ONE PARAMETER ... NOW MORE THAN ONE PARAMETER.

H. TASK TWO(n, p):

IF $p < 7$ AND $n < 5$ THEN
RETURN $(3+p)$
ELSE
RETURN $(2 + \text{TWO}(n-5, p-4))$

1. $\text{TWO}(4, 2) = 3+2 = 5$
2. $\text{TWO}(7, 3) = 2 + \text{TWO}(2, -1) = 2 + (3 + (-1)) = 4$
3. $\text{TWO}(14, 10) = 2 + \text{TWO}(9, 6) =$
$2 + (2 + \text{TWO}(4, 2)) = 2 + (2 + (3 + 2)) = 9$

I. HOMEWORK (OIS)

1. CONSIDER: TASK FIND(m)
IF $m < 4$ THEN
RETURN (5)
ELSE
RETURN $(6 + \text{FIND}(m-2))$

WHAT ARE FIND(2), FIND(7), AND FIND(10)?

2. CONSIDER: TASK IT(m, q):
IF $m < 8$ AND $q < 6$ THEN
RETURN $(2 + mq)$
ELSE
RETURN $(3 - \text{IT}(m-5, q-4)$

WHAT ARE IT(4,5), IT(2,7), AND IT(14,10)?

(CONTINUED)

3. WRITE A RECURSIVE TASK POWER(m,p) THAT RETURNS m^p FOR $m \neq 0$ AND p, A NONNEGATIVE INTEGER.

J. PARAMETERS NEED NOT BE NUMBERS.

1. ARRAY: AN ORDERED LIST OF ELEMENTS OF THE SAME TYPE, WITH THE POSITIONS NAMED.

A	7	3	12	5
	A(1)	A(2)	A(3)	A(4)

ARRAY NAME: A

FIRST POSITION A(1) A(1) = 7

SECOND POSITION A(2) A(2) = 3

A(2) IS READ "A OF 2"

IN A(1), A(2), A(3), A(4), THE 1,2,3,4 ARE CALLED SUBSCRIPTS OR INDICES

2. PLAN FOR IndxOfMin(A,n) A RECURSIVE TASK THAT HAS AN ARRAY AS A PARAMETER AND RETURNS THE INDEX WHERE THE MINIMUM VALUE IS STORED (IN THIS CASE THE INDEX IS 2 WHERE THE MINIMUM VALUE 3 IS STORED). THE n MEANS CONSIDER POSITIONS 1 THROUGH n.

IF CONSIDERING ONLY POSITION 1,

THEN RETURN INDEX 1, ELSE GET THE INDEX WHERE THE MINIMUM OCCURS IN THE FIRST N-1 POSITIONS; COMPARE WITH THE N^{TH} POSITION; PICK THE INDEX WHERE THE SMALLER VALUE IS.

3. <u>TASK</u> IndxOfMin (A,n):

```
IF n = 1 THEN
   RETURN (1)
ELSE
  IF A(n) < A(IndxOfMin(A,n-1)) THEN
    RETURN(n)
  ELSE
    RETURN(IndxOfMin(A,n-1)).
```

K. HOMEWORK (OIS)

1. WRITE A RECURSIVE TASK IndxOfMax (A,n) THAT RETURNS THE INDEX OF THE MAXIMUM VALUE IN THE ARRAY A IN POSITIONS 1 THROUGH n.

2. WRITE A RECURSIVE TASK Indx2ndMin(A,n) THAT FINDS THE INDEX OF THE 2nd SMALLEST ENTRY IN THE ARRAY A IN POSITIONS 1 THROUGH n, WHERE NO TWO ENTRIES IN A ARE EQUAL. IT IS PERMISSIBLE TO USE THE TASK IndxOfMin WITHIN THE TASK Indx2ndMin.

3. WRITE A RECURSIVE TASK SUM(A,n) THAT FINDS THE SUM $A(1)+A(2)+A(3)+\cdots+A(n)$.

L. IN COMPUTER SCIENCE RECURSION IS A POWERFUL TOOL TO HELP SOLVE BY THE DIVIDE AND CONQUER PRINCIPLE. A COMPLEX PROBLEM IS DIVIDED INTO SMALLER PROBLEMS, EACH OF WHICH IS SIMPLER TO SOLVE THAN THE ORIGINAL. THE PROGRAMMER DOES THE HIGH LEVEL THINKING. THE COMPUTER DOES THE WORK.

[CHAPTER 35]

COUNTING TECHNIQUES

A. HELPFUL TO KNOW COUNTING TECHNIQUES:

1. TO COUNT HOW MANY DIFFERENT PIECES OF INFORMATION CAN BE STORED IN A COMPUTER DEVICE.
2. TO COUNT HOW MANY STEPS A COMPUTER ALGORITHM COULD TAKE TO PERFORM A TASK.

B. <u>MULTIPLICATION</u> <u>PRINCIPLE</u>:

A PROCESS HAS 2 STAGES. THE FIRST CAN BE DONE m WAYS THE SECOND CAN BE DONE p WAYS. THE PROCESS CAN BE DONE $m \cdot p$ WAYS.

C. EXAMPLE: SAM HAS 2 PAIRS OF PANTS AND 3 SHIRTS. HOW MANY DIFFERENT OUTFITS CAN SAM HAVE?

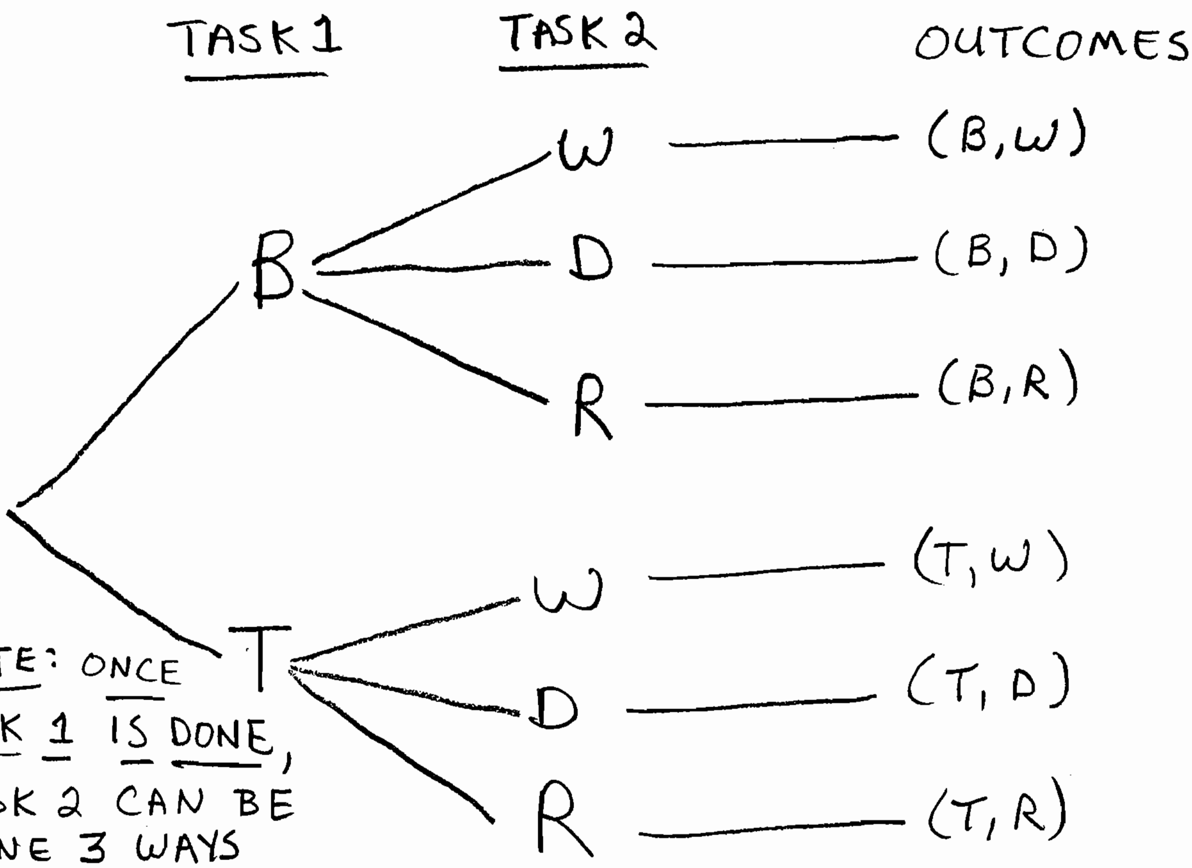
TASK 1: PANTS SELECTION		TASK 2: SHIRT SELECTION		PROCESS: OUTFIT SELECTION
2	$\cdot$	3	=	6

35 - 345

TREE DIAGRAM

2 PAIRS OF PANTS: B = BLUE T = TAN

3 SHIRTS: W = WHITE D = DARK R = RED



NOTE: ONCE TASK 1 IS DONE, TASK 2 CAN BE DONE 3 WAYS

TO COUNT THE OUTCOMES, COUNT THE LEAVES UNDER THE TASK 2 COLUMN.

THIS IS AN EXAMPLE OF A PROCESS MODEL (MORE WILL COME ON THE NEXT PAGE)

D. <u>PROCESS MODEL</u>: A MATHEMATICAL MODEL WILL BE MADE OF A PROCESS. WE WILL BE ABLE TO COUNT THE NUMBER OF OUTCOMES IN THE MODEL. WE WILL THEN BE ABLE TO DETERMINE THE NUMBER OF OUTCOMES IN REALITY (THIS IS A MODIFICATION OF TREE DIAGRAMS SEEN IN MANY BOOKS.)

APPLICATION TO PREVIOUS EXAMPLE.

PANTS: B = BLUE   T = TAN

SHIRTS: W = WHITE   D = DARK   R = RED

TASK 1	TASK 2	OUTCOMES
B	W	(B, W)
	D	(B, D)
	R	(B, R)
T	W	(T, W)
	D	(T, D)
	R	(T, R)

(B, W) IN THE MODEL REPRESENTS AN OUTFIT OF BLUE PANTS AND WHITE SHIRT IN REALITY. WE CAN COUNT 6 OUTCOMES IN THE MODEL. REALITY MATCHES ONE-TO-ONE WITH THE MODEL SO 6 OUTFITS IN REALITY

E. APPLICATION TO COMPUTER SCIENCE:

BIT: HOLDS THE VALUES 0, 1

MEMORY LOCATION: 16 BITS

HOW MANY DIFFERENT VALUES CAN BE STORED IN A MEMORY LOCATION?

EXAMPLE: 1011 1100 1001 1111

TASK i: SET THE BIT AT POSITION i

$$\underbrace{2}_{\text{WAYS TO FILL POSITION 1}} \cdot \underbrace{2}_{\text{WAYS TO FILL POS. 2}} \cdots \underbrace{2}_{\text{WAYS TO FILL POS. 16}} = \underbrace{2^{16}}_{\text{OUTCOMES}}$$

<u>IDEA</u>: RELATE BITS TO BINARY NOTATION AND POSITION 1 AS A SIGN BIT (0 = POSITIVE, 1 = NEGATIVE)

0111 1111 1111 1111 $= 2^{15} - 1 = 32767$ THE VALUE FOR MAXIMUM POSITIVE INTEGER

0000 0000 0000 0000 $= 0$

LEAVE REMAINING $2^{15} = 32768$ FOR THE NEGATIVE INTEGERS. SO -32768 IS THE MINIMUM NEGATIVE INTEGER.

(FOR 32 BIT MEMORY LOCATIONS:

$2^{31} - 1 = 2147483647$ MAXIMUM

$-2^{31} = -2147483648$ MINIMUM)

F. LET $M = \{1,2,3,4,5,6,7\}$

1. HOW MANY 3 DIGIT NUMBERS CAN BE FORMED USING ELEMENTS OF M WITH REPETITION OF DIGITS ALLOWED?

   EXAMPLES: 233, 333, 713

   THE MULTIPLICATION PRINCIPLE CAN BE EXTENDED TO MORE THAN 2 STAGES

STAGE 1		STAGE 2		STAGE 3		OUTCOMES
$\underline{7}$	$\cdot$	$\underline{7}$	$\cdot$	$\underline{7}$	$=$	7^3
# OF WAYS TO FILL IN 1ST DIGIT		# OF WAYS TO FILL IN 2ND DIGIT		# OF WAYS TO FILL IN 3RD DIGIT		

2. HOW MANY 3 DIGIT NUMBERS CAN BE FORMED USING ELEMENTS OF M WITH <u>NO</u> DIGITS REPEATED? (313 IS BAD.)

STAGE 1		STAGE 2		STAGE 3		OUTCOMES
$\underline{7}$	$\cdot$	$\underline{6}$	$\cdot$	$\underline{5}$	$=$	$7 \cdot 6 \cdot 5$
# OF WAYS TO FILL IN 1ST DIGIT		# OF WAYS TO FILL IN 2ND DIGIT		# OF WAYS TO FILL IN 3RD DIGIT		

<u>NOTE:</u> ONCE STAGE 1 IS DONE, PICKED, SETTLED UPON, STAGE 2 CAN BE DONE 6 WAYS... RECALL THE TREE DIAGRAM

3. HOW MANY 3 DIGIT <u>ODD</u> NUMBERS CAN BE FORMED USING ELEMENTS OF M WITH <u>NO</u> REPETITION OF DIGITS?

STAGE 2	STAGE 3	STAGE 1	OUTCOMES
6 ·	5 ·	4 =	$6 \cdot 5 \cdot 4$
# OF WAYS TO FILL IN 1ST DIGIT	# OF WAYS TO FILL IN 2ND DIGIT	# OF WAYS TO FILL IN 3RD DIGIT	

G. LET $T = \{0, 1, 2, 3, 4, 5\}$

1. HOW MANY 3 DIGIT NUMBERS CAN BE FORMED USING ELEMENTS OF T WITH REPETITION OF DIGITS ALLOWED?

NOTE: 043 IS <u>NOT</u> CONSIDERED A 3 DIGIT NUMBER, BUT THE 2 DIGIT NUMBER, 43.

STAGE 1	STAGE 2	STAGE 3	OUTCOMES
5 ·	6 ·	6 =	$5 \cdot 6^2$
# OF WAYS TO FILL IN 1ST DIGIT	# OF WAYS TO FILL IN 2ND DIGIT	# OF WAYS TO FILL IN 3RD DIGIT	

2. HOW MANY 3 DIGIT NUMBERS CAN BE FORMED USING ELEMENTS OF T WITH <u>NO</u> REPETITION OF DIGITS?

STAGE 1	STAGE 2	STAGE 3	OUTCOMES
5 $\cdot$	5 $\cdot$	4	$= 5^2 \cdot 4$
# OF WAYS TO FILL IN 1ST DIGIT	# OF WAYS TO FILL IN 2ND DIGIT	# OF WAYS TO FILL IN 3RD DIGIT	

3. <u>COMPLEMENT RULE</u>: NUMBER OF <u>DESIRED</u> OUTCOMES = TOTAL NUMBER OF POSSIBLE OUTCOMES <u>MINUS</u> NUMBER OF <u>UNDESIRED</u> OUTCOMES.

HOW MANY 3 DIGIT NUMBERS CAN BE FORMED USING ELEMENTS OF T WITH SOME DIGITS REPEATED? THE RESULTS OF THE PAST 2 PROBLEMS AND THE COMPLEMENT RULE WILL BE USED.

# OF WAYS 3 DIGIT # WITH REPETITIONS ALLOWED	MINUS	# OF WAYS 3 DIGIT # WITH <u>NO</u> REPETITIONS	=	# OF WAYS 3 DIGIT # SOME DIGITS REPEATED

$$5 \cdot 6^2 - 5^2 \cdot 4 = 80$$

H. NUMBER OF SUBSETS OF $\{1,2,3,\ldots,n\}$ REVISITED. (CHAPTER 33, P, 2)

ILLUSTRATION WITH $n=4$

NUMBER OF SUBSETS OF $\{1,2,3,4\}$

MATH MODEL: ORDERED 4-TUPLE, LIKE (Y, N, Y, Y), WOULD CORRESPOND TO $\{1,3,4\}$ Y = YES   N = NO

Y IN POSITION 1 = 1 IN THE SET
N IN POSITION 1 = 1 NOT IN THE SET
Y IN POSITION 2 = 2 IN THE SET
N IN POSITION 2 = 2 NOT IN THE SET, ETC.

(N, N, N, Y) CORRESPONDS TO $\{4\}$

EQUIVALENT PROBLEM: FIND THE NUMBER OF ORDERED 4-TUPLES WITH ONLY Y OR N IN EACH POSITION.

EACH POSITION CAN BE FILLED IN 2 WAYS.
THERE ARE 4 POSITIONS.

ANSWER: 2^4

GENERALIZATION: THERE ARE 2^n SUBSETS OF $\{1,2,3,\ldots,n\}$. IF A HAS n ELEMENTS, P(A) HAS 2^n ELEMENTS.

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I. A CLUB IS MADE UP OF 8 PEOPLE NAMED A, B, C, D, E, F, G, AND H. THE OFFICES ARE PRESIDENT, VP, AND REPORTER. NO PERSON CAN HOLD 2 OFFICES.

1. HOW MANY SLATES OF OFFICERS ARE POSSIBLE?

STAGE 1		STAGE 2		STAGE 3	OUTCOMES
8	$\cdot$	7	$\cdot$	6	$= 8 \cdot 7 \cdot 6$
# OF PRES.		# OF VP		# OF REPORTERS	

2. HOW MANY SLATES OF OFFICERS ARE POSSIBLE WHERE EITHER F OR G MUST BE PRESIDENT?

STAGE 1		STAGE 2		STAGE 3	OUTCOMES
2	$\cdot$	7	$\cdot$	6	$= 2 \cdot 7 \cdot 6$
# OF PRES.		# OF VP		# OF REPORTERS	

3. HOW MANY SLATES OF OFFICERS ARE THERE WHERE G MUST HOLD OFFICE?

# OF WAYS TO HAVE OFFICERS	MINUS	# OF WAYS G CANNOT HOLD OFFICE	=

$8 \cdot 7 \cdot 6 \quad - \quad 7 \cdot 6 \cdot 5$

4. HOW MANY SLATES OF OFFICERS ARE THERE WHERE G AND H MUST HOLD OFFICE? A DIFFERENT LOOK FOR THE STAGES.

STAGE 1		STAGE 2		STAGE 3	OUTCOMES
3	$\cdot$	2	$\cdot$	6	$= 3 \cdot 2 \cdot 6$
# OF OFFICES G CAN HOLD		# OF OFFICES H CAN HOLD		# OF PEOPLE FOR 3RD OFFICE	

MODEL:

(V, R, B)

V → G is VP

R → H is Reporter

B → B is President

J. HOMEWORK (OIS) YOU DO NOT HAVE TO MULTIPLY THE NUMBERS OUT.

FOR PROBLEMS 1 THROUGH 7 LICENSE PLATES ARE TO HAVE EXACTLY <u>FIVE</u> CHARACTERS FROM $\{A, B, C, \ldots, Z, 0, 1, 2, 3, \ldots, 9\}$

1. HOW MANY PLATES ARE THERE WHERE THE CHARACTERS ARE ALL LETTERS AND REPETITIONS ARE ALLOWED?
2. HOW MANY PLATES ARE THERE IF THE CHARACTERS ARE ALL LETTERS AND <u>NO</u> REPETITIONS ARE ALLOWED?
3. HOW MANY PLATES ARE THERE IF THE FIRST TWO CHARACTERS ARE LETTERS AND THE LAST 3 ARE EITHER A DIGIT OR A LETTER, REPETITIONS ALLOWED?
4. HOW MANY PLATES ARE THERE IF THE FIRST CHARACTER MUST BE M AND THE OTHER CHARACTERS ARE DIGITS, REPETITIONS ALLOWED?

5. HOW MANY PLATES ARE THERE IF THE CHARACTERS ARE ALL LETTERS AND NONE OF THE CHARACTERS IS M, REPETITIONS ALLOWED?

6. HOW MANY PLATES ARE THERE IF THE CHARACTERS ARE ALL LETTERS, REPETITIONS ARE ALLOWED, AND THE PLATE CONTAINS THE LETTER M?

7. HOW MANY PLATES ARE THERE IF THE CHARACTERS ARE ALL LETTERS, REPETITIONS ARE NOT ALLOWED, AND THE PLATE CONTAINS THE LETTER M?

NON-LICENSE PLATE PROBLEMS:

8. IN A NON LEAP YEAR, HOW MANY WAYS CAN 5 PEOPLE HAVE NO TWO WITH THE SAME BIRTHDAY? (YEAR NOT INCLUDED IN BIRTHDAY)

9. IN A NON LEAP YEAR, HOW MANY WAYS, IN A GROUP OF 5 PEOPLE, CAN AT LEAST 2 PEOPLE HAVE THE SAME BIRTHDAY? (YEAR NOT INCLUDED IN BIRTHDAY)

K. <u>PERMUTATIONS</u>: AN ORDERING OF n OBJECTS IS CALLED A <u>PERMUTATION</u> OF THE OBJECTS.

1. CONSIDER THE SET OF 3 OBJECTS $\{A,B,C\}$. THE PERMUTATIONS ARE ABC, ACB, BAC, BCA, CAB, CBA (NOTE $6=3!$)

2. MULTIPLICATION PRINCIPLE VERIFIES $\exists$ 3! PERMUTATIONS OF 3 OBJECTS

STAGE 1		STAGE 2		STAGE 3	
3	$\cdot$	2	$\cdot$	1	$= 3!$
# OF WAYS TO FILL POSITION 1		# OF WAYS TO FILL POSITION 2		# OF WAYS TO FILL POSITION 3	

3. GENERALIZATION: THERE ARE $n!$ PERMUTATIONS OF n DISTINCT OBJECTS.

4. KEY IDEAS FOR A PERMUTATION: ORDER, NO REPETITIONS

5. <u>r-PERMUTATION</u>: AN ORDERING OF r DISTINCT OBJECTS TAKEN FROM n DISTINCT OBJECTS. (EXAMPLES OF 2-PERMUTATIONS TAKEN FROM $\{A,B,C\}$ ARE: AB, BA, AC, CA, BC, CB)

6. NOTATION: $P(n,r)$ IS READ "THE NUMBER OF PERMUTATIONS OF n THINGS TAKEN r AT A TIME".

   a. THIS IS THE COUNT OF r-PERMUTATIONS.

   b. $P(n,r)$ IS SOMETIMES DENOTED ${}_nP_r$

7. FORMULA DERIVATION FOR $P(n,r)$

   (FIRST FOR $n=7$, $r=3$)

   MULTIPLICATION PRINCIPLE FOR THE NUMBER OF 3-PERMUTATIONS

STAGE 1		STAGE 2		STAGE 3	
7	$\cdot$	6	$\cdot$	5	=
# OF WAYS TO FILL POSITION 1		# OF WAYS TO FILL POSITION 2		# OF WAYS TO FILL POSITION 3	

$$7\cdot6\cdot5 = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1} = \frac{7!}{4!} = \frac{7!}{(7-3)!}$$

$$P(7,3) = \frac{7!}{(7-3)!}$$

IN GENERAL: $P(n,r) = \dfrac{n!}{(n-r)!}$

8. EXAMPLE: HOW MANY SLATES OF PRESIDENT, VP, REPORTER ARE THERE IN A CLUB OF 8 MEMBERS?

ORDER, NO REPETITION

$$P(8,3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1}$$

$$= 8\cdot7\cdot6$$

L. <u>CIRCLE</u> <u>PERMUTATION</u>: AN ORDERING OF OBJECTS ARRANGED IN A CIRCLE WHERE A ROTATION OF AN ORDERING IS NOT CONSIDERED A DIFFERENT CIRCLE PERMUTATION.

CIRCLE PERMUTATIONS OF $\{A, B, C, D, E, F\}$

D
B E
C A
F

SAME PERMUTATION AS →

B
C D
F E
A

DIFFERENT CIRCLE PERMUTATION FROM ABOVE →

B
E A
F C
D

M. COUNTING CIRCLE PERMUTATIONS:

EXAMPLE FOR $\{A, B, C, D, E, F\}$

B — FIX TOP ELEMENT

POSITION 5 __   __ POSITION 1

POSITION 4 __   __ POSITION 2

__ POSITION 3

ST = STAGE   POS = POSITION

ST 1		ST 2		ST 3		ST 4		ST 5	
5	$\cdot$	4	$\cdot$	3	$\cdot$	2	$\cdot$	1	$= 5!$
# WAYS FILL POS 1		# WAYS FILL POS 2		# WAYS FILL POS 3		# WAYS FILL POS 4		# WAYS FILL POS 5	

GENERAL: FOR A COLLECTION OF n DISTINCT OBJECTS THERE ARE $(n-1)!$ CIRCLE PERMUTATIONS.

N. HOMEWORK (OIS)

1. HOW MANY PERMUTATIONS ARE THERE OF THE ELEMENTS OF $\{A, B, C, D, E\}$?
2. HOW MANY 3-PERMUTATIONS ARE THERE OF THE ELEMENTS OF $\{A, B, C, D, E\}$?
3. $P(10,6) =$
4. $P(5,5) =$
5. HOW MANY DIFFERENT LICENSE PLATES ARE THERE WHERE THERE ARE EXACTLY 6 CHARACTERS THAT ARE ALL CAPITAL LETTERS WITH <u>NO</u> REPETITIONS? WITH REPETITIONS?
6. HOW MANY DIFFERENT WAYS CAN A ROW OF 3 CHAIRS BE FILLED BY 3 PEOPLE FROM THE SET OF 5 PEOPLE $\{A, B, C, D, E\}$?
7. HOW MANY DIFFERENT SEATING ARRANGEMENTS ARE THERE FOR THE 5 PEOPLE A, B, C, D, E IN A ROW OF 8 CHAIRS?
8. 7 PEOPLE WANT TO SIT AROUND A CIRCULAR TABLE. HOW MANY DIFFERENT SEATING ARRANGEMENTS ARE THERE? (A DIFFERENT ROTATION IS NOT A DIFFERENT SEATING ARRANGEMENT.)

O. <u>COMBINATIONS</u>: AN r- COMBINATION OF A SET T OF n OBJECTS IS A SUBSET OF T WITH EXACTLY r OBJECTS.

1. CONSIDER $\{A, B, C, D, E, F, G, H\} = T$
   $\{A, D, E\}$, $\{B, F, A\}$, $\{G, H, A\}$ ARE ALL 3- COMBINATIONS OF SET T
2. SINCE $\{A, D, E\} = \{D, A, E\} = \{A, A, D, E\}$, KEY IDEAS FOR A COMBINATION ARE:
   ORDER <u>NOT</u> IMPORTANT
   REPETITIONS <u>NOT</u> ALLOWED

P. NOTATION: $C(n, r)$ IS READ "THE NUMBER OF COMBINATIONS OF n THINGS TAKEN r AT A TIME".

a. THIS IS THE COUNT OF r- COMBINATIONS FROM A SET OF n OBJECTS.
b. $C(n, r)$ IS SOMETIMES DENOTED ${}_nC_r$
   $\binom{n}{r}$ "n choose r"

Q GENERAL FORMULA FOR $C(n,r)$

ILLUSTRATE FOR $C(8,3)$

LET $T = \{A, B, C, D, E, F, G, H\}$

3-PER = 3-PERMUTATIONS

3-COM = 3-COMBINATIONS

3-PER	3-COM
ABC	{A,B,C}
ACB	
BAC	
BCA	
CAB	
CBA	
ABD	{A,B,D}
ADB	
BAD	
BDA	
DAB	
DBA	
ABE	{A,B,E}
...	...

THE COUNT IN THE COLUMN 3-PER IS KNOWN, $P(8,3)$. WE DESIRE THE COUNT IN THE COLUMN 3-COM.

NOTE: {A,B,C} GENERATES 3! PERMUTATIONS IN THE 3-PER COLUMN. SO EACH ENTRY IN COLUMN 3-COM GENERATES 3! IN THE 3-PER COLUMN. SO DIVIDE $P(8,3)$ BY 3! TO GET $C(8,3)$.

$$C(8,3) = \frac{P(8,3)}{3!} = \frac{\frac{8!}{(8-3)!}}{3!} = \frac{8!}{3!\,(8-3)!}$$

GENERAL: $C(n,r) = \dfrac{n!}{r!\,(n-r)!}$

R. HOW MANY COMMITTEES OF 3 CAN BE FORMED FROM A CLUB OF 8 PEOPLE, WHERE COMMITTEE MEMBERS ARE GIVEN NO SPECIAL DESIGNATION?

NO ORDER, NO REPETITIONS, SO USE COMBINATIONS

$$C(8,3) = \frac{8!}{3!(8-3)!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= 8\cdot 7 = 56$$

S. INCORPORATING PREVIOUS COUNTING TECHNIQUES WITH COMBINATIONS.

A BASKETBALL SQUAD HAS 7 BIG MEN AND 8 SMALL MEN. A LINEUP CONSISTS OF 2 BIG MEN AND 3 SMALL MEN. HOW MANY LINEUPS ARE THERE?

$$\underset{\text{STAGE 1}}{\frac{C(7,2)}{\begin{array}{c}\#\text{ WAYS}\\ \text{2 BIG}\end{array}}} \cdot \underset{\text{STAGE 2}}{\frac{C(8,3)}{\begin{array}{c}\#\text{ WAYS}\\ \text{3 SMALL}\end{array}}} = \frac{7!}{2!(7-2)!}\cdot\frac{8!}{3!(8-3)!}$$

$$= \frac{7!}{2!\,5!}\cdot\frac{8!}{3!\,5!} = \frac{7\cdot 6}{2\cdot 1}\cdot\frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = 7\cdot 3\cdot 8\cdot 7 = 1176$$

T. $\exists$ 4 TEAMS A, B, C, D. EACH TEAM HAS PLAYERS NUMBERED 1, 2, 3, 4, 5, 6, 7, 8, 9. HOW MANY GROUPS OF 5 CAN BE FORMED CONTAINING 4 OF ONE TEAM AND ONE OF ANOTHER TEAM?

MODEL: BUILD ORDERED TRIPLES WHERE EACH ORDERED TRIPLE REPRESENTS A GROUP OF 5. THE TRIPLE =
(SUBSET OF 4 DIGITS, TEAM, LEFTOVER)

$(\{2,3,6,9\}, A, C3) \longleftrightarrow \{A2, A3, A6, A9, C3\}$

$(\{1,5,7,8\}, B, D9) \longleftrightarrow \{B1, B5, B7, B8, D9\}$

COUNTING

$$\underbrace{C(9,4)}_{\text{\# OF SUBSETS OF 4}} \cdot \underbrace{4}_{\text{\# OF TEAMS}} \cdot \underbrace{(36-9)}_{\text{\# OF LEFTOVERS}} =$$

STAGE 1 — STAGE 2 — STAGE 3

$$\frac{9!}{4!(9-4)!} \cdot 4 \cdot 27 = \frac{9! \cdot 4 \cdot 27}{4!\,5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 4 \cdot 27}{4 \cdot 3 \cdot 2 \cdot 1} = 13608$$

NOTE: A MODEL WAS MADE. WE COULD COUNT THE MODEL ITEMS. THERE WAS A 1-1 MATCH WITH DESIRED ITEMS.

U. HOW MANY DIFFERENT SEATING ARRANGEMENTS ARE THERE FOR THE 5 PEOPLE A, B, C, D, E IN A ROW OF 8 CHAIRS (PROBLEM N,7 REVISITED)?

METHOD 1, USING COMBINATIONS:

MODEL: ORDERED PAIR OF (SEAT # PEOPLE SIT IN, THE SEATING PERMUTATION)

$(\{2,4,5,7,8\}, BCADE)$ MATCHES TO

	B		C	A		D	E
1	2	3	4	5	6	7	8

STAGE 1 · STAGE 2 = OUTCOMES

$$\underbrace{C(8,5)}_{\text{\# OF SUBSETS OF 5}} \cdot \underbrace{5!}_{\text{\# OF PERMUTATIONS OF 5}} = \frac{8!}{5!(8-5)!} \cdot 5! = \frac{8!}{3!}$$

METHOD 2 USING PERMUTATIONS ONLY

MODEL: ORDERED 5-TUPLE OF

(A SEAT #, B SEAT #, C SEAT #, D SEAT #, E SEAT #)

(5, 2, 4, 7, 8) MATCHES TO

	B		C	A		D	E
1	2	3	4	5	6	7	8

ORDER, NO REPETITIONS, SO PERMUTATIONS

$$P(8,5) = \frac{8!}{(8-5)!} = \frac{8!}{3!}$$

V. HOW MANY WAYS CAN 5 MEDIATORS AND 3 FIGHTERS LINE UP SO THAT NO TWO FIGHTERS ARE SIDE BY SIDE?

<u>PROBLEM REPHRASED</u>: HOW MANY WAYS CAN THE CHARACTERS a, b, c, J, K, L, M, N BE LINED UP SO NO TWO LOWER CASE LETTERS ARE TOGETHER.

<u>MODEL</u>: NUMBER THE SLOTS IN FRONT OF, BETWEEN, AND AFTER THE CAPITAL LETTERS 1, 2, 3, 4, 5, 6. A 3-PERMUTATION OF 1, 2, 3, 4, 5, 6 TELLS THE SLOT NUMBERS FOR a, b, c RESPECTIVELY. AN ORDERED PAIR (PERMUTATION OF CAPITAL LETTERS, SLOT NUMBER PERMUTATION FOR a, b, c) WILL CORRESPOND TO A LINEUP.

(MJLKN, 624) MATCHES TO

$\underset{1}{\underline{\quad}}\ M\ \underset{2}{\underline{\ b\ }}\ J\ \underset{3}{\underline{\quad}}\ L\ \underset{4}{\underline{\ c\ }}\ K\ \underset{5}{\underline{\quad}}\ N\ \underset{6}{\underline{\ a\ }}$

624 IS CODE FOR: a GOES IN SLOT 6, b GOES IN SLOT 2, c GOES IN SLOT 4

STAGE 1		STAGE 2	OUTCOMES
$\underbrace{5!}$ # OF PERMUTATIONS OF J, K, L, M, N	$\cdot$	$\underbrace{P(6,3)}$ # OF 3 PERMUTATIONS	$= \dfrac{5!\cdot 6!}{3!} = 14400$

THE ORDERED PAIRS MATCH 1-1 WITH THE REPHRASED PROBLEM WHICH MATCH 1-1 WITH THE ORIGINAL PROBLEM.

NOTE: WE SAW 2 PROBLEM SOLVING METHODS: 1) REPHRASE TO AN EQUIVALENT PROBLEM 2) MAKE A MATH MODEL

W. HOMEWORK (OIS)

1. TEAM A CONSISTS OF 8 PEOPLE AND TEAM B CONSISTS OF 6 PEOPLE. THE CLUB CONSISTS OF EXACTLY THE 14 PEOPLE IN TEAM A AND TEAM B.

   a. A COMMITTEE IS TO BE MADE UP OF 2 PEOPLE FROM TEAM A AND TWO PEOPLE FROM TEAM B. HOW MANY WAYS CAN THIS HAPPEN?

   b. THE CLUB WANTS A COMMITTEE OF 4 PEOPLE WHERE THERE IS AT LEAST ONE PERSON FROM TEAM B ON THE COMMITTEE. HOW MANY WAYS CAN THIS HAPPEN?

   c. HOW MANY DIFFERENT COMMITTEES OF 4 CAN BE FORMED?

   d. A COMMITTEE OF 4 IS TO HAVE AT MOST ONE MEMBER OF TEAM A. HOW MANY WAYS CAN THIS HAPPEN?

2. THERE ARE 3 TEAMS A, B, AND C. EACH PLAYER ON EACH TEAM HAS ONE OF THE NUMBERS 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. THERE ARE 10 PLAYERS ON EACH TEAM. THE CLUB HAS EXACTLY 30 MEMBERS, MADE UP OF

(CONTINUED)

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THE MEMBERS OF TEAMS A, B, AND C. COMMITTEES ARE TO CONSIST OF 4 PEOPLE EACH.

a. HOW MANY COMMITTEES CONTAIN 3 PEOPLE WITH THE NUMBER 9?

b. HOW MANY COMMITTEES CONTAIN 3 PEOPLE WITH THE SAME NUMBER?

c. HOW MANY COMMITTEES CONTAIN ONLY MEMBERS OF TEAM A?

d. HOW MANY COMMITTEES CONTAIN MEMBERS FROM ONLY EXACTLY TWO TEAMS?

e. HOW MANY COMMITTEES CONTAIN MEMBERS FROM ALL TEAMS?

3. EIGHT COINS ARE TOSSED

a. HOW MANY TIMES WILL THERE BE EXACTLY 5 TAILS?

b. HOW MANY TIMES WILL THERE BE AT MOST 2 TAILS?

REFERENCE: <u>DISCRETE MATHEMATICS</u> by Richard Johnsonbaugh, Third Edition, Macmillan, 1993

[CHAPTER 36] 36-370

BOOLEAN ALGEBRA

A. LET U BE A NONEMPTY SET. LET $S = P(U)$, THE POWER SET OF U, THE SET OF ALL SUBSETS OF U.
$\forall H \subseteq U$, DEFINE $H' = U - H$. H' IS CALLED THE COMPLEMENT OF H.

NOTICE: S IS A SET SUCH THAT THERE ARE OPERATORS $'$, $\cup$, AND $\cap$ SUCH THAT

1. $\forall H, K \in S$, H', $H \cup K$ AND $H \cap K$ ARE IN S.
2. $\forall H, K \in S$, $H \cup K = K \cup H$ AND $H \cap K = K \cap H$
COMMUTATIVITY
3. $\forall H, K, L \in S$, $H \cup (K \cup L) = (H \cup K) \cup L$ AND $(H \cap K) \cap L = H \cap (K \cap L)$. ASSOCIATIVITY
4. $\forall H, K, L \in S$, $H \cap (K \cup L) = (H \cap K) \cup (H \cap L)$ AND $H \cup (K \cap L) = (H \cup K) \cap (H \cup L)$. DISTRIBUTIVITY OF $\cap$ OVER $\cup$ AND VICE VERSA.
5. $\exists \phi \in S \ni \forall H \in S$, $H \cup \phi = H$
ϕ IS THE IDENTITY FOR $\cup$
6. $\exists U \in S \ni \forall H \in S$, $H \cap U = H$
U IS THE IDENTITY FOR $\cap$

7. $\forall H \in S$, $H \cup H' = U$ AND $H \cap H' = \phi$

THE PRECEDING WAS TO SHOW A BOOLEAN ALGEBRA EXISTS.

B. DEFINITION OF A <u>BOOLEAN</u> <u>ALGEBRA</u>:

B IS A BOOLEAN ALGEBRA IFF B IS A SET WITH OPERATORS $'$, $+$, AND $\cdot$ SUCH THAT

1. $\forall x, y \in B$, x', $x+y$, AND $x \cdot y$ ARE IN B
2. $\forall x, y \in B$, $x+y = y+x$ AND $x \cdot y = y \cdot x$
   COMMUTATIVITY
3. $\forall x, y, z \in B$, $x+(y+z) = (x+y)+z$ AND
   $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ ASSOCIATIVITY
4. $\forall x, y, z \in B$, $x \cdot (y+z) = x \cdot y + x \cdot z$ AND
   $x+(y \cdot z) = (x+y) \cdot (x+z)$ DISTRIBUTIVITY OF $\cdot$ OVER $+$ AND VICE VERSA.
5. $\exists\, 0 \in B \ni \forall x \in B$, $x+0 = x$
   0 IS THE IDENTITY FOR $+$
6. $\exists\, 1 \in B \ni \forall x \in B$, $x \cdot 1 = x$
   1 IS THE IDENTITY FOR $\cdot$
7. $\forall x \in B$, $x + x' = 1$ AND $x \cdot x' = 0$

(IN BOOLEAN ALGEBRA, x' IS CALLED THE COMPLEMENT OF x.)

C. FOR SOMETHING TO BE A BOOLEAN ALGEBRA, IT MUST HAVE AN ELEMENT THAT WE ARE DENOTING BY THE SYMBOL 0. FOR EXAMPLE, THE BOOLEAN ALGEBRA IN PART A, 0 IS THE SYMBOL FOR THE EMPTY SET, ϕ. ALL THE CORRESPONDENCES NOW FOLLOW: $0 = \phi$, $1 = U$, $+ = \cup$, $\cdot = \cap$, $' = '$, $B = S$.

D. <u>THE 2-VALUED BOOLEAN ALGEBRA</u>: THE BOOLEAN ALGEBRA IN PART A COULD HAVE INFINITELY MANY ELEMENTS. WE NOW SHOW YOU THE BOOLEAN ALGEBRA WITH ONLY TWO VALUES

LET $B = \{0, 1\}$. $0' = 1$. $1' = 0$.

+ AND $\cdot$ ARE DEFINED AS FOLLOWS:

+	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

E. HOMEWORK (OIS)

1. FOR 2-VALUED BOOLEAN ALGEBRA PROVE $\forall x, y, z \in B$, $x + (y \cdot z) = (x + y) \cdot (x + z)$

F. THERE IS A STATEMENT ALGEBRA COUNTERPART TO 2 VALUED BOOLEAN ALGEBRA WITH THE CORRESPONDENCES "IS EQUAL TO" CORRESPONDS TO "IS EQUIVALENT TO, $\equiv$", $0 = F$, $1 = T$, $+ = \vee$, $\cdot = \wedge$, AND $x' = \sim x$

$B = \{T, F\}$ IS THE SET WITH OPERATORS $\sim$, $\vee$, AND $\wedge$ SUCH THAT

1. $\forall p, q \in B$, $\sim p$, $p \vee q$, AND $p \wedge q$ ARE IN B.
2. $\forall p, q \in B$, $p \vee q \equiv q \vee p$ AND $p \wedge q \equiv q \wedge p$
   COMMUTATIVITY
3. $\forall p, q, r \in B$ $p \vee (q \vee r) \equiv (p \vee q) \vee r$ AND $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ ASSOCIATIVITY
4. $\forall p, q, r \in B$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ AND $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ DISTRIBUTIVITY OF $\wedge$ OVER $\vee$ AND VICE VERSA
5. $\exists F \in B \ni \forall p \in B$, $p \vee F \equiv p$
   F IS THE IDENTITY FOR $\vee$
6. $\exists T \in B \ni \forall p \in B$, $p \wedge T \equiv p$
   T IS THE IDENTITY FOR $\wedge$
7. $\forall p \in B$, $p \vee \sim p \equiv T$ AND $p \wedge \sim p \equiv F$

G. BOOLEAN ALGEBRA THEOREMS

1. THEOREM: IF B IS A BOOLEAN ALGEBRA, THEN $\forall x \in B$, $x+1=1$.

   a. ASSUME B IS A BOOLEAN ALGEBRA AND $x \in B$ (SHOW $x+1=1$)

   b. $x+1=(x+1)\cdot 1$ — 6 BOOL. ALG. DEF.

   c. $=(x+1)\cdot(x+x')$ — b, 7 BOOL. ALG. DEF. $x+x'=1$

   d. $=x+(1\cdot x')$ — c, 4 BOOL. ALG. DEF. DISTRIBUTIVITY OF + OVER •

   e. $=x+(x'\cdot 1)$ — d, 2 BOOL. ALG. DEF. COMMUTATIVITY

   f. $=x+x'$ — e, 6 BOOL. ALG. DEF. IDENTITY

   g. $=1$ — f, 7 BOOL. ALG. DEF.

2. THEOREM: IF B IS A BOOLEAN ALGEBRA, THEN $\forall x \in B$, $x\cdot 0=0$.

   a. ASSUME B IS A BOOLEAN ALGEBRA AND $x \in B$. (SHOW $x\cdot 0=0$)

   b. $x\cdot 0=(x\cdot 0)+0$ — 5 BOOL AL. DEF. IDENTITY

   c. $=(x\cdot 0)+(x\cdot x')$ — 7 BOOL. AL. DEF., b

   d. $=x\cdot(0+x')$ — DIST., 4 BOOL. AL. DEF., c

   e. $=x\cdot(x'+0)$ — COMM., 2 BOOL. AL. DEF., d

   f. $=x\cdot x'$ — IDENTITY, 5 BOOL. AL. DEF., e

   g. $=0$ — 7 BOOL. ALG. DEF., f

H <u>DUAL</u> OF A BOOLEAN ALGEBRA STATEMENT: REPLACE 0 BY 1, 1 BY 0, + BY $\cdot$, $\cdot$ BY +

A. THE DUAL OF "$\forall x \in B$, $x+1=1$" IS "$\forall x \in B$ $x \cdot 0 = 0$". (NOTICE THESE WERE THE LAST TWO THEOREMS WE PROVED.)

B. THEOREM: THE DUAL OF A BOOLEAN ALGEBRA THEOREM IS ALSO A THEOREM!

1. TO GET A PROOF OF THE DUAL, TAKE THE DUAL OF ALL THE STATEMENTS IN THE PROOF OF THE ORIGINAL THEOREM.
2. NOTICE, THIS IS WHAT WAS DONE IN THE PRECEDING TWO THEOREMS: THE STATEMENTS IN THE SECOND THEOREM ARE THE DUALS OF THE STATEMENTS IN THE FIRST THEOREM.

I. HOMEWORK (OIS)

1. FIND THE DUAL OF

   a. $\forall x \in B,\ x + x = x$

   b. $\forall x, y \in B,\ x + (x \cdot y) = x$

   c. $\forall x \in B,\ x + x' = 1$

2. ASSUME B IS A BOOLEAN ALGEBRA. PROVE:

   a. $\forall x \in B,\ x + x = x$

   b. $\forall x \in B,\ x \cdot x = x$

   c. $\forall x, y \in B,\ x + (x \cdot y) = x$

   d. $\forall x, y \in B,\ x \cdot (x + y) = x$

J. DISJUNCTIVE NORMAL FORM

1. THIS WILL BE DEFINED FOR 3 VARIABLES. (THE DEFINITION FOR 2 OR MORE SHOULD BE CLEAR FROM THIS DEFINITION.)

2. A MINTERM IN p, q, r IS AN EXPRESSION OF THE FORM

$$x_1 \wedge x_2 \wedge x_3$$

WHERE x_1 IS EITHER p OR $\sim p$, x_2 IS EITHER q OR $\sim q$, AND x_3 IS EITHER r OR $\sim r$.

3. EXAMPLES OF MINTERMS IN p, q, r

   a. $p \wedge \sim q \wedge r$

   b $\sim p \wedge q \wedge \sim r$

4. A REASON FOR THE NAME MINTERM: CONSIDER $0 = F$ $1 = T$. THE VALUE OF $X_1 \wedge X_2 \wedge X_3$ IS THE MINIMUM OF EACH OF THE VALUES X_1, X_2, X_3.

5. AN EXPRESSION D INVOLVING $p, q,$ AND r IS IN DISJUCTIVE NORMAL FORM IFF $\exists$ A POSITIVE INTEGER K, $\exists$ MINTERMS $M_1, M_2, \ldots, M_K$ IN $p, q,$ AND r SUCH THAT

$$D = M_1 \vee M_2 \vee M_3 \vee \cdots \vee M_K$$

6. AN EXAMPLE OF DISJUNCTIVE NORMAL FORM IN $p, q,$ AND r.

$$(\sim p \wedge q \wedge \sim r) \vee (p \wedge q \wedge \sim r)$$

7. TIP FOR GETTING AN EXPRESSION INTO DISJUNCTIVE NORMAL FORM. ($\equiv$ MEANS "IS EQUIVALENT TO")

NOTE: $B \equiv B \wedge (q \vee \sim q)$

$\equiv (B \wedge q) \vee (B \wedge \sim q)$ DIST.

(SUPPOSE $B = p \wedge r$)

$\equiv (p \wedge r \wedge q) \vee (p \wedge r \wedge \sim q)$

$\equiv (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r)$ COMM.

THUS IF AN EXPRESSION IS A CONJUCTION INVOLVING 2 OF THE VARIABLES $p, q, r,$ CONJUNCT IT WITH

THIRD VARIABLE $\vee \sim$ THIRD VARIABLE

THEN

DISTRIBUTE AND COMMUTE

8. PUT $(p \vee q) \wedge r$ IN DISJUNCTIVE NORMAL FORM

a. $(p \vee q) \wedge r \equiv r \wedge (p \vee q)$ COMMUTATIVE

b. $\equiv (r \wedge p) \vee (r \wedge q)$ DISTRIBUTIVE

c. $\equiv ((r \wedge p) \wedge (q \vee \sim q)) \vee ((r \wedge q) \wedge (p \vee \sim p))$ TIP

d. $\equiv (r \wedge p \wedge q) \vee (r \wedge p \wedge \sim q) \vee (r \wedge q \wedge p) \vee (r \wedge q \wedge \sim p)$ DISTRIBUTIVE

e. $\equiv (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$ COMMUTATIVE

f. $\equiv (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r)$

DROP DUPLICATE $p \wedge q \wedge r$

9. HOMEWORK (OIS)

a. GIVE THE DEFINITION OF AN EXPRESSION INVOLVING p AND q BEING IN DISJUNCTIVE NORMAL FORM.

b. GIVE THE DEFINITION OF AN EXPRESSION INVOLVING $P_1, P_2, P_3, \cdots, P_n$ BEING IN DISJUNCTIVE NORMAL FORM.

c. PUT $(p \vee q) \wedge (q \vee r)$ IN DISJUNCTIVE NORMAL FORM. (3 VARIABLE CASE)

d. PUT $p \vee \sim q$ IN DISJUNCTIVE NORMAL FORM. (2 VARIABLE CASE)

e. PUT $p \vee \sim q$ IN DISJUNCTIVE NORMAL FORM. (3 VARIABLE CASE)

K. CONJUNCTIVE NORMAL FORM

1. A MAXTERM IN p, q, r IS AN EXPRESSION OF THE FORM

$$x_1 \vee x_2 \vee x_3$$

WHERE x_1 IS EITHER p OR $\sim p$, x_2 IS EITHER q OR $\sim q$, AND x_3 IS EITHER r V $\sim r$

2. EXAMPLES OF MAXTERMS IN p, q, r
   a. $p \vee \sim q \vee r$
   b. $\sim p \vee q \vee \sim r$

3. A REASON FOR THE NAME MAXTERM: CONSIDER $0 = F$ AND $1 = T$. THE VALUE OF $x_1 \vee x_2 \vee x_3$ IS THE MAXIMUM OF EACH OF THE VALUES x_1, x_2, x_3

4. AN EXPRESSION D INVOLVING p, q, r IS IN CONJUNCTIVE NORMAL FORM IFF $\exists$ A POSITIVE INTEGER k, $\exists$ MAXTERMS $M_1, M_2, \ldots, M_k$ IN p, q, AND r SUCH THAT

$$D = M_1 \wedge M_2 \wedge M_3 \wedge \cdots \wedge M_k$$

5. THE DUAL OF AN EXPRESSION IN DISJUNCTIVE NORMAL FORM IS IN CONJUNCTIVE NORMAL FORM.

6. DUAL OF PREVIOUS TIP: A WAY TO GET INTO CONJUNCTIVE NORMAL FORM:

NOTE: $B \equiv B \vee \text{FALSE}$

$B \equiv B \vee (r \wedge \sim r)$

$\equiv (B \vee r) \wedge (B \vee \sim r)$ DISTRIBUTIVE

(SUPPOSE $B = p \vee q$)

$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r)$

TIP: DISJUNCT B WITH $r \wedge \sim r$

7. PUT $(p \vee q) \wedge (\sim p \vee r)$ IN CONJUNCTIVE NORMAL FORM.

$(p \vee q) \wedge (\sim p \vee r) \equiv$

$((p \vee q) \vee (r \wedge \sim r)) \wedge ((\sim p \vee r) \vee (q \wedge \sim q)) \equiv$ TIP

$(p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee r \vee q) \wedge (\sim p \vee r \vee \sim q) \equiv$

DISTRIBUTIVITY

$(p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee r)$

COMMUTATIVITY

8. HOMEWORK (OIS)

a. GIVE THE DEFINITION OF AN EXPRESSION INVOLVING $P_1, P_2, P_3, \ldots, P_n$ BEING IN CONJUNCTIVE NORMAL FORM.

b. PUT $(\sim p \vee q) \wedge (p \vee \sim r)$ IN CONJUNCTIVE NORMAL FORM. (3 VARIABLE CASE)

c. PUT $\sim p \wedge q$ IN CONJUNCTIVE NORMAL FORM. (2 VARIABLE CASE)

d. PUT $\sim p \wedge q$ IN CONJUNCTIVE NORMAL FORM. (3 VARIABLE CASE).

L. APPLICATION OF BOOLEAN ALGEBRA TO ELECTRICAL ENGINEERING:

CIRCUITS CAN BE REPRESENTED BY BOOLEAN EXPRESSIONS

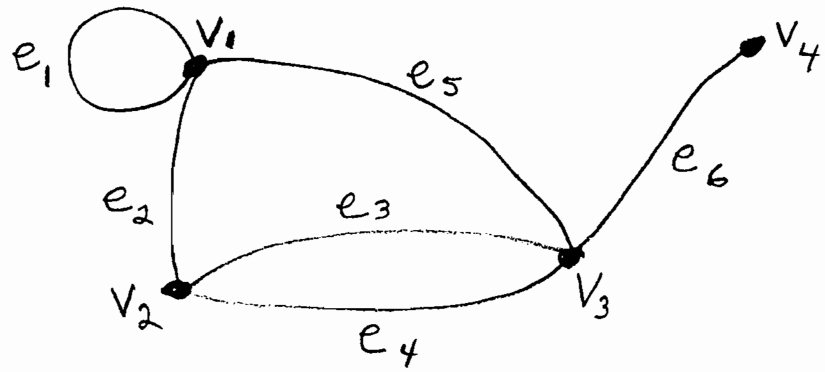
M. REFERENCES

1. DISCRETE MATHEMATICS by DOSSEY,..; HARPER COLLINS, ISBN 0-673-46287-0
2. LOGIC AND DISCRETE MATHEMATICS by GRASSMANN,..; PRENTICE HALL, ISBN 0-13-501206-6
3. DISCRETE MATHEMATICS by Johnsonbaugh; Macmillan, ISBN 0-02-360721-1

[CHAPTER 37] 37-383

GRAPHS AND TREES

A. A PICTURE OF A GRAPH:



G_A

B. DEFINITION: A GRAPH G CONSISTS OF A SET V OF VERTICES AND A SET E OF EDGES ∋ EACH EDGE E HAS EXACTLY 2 END POSITIONS ∋ ASSOCIATED WITH EACH END POSITION IS A VERTEX, SAID TO BE INCIDENT TO THE EDGE AND VICE VERSA.

C. IN THE EXAMPLE ABOVE: GRAPH G_A

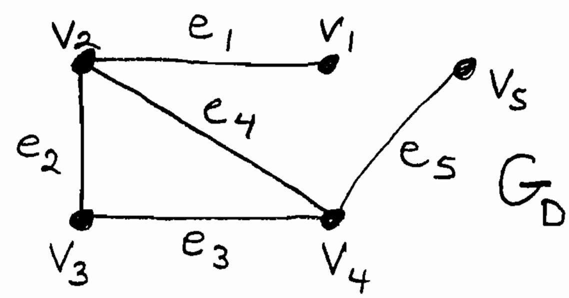
$V = \{V_1, V_2, V_3, V_4\}$ $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

V_3 AND V_4 ARE INCIDENT TO e_6

e_6 IS INCIDENT TO V_3 AND V_4

V_1 IS INCIDENT TO e_1

D. PATH: FOR GRAPH G_D, $V_1, e_1, V_2, e_4, V_4, e_3, V_3$ IS A PATH.



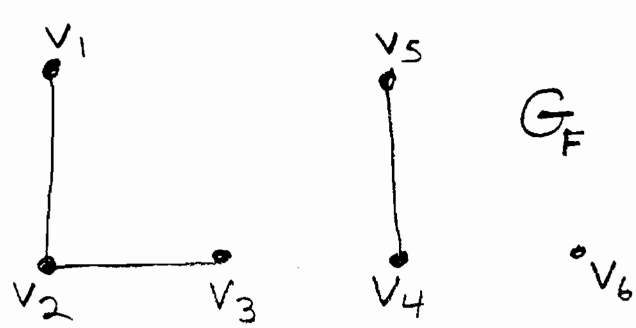
IN GENERAL, A PATH IS A SEQUENCE $w_0, f_1, w_1, f_2, w_2, f_3, w_3, \ldots, f_n, w_n$ WHERE FOR EACH $i \in \{1, 2, \ldots, n\}$ w_{i-1} AND w_i ARE VERTICES INCIDENT TO EDGE f_i

E. A CYCLE IS A PATH WITH AT LEAST 2 EDGES THAT STARTS AND STOPS AT THE SAME VERTEX AND NO EDGE IS REPEATED. ($V_2, e_4, V_4, e_3, V_3, e_2, V_2$ IS A CYCLE IN GRAPH G_D ABOVE.)

F. G IS A CONNECTED GRAPH IFF G IS A GRAPH SUCH THAT FOR ALL VERTICES v, w OF G, THERE IS A PATH FROM v TO w.

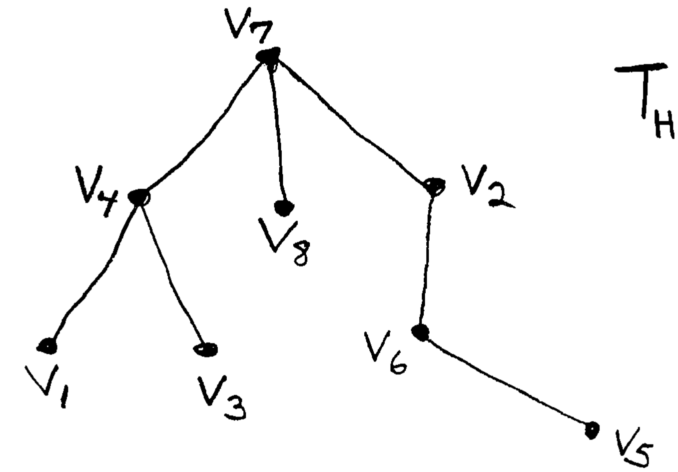
GRAPH G_F IS NOT CONNECTED.

GRAPH G_D IS CONNECTED.



G. AN <u>ACYCLIC</u> GRAPH IS A GRAPH WITH NO CYCLES.

H. A <u>TREE</u> IS A CONNECTED ACYCLIC GRAPH.



T_H IS A TREE. V_7 IS DESIGNATED THE <u>ROOT</u>

V_7 IS THE <u>PARENT</u> OF V_4 AND V_2 AND V_8

V_4 IS THE PARENT OF V_1 AND V_3

V_1 AND V_3 ARE <u>SIBLINGS</u>: CHILDREN OF THE SAME PARENT

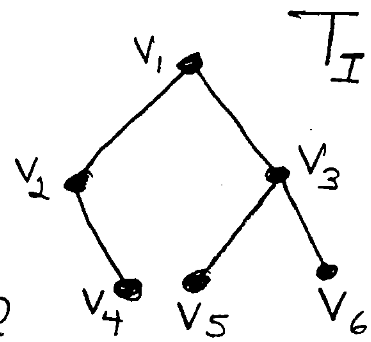
V_1 AND V_3 ARE <u>CHILDREN</u> OF V_4.

V_7 IS AN <u>ANCESTOR</u> OF V_4, V_2, V_1, V_3, V_8 V_6 AND V_5. V_2 IS AN ANCESTOR OF V_6 AND V_5. V_5 AND V_6 ARE <u>DESCENDANTS</u> OF V_2

A <u>ROOTED TREE</u> IS A TREE IN WHICH ONE VERTEX IS DESIGNATED THE <u>ROOT</u>

I. A BINARY TREE IS A ROOTED TREE WITH EACH VERTEX HAVING AT MOST 2 CHILDREN (CALLED THE LEFT CHILD AND THE RIGHT CHILD).

T_I IS A BINARY TREE. V_2 IS THE LEFT CHILD OF V_1. V_3 IS THE RIGHT CHILD OF V_1. V_2 HAS NO LEFT CHILD. V_4 IS THE RIGHT CHILD OF V_2.



J. TRAVERSING A TREE: VISITING (I.E. PERFORMING A TASK*) AT EACH VERTEX OF A TREE ONCE AND ONLY ONCE.

1. WE WILL GIVE 3 TRAVERSALS OF A BINARY TREE: PREORDER, INORDER, POSTORDER. THEY WILL BE RECURSIVELY DEFINED.
2. A SIMPLER PICTURE OF TREES WILL FOLLOW.

* DISCRETE MATHEMATICAL STRUCTURES BY KOLMAN
ISBN 0-13-320912-1 PRENTICE HALL

K. <u>TASK</u> PRE (V): (PREORDER TRAVERSAL)

```
IF V IS NOT THE EMPTY TREE THEN
  VISIT(V)
  PRE (LEFT CHILD OF V)
  PRE (RIGHT CHILD OF V)
```

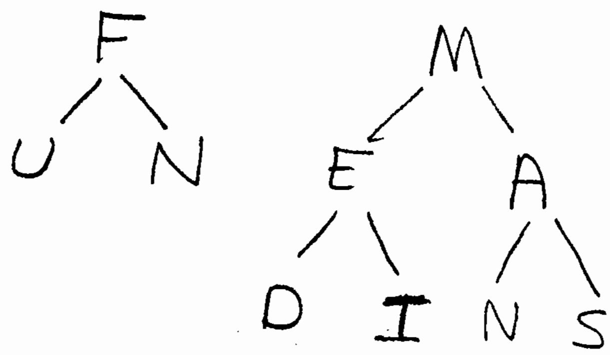
L. <u>TASK</u> IN(V): (INORDER TRAVERSAL)

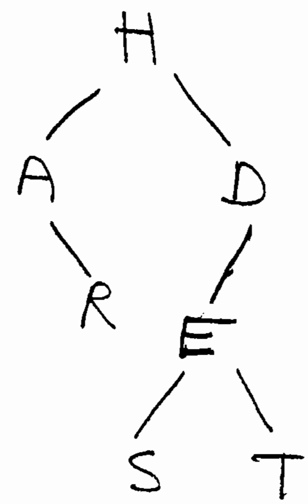
```
IF V IS NOT THE EMPTY TREE THEN
  IN(LEFT CHILD OF V)
  VISIT (V)
  IN(RIGHT CHILD OF V)
```

M. <u>TASK</u> POST(V): (POSTORDER TRAVERSAL)

```
IF V IS NOT THE EMPTY TREE THEN
  POST(LEFT CHILD OF V)
  POST(RIGHT CHILD OF V)
  VISIT(V)
```

N. NOTE: A TREE IS NAMED BY THE VERTEX THAT IS ITS ROOT. A SUBTREE CONSISTS OF A VERTEX AND ALL OF ITS DESCENDANTS θ. 3 TREES TO TRAVERSE (VISIT MEANS WRITE VERTEX NAME)





1. PREORDER TRAVERSALS

PRE(F) = FUN

PRE(M) = MEDIANS

PRE(H) = HARDEST

2. INORDER TRAVERSALS

IN(F) = UFN

IN(M) = DEIMNAS

IN(H) = ARHSETD

3. POSTORDER TRAVERSALS

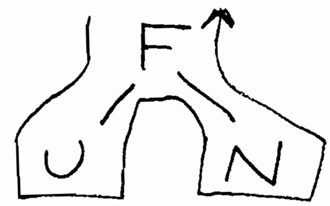
POST(F) = UNF

POST(M) = DIENSAM

POST(H) = RASTEDH

P. BINARY TREE TRAVERSALS: ALTERNATE WAY

GO AROUND THE TREE STARTING AT THE TOP OF THE ROOT



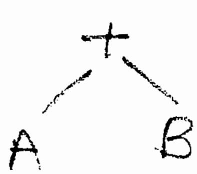
1. PREORDER TRAVERSAL: FOR EVERY DOWN STROKE ↓ BESIDE A VERTEX, CONSIDER THAT A VISIT. FUN

2. INORDER TRAVERSAL: FOR EVERY HORIZONTAL STROKE ⟶ UNDER A VERTEX, CONSIDER THAT A VISIT. UFN

3. POSTORDER TRAVERSAL: FOR EVERY UP STROKE ↑ BESIDE A VERTEX, CONSIDER THAT A VISIT: UNF

Q. EXPRESSION TREES: ARITHMETIC EXPRESSIONS CAN BE WRITTEN IN THE FORM OF A TREE.

1. A+B



PUT THE OPERATOR AT THE TOP AND THE OPERANDS AS ITS CHILDREN.

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2. (A+B)*E − D/C

POSTORDER TRAVERSAL: AB+E*DC/−

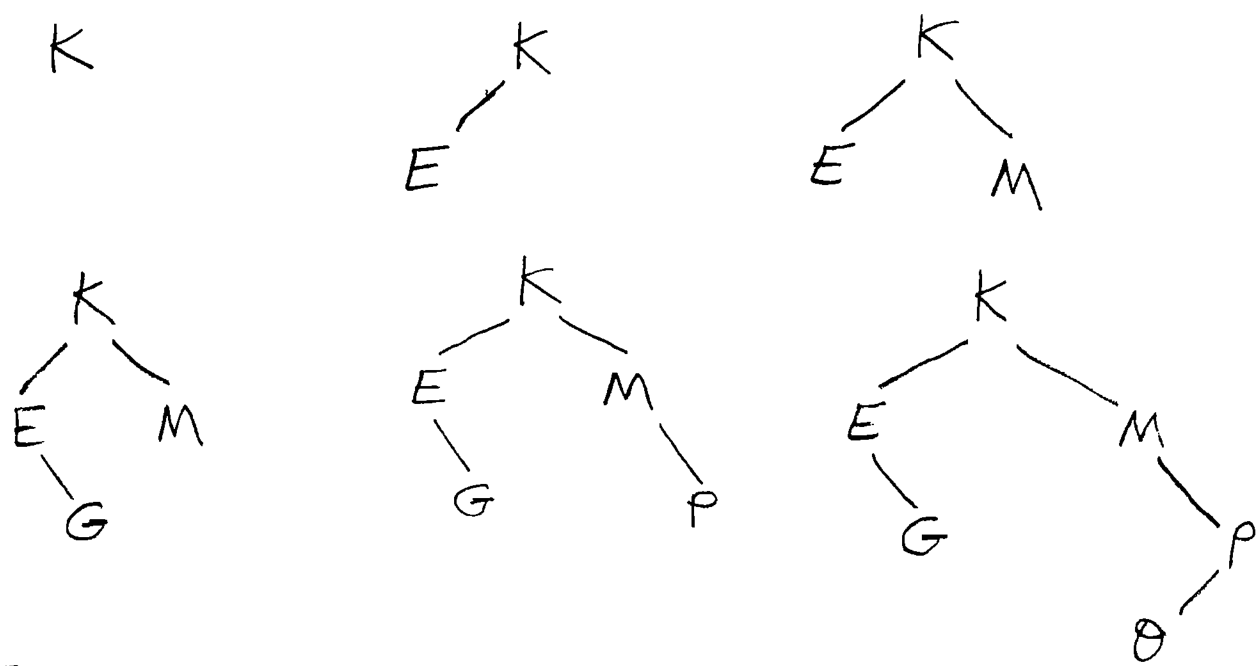
A POST ORDER TRAVERSAL OF AN EXPRESSION TREE IS ALSO CALLED <u>REVERSE POLISH NOTATION</u>.

a. SOME CALCULATORS REQUIRE REVERSE POLISH NOTATION

b. ADVANTAGE OF REVERSE POLISH NOTATION: NO PARENTHESES NEEDED.

R.. A TREE STRUCTURE CAN BE A HELPFUL DATA STRUCTURE IN COMPUTER SCIENCE TO STORE DATA. SEARCHING FOR DATA CAN BE VERY QUICK USING THE TREE DATA STRUCTURE

S. BINARY SORT: AS A LIST OF DATA COMES IN, A BINARY TREE CAN BE "GROWN". AN ALPHABETICAL PREDECESSOR GROWS A LEFT CHILD, A SUCCESSOR A RIGHT CHILD: AN INORDER TRAVERSAL WILL PRODUCE A SORT ON THE DATA. SUPPOSE THE DATA CAME IN THIS SEQUENCE K, E, M, G, P, O. THE STEP BY STEP TREE GROWTH FOLLOWS:



INORDER TRAVERSAL: E, G, K, M, O, P

NOTE: IT IS SORTED.

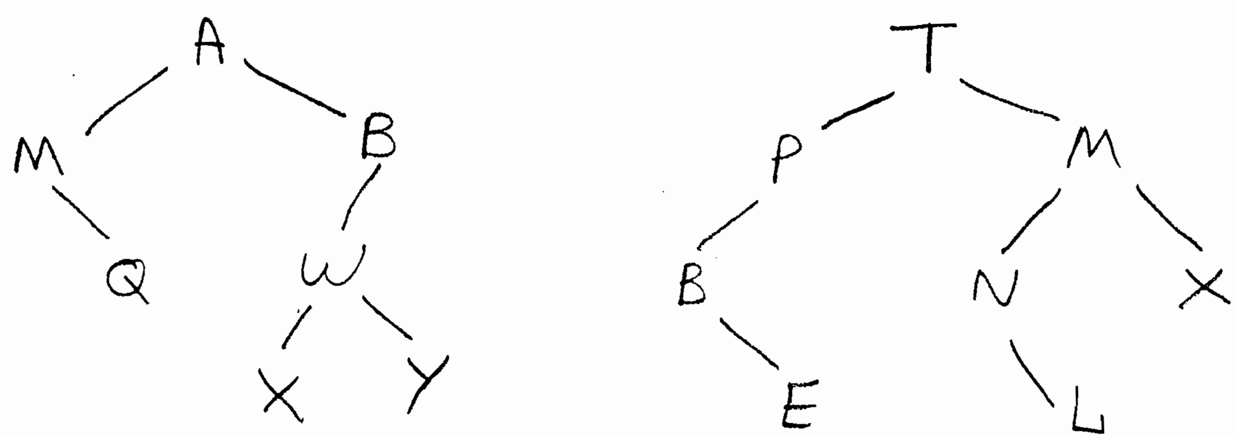
T. HOMEWORK (OIS)

1. DRAW AN ACYCLIC GRAPH THAT IS NOT CONNECTED.

2. DRAW GRAPH WITH A CYCLE SUCH THAT A VERTEX BESIDES THE START/STOP VERTEX IS REPEATED; NAME THE CYCLE.

3. DRAW A GRAPH WITH A PATH THAT STARTS AND STOPS AT THE SAME VERTEX, BUT IS NOT A CYCLE.

4. GIVE PREORDER, INORDER, AND POST-ORDER TRAVERSALS OF THE TREES BELOW:



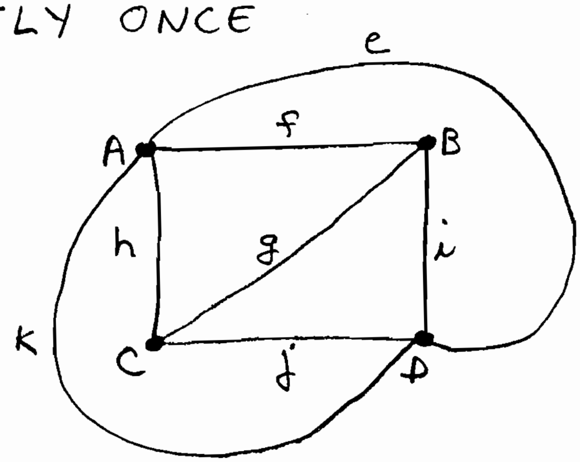
5. FIRST BUILD EXPRESSION TREES, THEN GIVE REVERSE POLISH NOTATION FOR
   a. $(4+(A-B))*(F-G)$
   b. $(3+(5-B))*(T-E)$

6. GROW A BINARY TREE FOR THE SEQUENCE T, W, E, B, Z, X, A, F, H. INORDER TRAVERSE.

[CHAPTER 38]

MORE ABOUT GRAPHS

A. AN EULER (PRONOUNCED OIL-ER) PATH FOR A GRAPH G IS A PATH THAT INCLUDES EVERY EDGE OF THE GRAPH EXACTLY ONCE



G_A

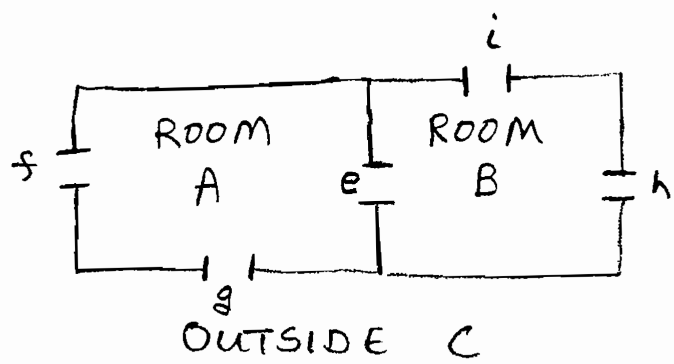
B, g, C, h, A, f, B, i, D, e, A, K, D, j, C IS AN EULER PATH. NOTE: IT IS PERMISSIBLE TO REPEAT VERTICES, BUT NOT EDGES IN AN EULER PATH.

1. THE ORDER OF A VERTEX IS THE NUMBER OF EDGES INCIDENT TO THE VERTEX. ORDER(A) = 4, ORDER(B) = 3, ORDER (C) = 3, ORDER (D) = 4.

2. THEOREM: AN EULER PATH EXISTS IFF THERE ARE ONLY 0 OR 2 ODD VERTICES. (FOR CONNECTED GRAPHS)

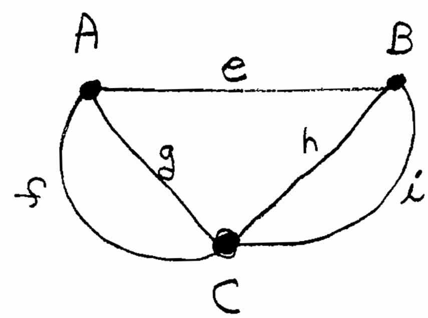
B. RECOGNIZING GRAPH PROBLEMS

1. WALKING THROUGH DOORS PROBLEM:
   IS IT POSSIBLE TO TAKE A WALK AND GO THROUGH EACH DOOR OF THE HOUSE ONCE AND ONLY ONCE?



2. CORRESPONDING GRAPH PROBLEM:

   LET A ROOM OR OUTSIDE BE A VERTEX.
   LET A DOOR BE AN EDGE.



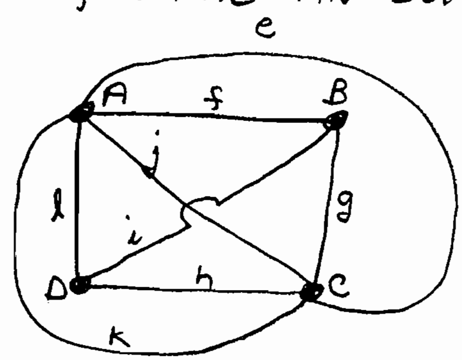
EVEN NUMBER OF ODD VERTICES, SO AN EULER PATH EXISTS:

A, f, C, g, A, e, B, h, C, i, B

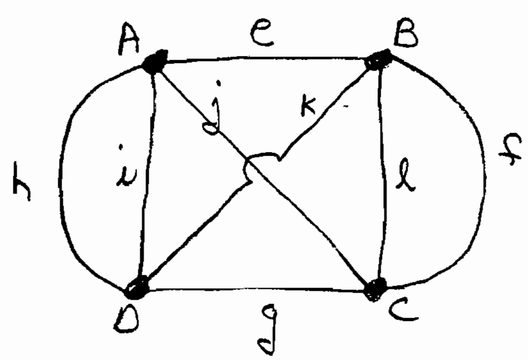
NOW TRANSLATE THIS SOLUTION BACK TO THE ORIGINAL PROBLEM.

C. HOMEWORK (OIS)

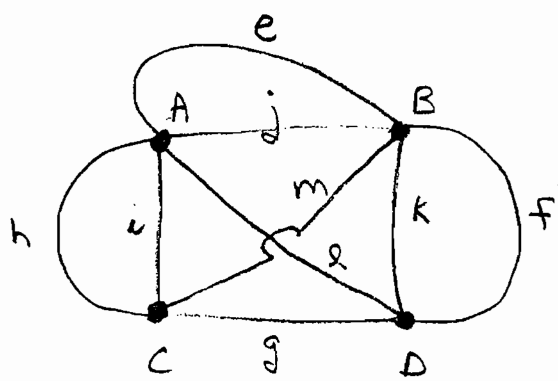
1. WHICH OF THE FOLLOWING GRAPHS HAVE AN EULER PATH? FOR THOSE THAT DO, NAME AN EULER PATH.



G_1



G_2



G_3

2. FOR A GRAPH WITH 2 ODD VERTICES, WHERE CAN YOU START AN EULER PATH? WHERE MUST IT END? NOW THE SAME QUESTION FOR A GRAPH WITH 0 ODD VERTICES.

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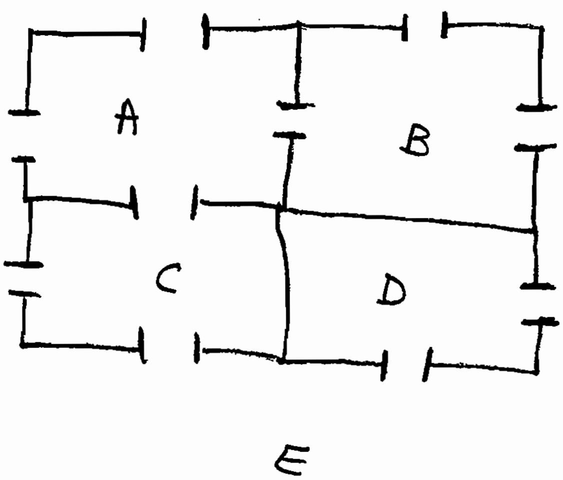
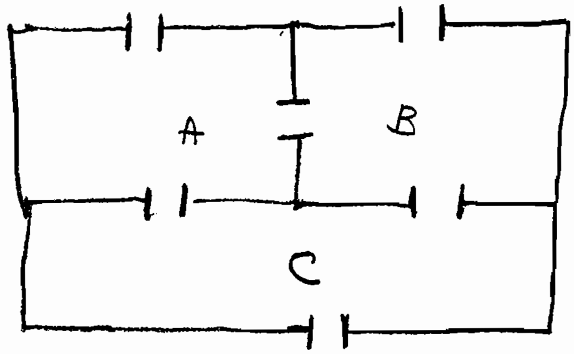
3. FOR EACH OF THE TWO HOUSES BELOW, IS IT POSSIBLE TO TAKE A WALK THROUGH EACH DOOR ONCE AND ONLY ONCE? CHANGE EACH TO A GRAPH PROBLEM. FOR ANY GRAPH WITH AN EULER PATH, STATE ONE, AND THEN SHOW THE CORRESPONDING WALK THROUGH THE DOORS OF THE HOUSE.

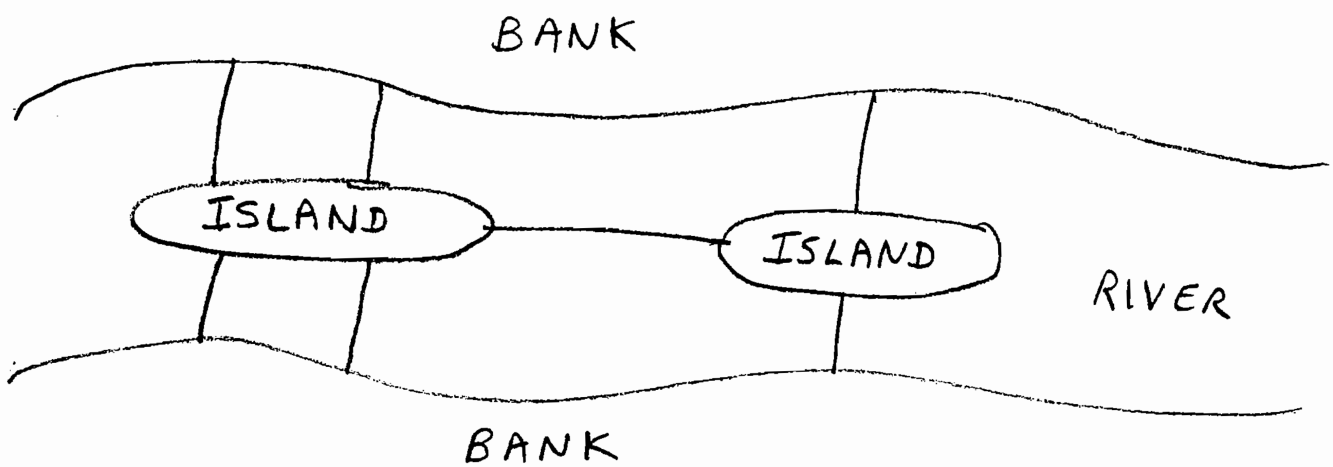


H_1

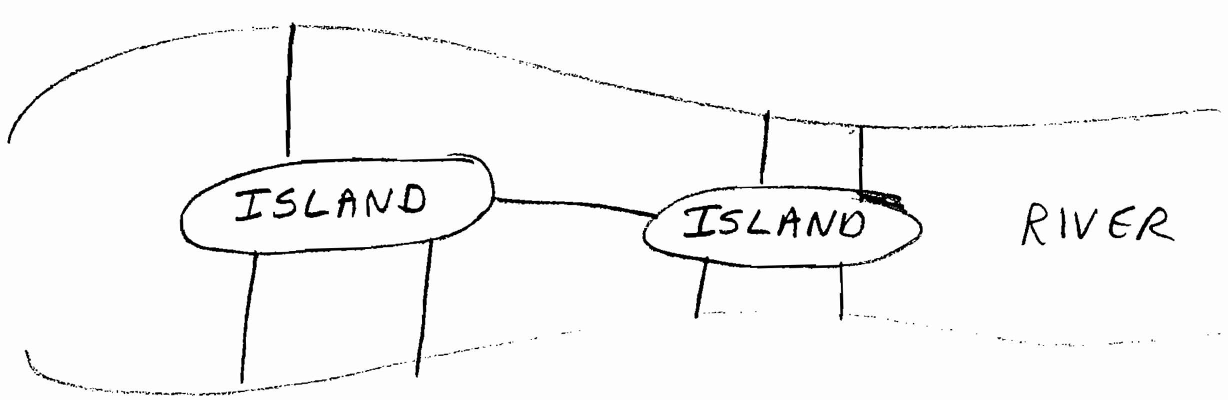


H_2

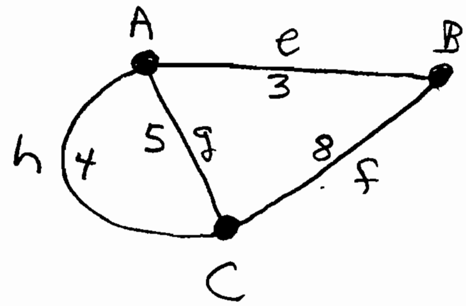
4. CHANGE TO A GRAPH PROBLEM AND SOLVE: A RIVER FLOWS BY TWO ISLANDS. THE BRIDGES AND RIVER BANKS ARE SHOWN. IS IT POSSIBLE TO TAKE A WALK, WALKING OVER EACH BRIDGE ONCE AND ONLY ONCE?



5. SAME DIRECTIONS AS PROBLEM 4.



D. WEIGHTED GRAPH: A GRAPH WHERE THE EDGES HAVE A NUMBER ASSOCIATED WITH EACH EDGE.



G_D

VERTICES COULD REPRESENT CITIES. THE WEIGHT (NUMBERS) COULD BE DISTANCES BETWEEN CITIES OR COULD BE COST TO TRANSPORT GOODS BETWEEN CITIES.

E. <u>HAMILTONIAN</u> <u>CYCLE</u> FOR A GRAPH G:

A CYCLE THAT CONTAINS EACH VERTEX ONCE AND ONLY ONCE EXCEPT FOR THE THE START STOP VERTEX, WHICH APPEARS TWICE.

1. HAMILTON CYCLES FOR G_D ABOVE:

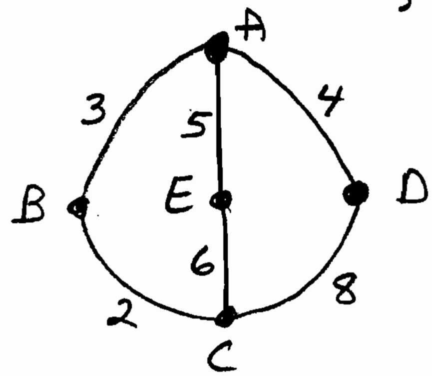
   B, e, A, h, C, f, B TOTAL WEIGHT 3+4+8

   B, e, A, g, C, f, B TOTAL WEIGHT 3+5+8

2. <u>TRAVELING</u> <u>SALESMAN</u> <u>PROBLEM</u>: CAN YOU FIND A MINIMUM LENGTH HAMILTONIAN CYCLE FOR A WEIGHTED GRAPH?

   B, e, A, h, C, f, B IS ONE FOR G_D.

3. SOME GRAPHS DO NOT CONTAIN A HAMILTONIAN CYCLE, LIKE THE ONE BELOW:

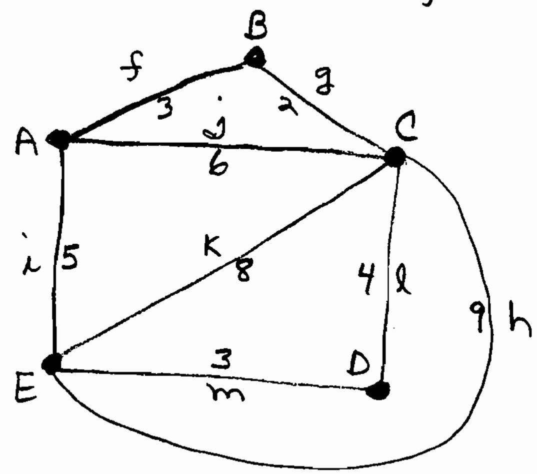


4. INTERPRETATION OF THE TRAVELING SALESMAN PROBLEM: VERTICES ARE CITIES. EDGES ARE ROADS. CAN THE SALESPERSON VISIT EACH CITY ONLY ONE TIME, STARTING AND STOPPING IN THE SAME CITY AND HAVE THE SHORTEST ROUTE DOING THIS?

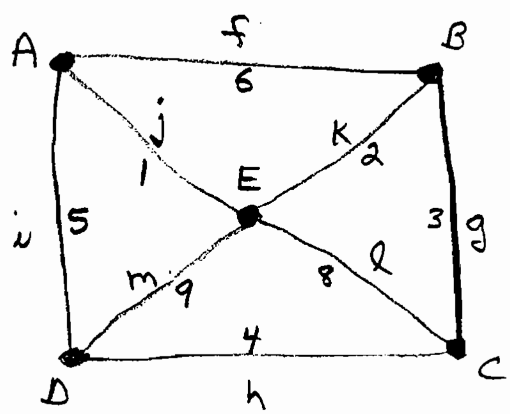
5. THERE IS NO KNOWN GENERAL METHOD TO SOLVE THE TRAVELING SALESMAN PROBLEM THAT TAKES A 'REASONABLE' AMOUNT OF TIME. THIS IS A FAMOUS UNSOLVED PROBLEM IN MATHEMATICS.

F. HOMEWORK (OIS)

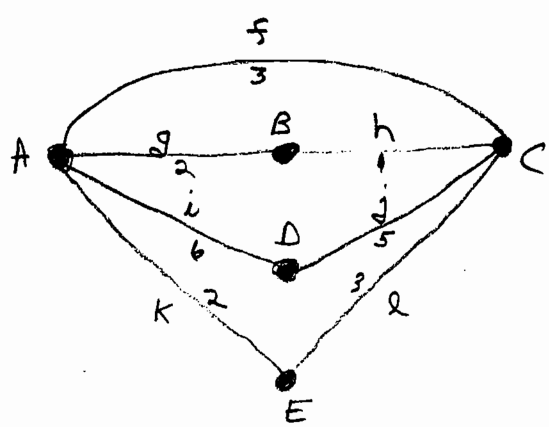
1. FOR EACH GRAPH BELOW, FIND A MINIMAL HAMILTONIAN CYCLE, IF POSSIBLE



G_a



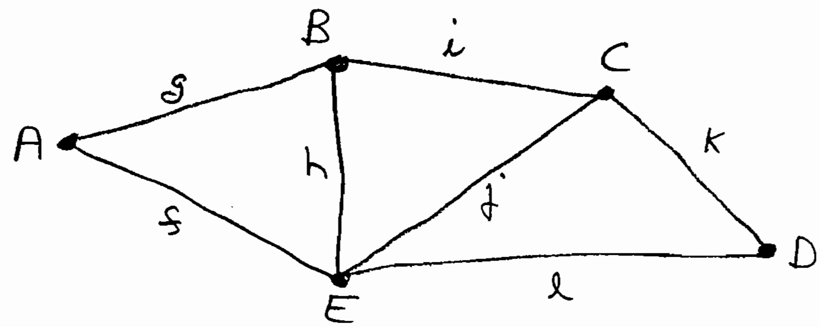
G_b



G_c

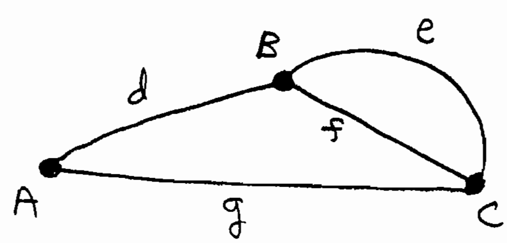
2. IF A HAMILTONIAN CYCLE EXISTS, DOES ONE EXIST WHERE YOU CAN START AT ANY VERTEX?

G. PATHS CAN BE NAMED BY THEIR SEQUENCE OF VERTICES ONLY IF THERE IS NO CONFUSION



THE PATH A, g, B, i, C, k, D, l, E COULD BE DENOTED A, B, C, D, E.

FOR THE GRAPH BELOW:



A, B, C WOULD NOT BE ENOUGH INFORMATION TO DENOTE A PATH. WOULD THE PATH INVOLVE EDJE e OR EDGE f ?

H. <u>DIJKSTRA'S</u> <u>ALGORITHM</u> FINDS A SHORTEST PATH BETWEEN 2 GIVEN VERTICES.

DIJKSTRA'S ALGORITHM

TO FIND SHORTEST PATH FROM VERTEX A TO VERTEX E:

PUT 0 AS A LABEL FOR VERTEX A.

PUT ∞ AS A LABEL FOR ALL OTHER VERTICES.

WHILE E IS NOT CIRCLED,

  CIRCLE AN UNCIRCLED VERTEX V WITH A MINIMUM LABEL.

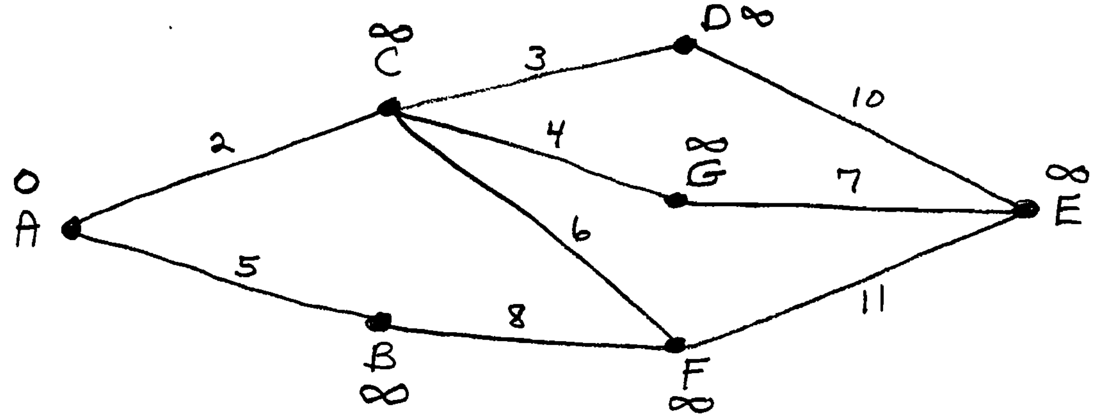
  FOR EACH UNCIRCLED VERTEX W THAT IS ADJACENT TO V,

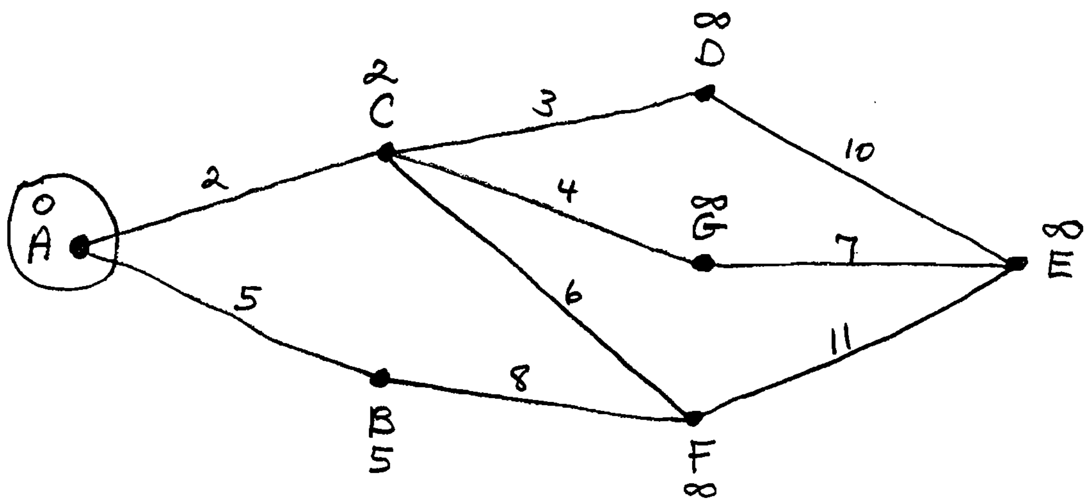
    MAKE THE LABEL FOR W THE MINIMUM OF:

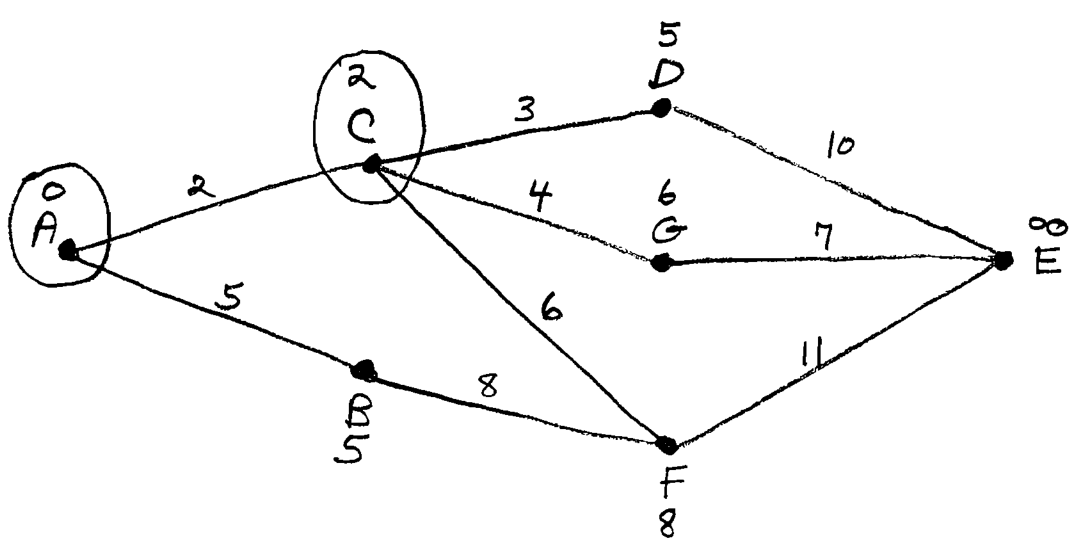
    1. THE CURRENT LABEL FOR W, AND
    2. THE LABEL AT V PLUS THE SMALLEST WEIGHT OF AN EDGE BETWEEN V AND W.

THE MINIMUM LENGTH OF A PATH FROM A TO E IS THE FINAL LABEL AT E. TO GET A PATH OF MINIMAL LENGTH, WORK BACKWARDS FROM E ALONG EDGES THAT PRODUCED THE MINIMAL LABEL, UNTIL YOU GET TO VERTEX A.

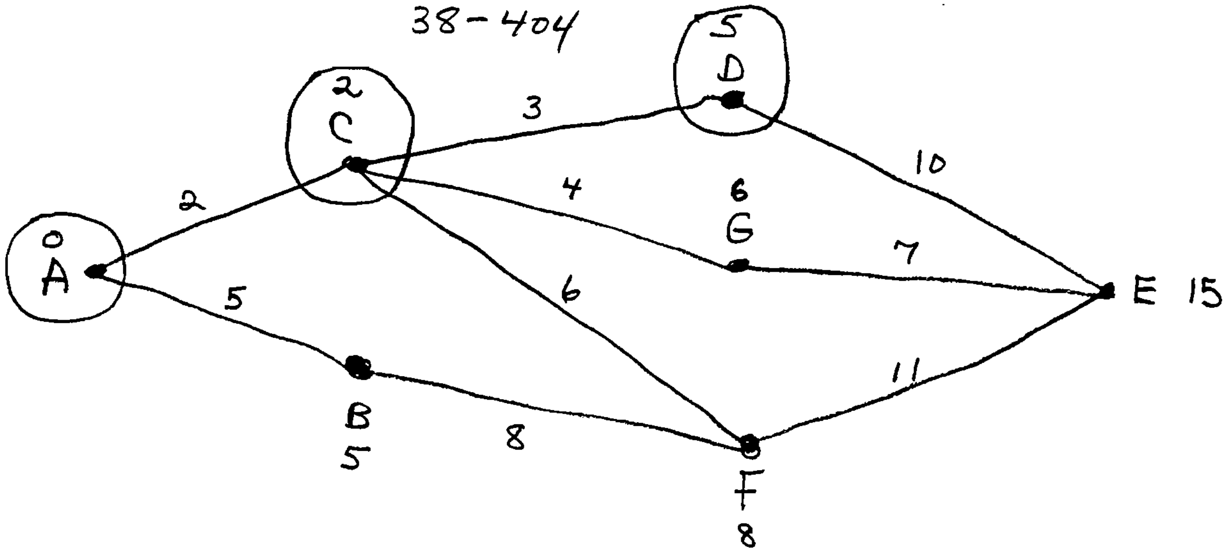
I. DIJKSTRA'S ALGORITHM ILLUSTRATED.







38-404


0
A
2
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E 15
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10
7
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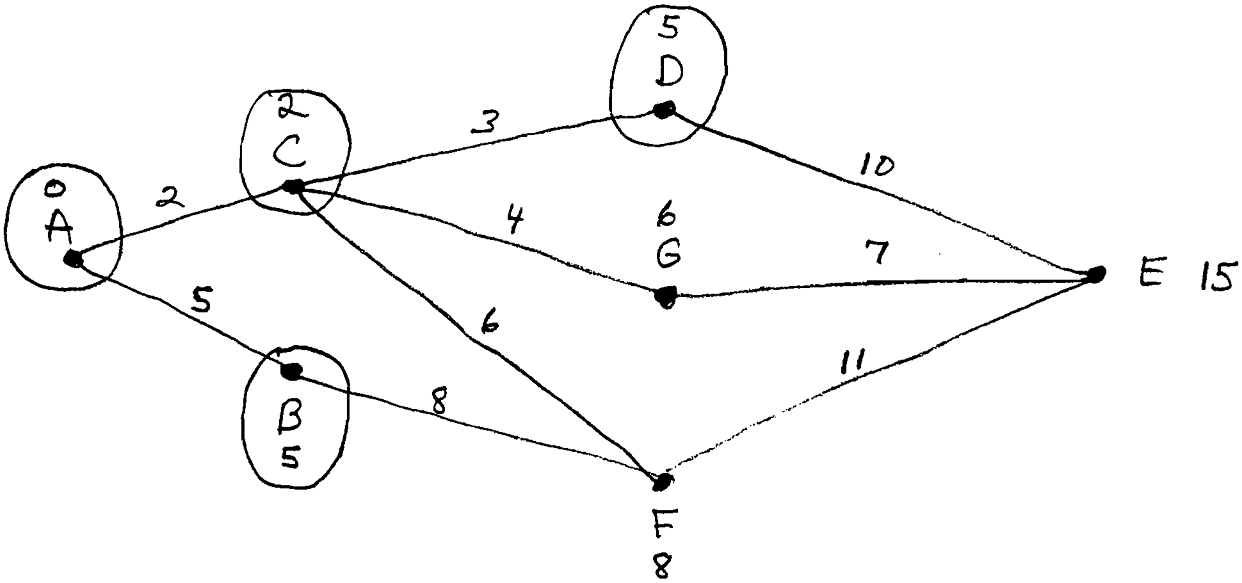
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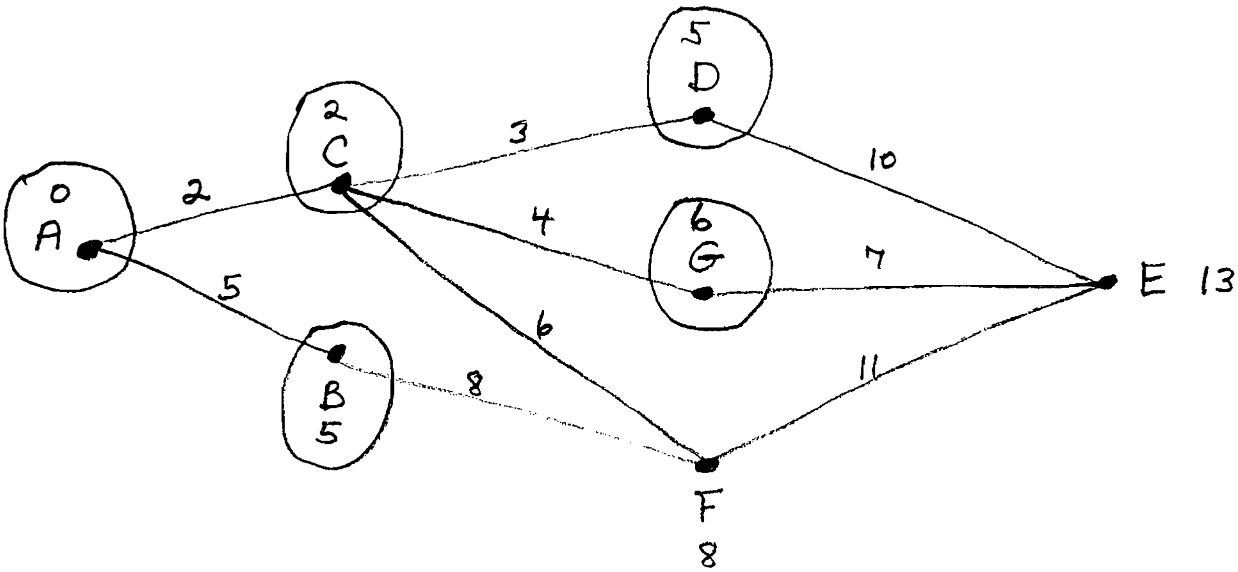
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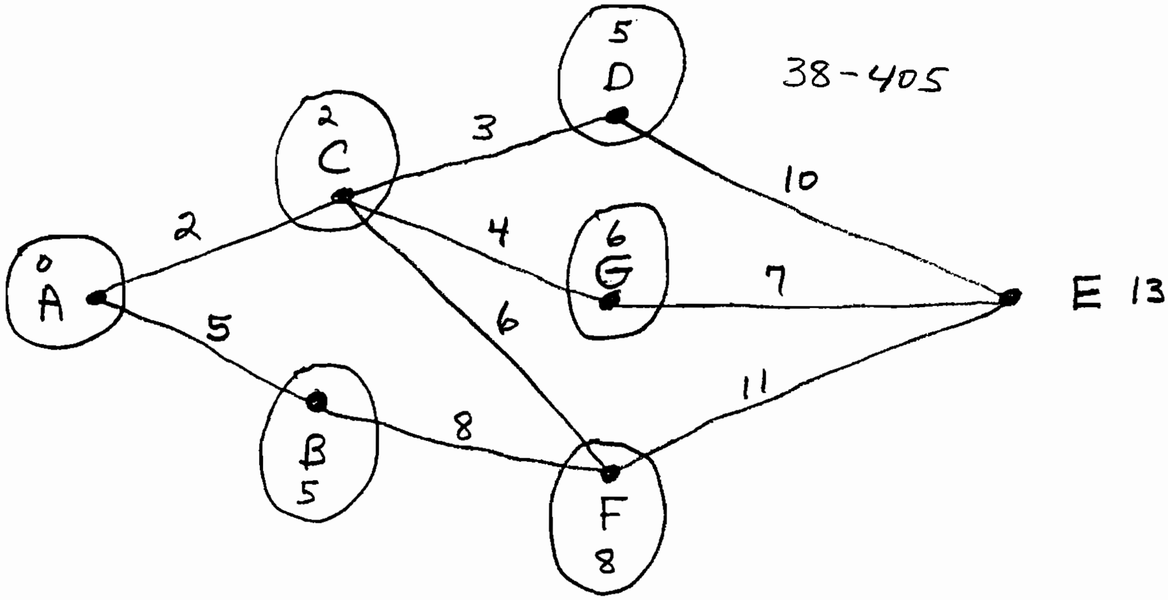
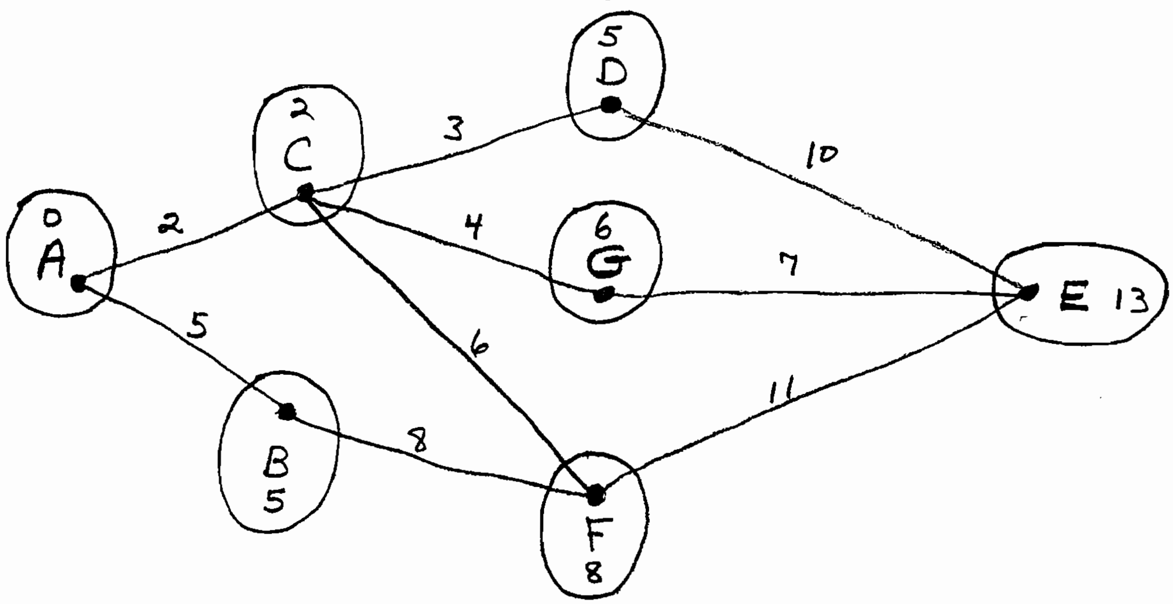
38-405





THE MINIMUM LENGTH OF A PATH FROM A TO E IS 13.

WHAT ARC MADE E LABEL 13 ? G-E

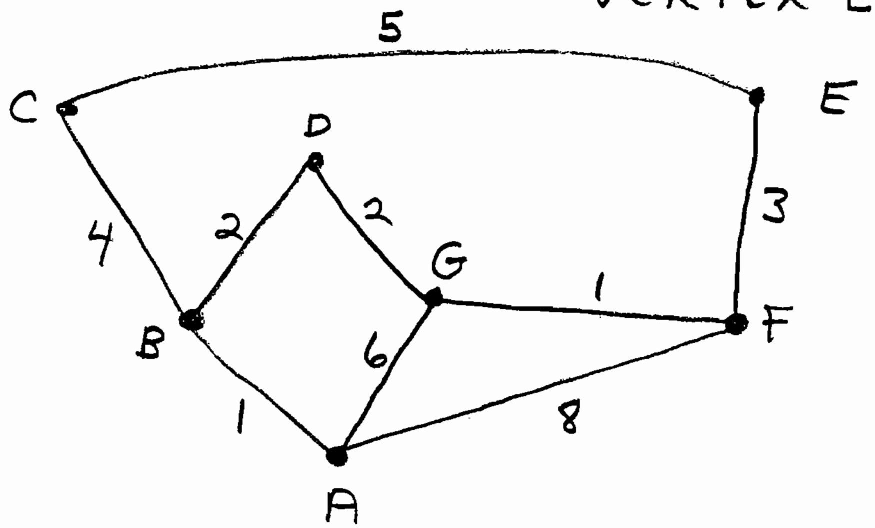
WHAT ARC MADE G LABEL 6 ? C-G

WHAT ARC MADE C LABEL 2 ? A-C

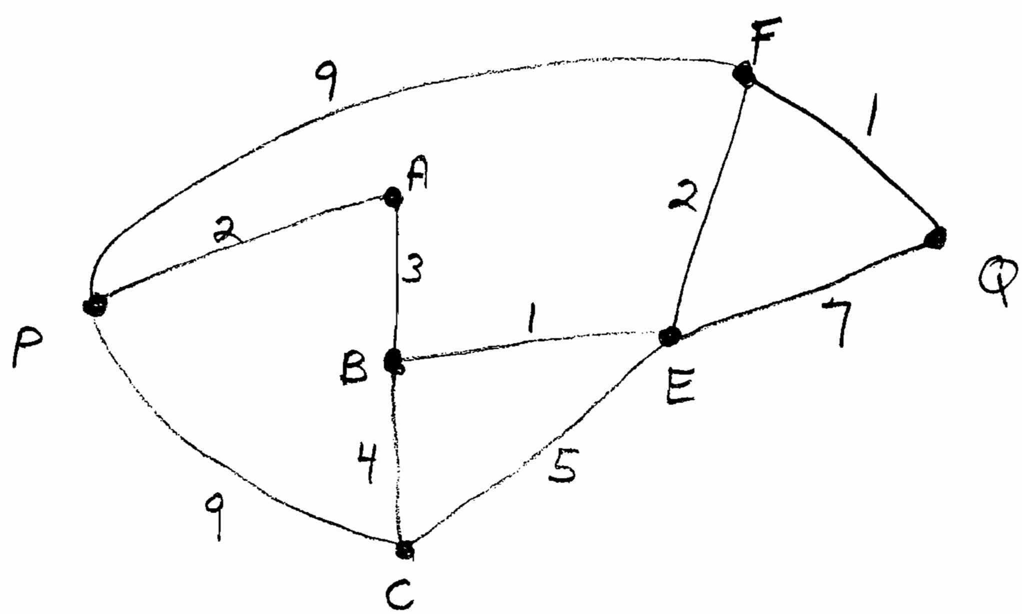
MINIMAL LENGTH PATH : A, C, G, E

J. HOMEWORK (OIS): FOR THE FOLLOWING GRAPHS SHOW THE STEP BY STEP APPLICATION OF DIJKSTRA'S ALGORITHM. ALSO, NAME A MINIMAL PATH, THEN NAME ITS LENGTH.

1. FROM VERTEX A TO VERTEX E.

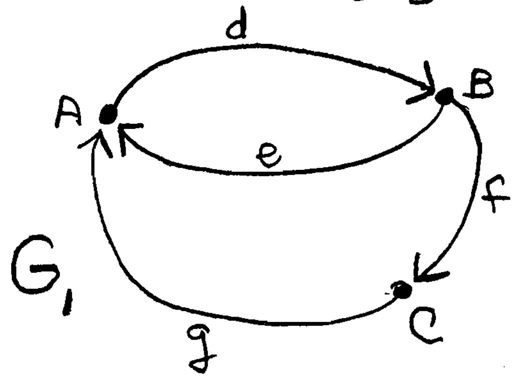


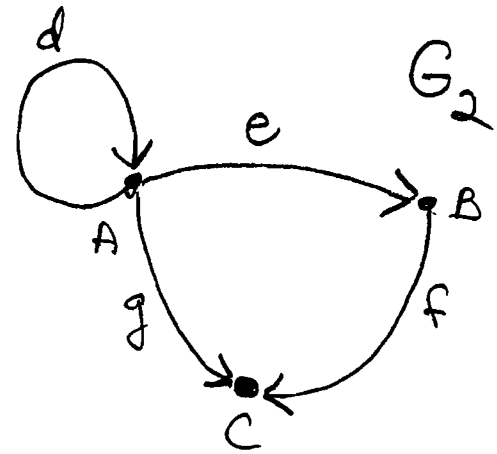
2. FROM VERTEX P TO VERTEX Q.



K. DIRECTED GRAPHS (DIGRAPHS)

1. PICTURES





2. DEFINITION: A DIGRAPH IS A GRAPH G SUCH THAT $\forall$ EDGE e, ONE VERTEX ASSOCIATED WITH AN END POSITION IS DESIGNATED AN <u>INITIAL VERTEX</u> AND THE VERTEX ASSOCIATED WITH THE OTHER END POSITION IS DESIGNATED THE TERMINAL VERTEX

3. FOR GRAPH G_1, EDGE d, A IS THE INITIAL VERTEX AND B IS THE TERMINAL VERTEX; FOR EDGE e, B IS THE INITIAL AND A IS THE TERMINAL VERTEX FOR GRAPH G_2, EDGE d, A IS BOTH THE INITIAL AND TERMINAL VERTEX.

L. APPLICATION FOR A DIGRAPH:

A GRAPH, LIKE G_1 PREVIOUSLY, COULD REPRESENT A CITY MAP. THE EDGE g COULD REPRESENT A ONE WAY STREET. EDGES d AND e TOGETHER COULD REPRESENT A TWO WAY STREET.

M. HOMEWORK (OIS)

1. APPLICATION: PICTURE OF A RELATION. DRAW A DIGRAPH THAT REPRESENTS A PICTURE OF RELATION r BELOW

$$r = \{(2,3), (3,4), (4,2), (2,4), (3,3)\}$$

HINT LET THE VERTICES BE 2,3,4

REFERENCES

DISCRETE MATHEMATICAL STRUCTURES BY KOLMAN,.. ISBN 0-13-320912-1

A

SELECTED ANSWERS

SELECTED ANSWERS

1.5 YES

1.7 YES

1.9 YES

1.11 NO

1.13 NO

1.15 YES

2.23 A SYMBOL THAT CAN BE REPLACED BY ANY STATEMENT.

2.25 NO

2.27 p, $\sim p$, $p \vee q$, $p \wedge q$, $p \rightarrow q$

2.29 c SAM DID NOT HAVE A 90 AVERAGE.

2.29 h SAM MADE AN A AND SAM HAD A 90 AVERAGE.

2.29 j IF SAM MADE AN A, THEN SAM HAD A 90 AVERAGE.

2.30 a $\sim p$

2.30 c $p \wedge q$

3.6a TRUE 2.1 WAS $1<2$ OR $2<3$, THIS IS $2<3$ OR $1<2$. WE ARE IN THE PROCESS OF CHECKING TO SEE IF <u>OR</u> IS COMMUTATIVE.

3.8 b. TRUE 2.11 WAS "$1<2$ IMPLIES $6<5$" WHICH WAS <u>FALSE</u>. THIS IS "$6<5$ IMPLIES $1<2$" WHICH IS <u>TRUE</u>. SO <u>IMPLIES</u> IS <u>NOT</u> COMMUTATIVE.

4.11 $[(p \rightarrow q) \wedge (\sim p)] \rightarrow (\sim q)$

[(	p	$\rightarrow$	q)	$\wedge$	($\sim$	p)]	$\rightarrow$	($\sim$	q)
	1	2	3	4	5	6	7	8	9

COLUMN NO.	TRUTH VALUES FOR INSTANCES OF
2	$p \rightarrow q$
5	$\sim p$
7	$[(p \rightarrow q) \wedge (\sim p)] \rightarrow (\sim q)$
9	q

TO FILL IN COLUMN	LOOK IN COLUMNS
2	1, 3
5	6
7	4, 8

4.12 YES

4.14 NO

4.16 $[(p \rightarrow q) \wedge p] \rightarrow q$

p	$\rightarrow$	q	$\wedge$	p	$\rightarrow$	q
T	T	T	T	T	T	T
T	F	F	F	T	T	F
F	T	T	F	F	T	T
F	T	F	F	F	T	F

4.18 $(p \rightarrow q) \wedge (q \rightarrow p)$

p	$\rightarrow$	q	$\wedge$	q	$\rightarrow$	p
T	T	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	F	T	F	F
F	T	F	T	F	T	F

4.19 THERE SHOULD BE 5 DIFFERENT STATEMENT PATTERNS.

4.20 a DISJUNCTION 4.20 c IMPLICATION, THE HYPOTHESIS IS $[(p \to q) \wedge (\sim p)]$. THE CONCLUSION IS $(\sim q)$

4.20 e CONJUNCTION 4.20 h NEGATION

4.21 (a)(1) $1 < 2$ OR $1 \not< 2$

4.21 (a)(3) $6 < 3 \longrightarrow 1 \not< 2$

4.21 (b) $(\sim p) \to (\sim q)$

5.7 (a) TAUTOLOGY 5.7 (c) TAUTOLOGY

5.7 (e) TAUTOLOGY 5.7 (g) NOT A TAUTOLOGY

5.7 (j) TAUTOLOGY

6.25 a $(p \leftrightarrow q) \leftrightarrow [(p \to q) \wedge (q \to p)]$

p	$\leftrightarrow$	q	$\leftrightarrow$	p	$\to$	q	$\wedge$	q	$\to$	p
T	T	T	T	T	T	T	T	T	T	T
T	F	F	T	T	F	F	F	F	T	T
F	F	T	T	F	T	T	F	T	F	F
F	T	F	T	F	T	F	T	F	T	F

$p \leftrightarrow q$ IS EQUIVALENT TO $(p \to q) \wedge (q \to p)$

6.25 f EQUIVALENT

6.30 b GIVEN $(\sim p) \to (\sim q)$ CONVERSE $(\sim q) \to (\sim p)$

INVERSE $p \to q$ CONTRAPOSITIVE $p \to q$

6.30 d. GIVEN: IF SAM DID NOT HAVE A POS. SKIN TEST, THEN SAM DID NOT HAVE TB.

CONVERSE: IF SAM DID NOT HAVE TB, THEN SAM DID NOT HAVE A POS. SKIN TEST

INVERSE: IF SAM HAD A POSITIVE SKIN TEST, THEN SAM HAD TB.

CONTRAPOSITIVE: IF SAM HAD TB, THEN SAM HAD A POSITIVE SKIN TEST.

6.32 $(p \to q) \leftrightarrow [(\sim p) \vee q]$

T	T	T	T	X	X	T	X
T	F	F	T	X	X	F	X
F	T	T	T	X	X	T	X
F	T	F	T	X	X	T	X

(6, T, 1) MEANS (CHAPTER 6, SECTION T, PART 1)

(6, T, 1) $(\sim p) \vee (\sim q)$ (6, T, 3) $(\sim M) \vee (\sim B)$

(6, T, 5) $[\sim(\sim A)] \vee (\sim B)$ OR $A \vee (\sim B)$

(6, T, 7) $A \vee B$ (6, T, 9) $(\sim A) \vee [(\sim B) \wedge (\sim C)]$

(6, T, 11) $(\sim A) \vee [(\sim B) \vee (\sim C)]$ (6, T, 13) $(\sim p) \wedge (\sim q)$

(6, T, 15) $(\sim A) \wedge B$ (6, T, 17) $A \wedge B$

(6, T, 19) $A \wedge [(\sim B) \vee (\sim C)]$ (6, T, 21) $p \wedge (\sim q)$

(6, T, 23) $M \wedge (\sim B)$ (6, T, 25) $M \wedge D$

(6, T, 27) $A \wedge [(\sim B) \wedge (\sim C)]$ (6, T, 29) $A \wedge [B \wedge (\sim C)]$

(6, T, 31) $(\sim B) \to (\sim A)$ OR $(\sim A) \vee B$

(6, T, 33) $B \to A$ OR $A \vee (\sim B)$

(6, T, 35) $[D \wedge (\sim W)] \to A$ OR $A \vee ((\sim D) \vee W)$

6.31 a A IS <u>NOT</u> A SUBSET OF B OR B IS <u>NOT</u> A SUBSET OF A.

6.31 c I AM <u>NOT</u> GOING TO PASS LOGIC AND I AM <u>NOT</u> GOING TO DIE TRYING.

6.31 e $p \wedge r$

(6, Z, 1) $(\sim A) \wedge B$ (6, Z, 3) $(\sim A) \wedge [B \wedge (\sim E)]$

(6, Z, 5) $(\sim A) \wedge B$ (6, Z, 7) $A \wedge [B \vee (\sim E)]$

7.20(a) IF $1 > 0$, THEN 1 IS POSITIVE

7.20(c) IF IT STARTED RAINING, THEN SAM PUT UP THE UMBRELLA.

7.20(e) IF SAM MADE OVER \$10,000, THEN SAM FILLED IN LINE 10-b ON THE INCOME TAX RETURN.

7.20(g) IF SAM PROVED A THEOREM, THEN SAM KNEW LOGIC.

7.20(i) $0 < 1$ IF AND ONLY IF $1 > 0$.

7.21 (a) $(\sim p) \to q$ 7.21 (c) $(\sim p) \wedge (\sim q)$

7.21 (e) $p \to [((\sim r) \to w) \wedge t]$

7.21 (g) $\sim\left(p \wedge [(\sim w) \to (\sim q)]\right)$

9.22(a) VALID, MODUS PONENS

9.22(c) INVALID, INVERSE REASONING

9.22(e) VALID, MODUS PONENS

9.22(g) INVALID, CONVERSE REASONING

9.22(i) INVALID, CONVERSE REASONING

9.22(k) VALID, MODUS TOLLENS

9.23(a) VALID 9.23(c) VALID

9.23(e) VALID 9.23(g) VALID

9.23(i) INVALID

9.24(a) VALID

9.24(c) INVALID

9.25 (a)

$$\begin{array}{l} A \rightarrow (\sim B) \\ A \\ \hline \therefore \sim B \end{array}$$

MP

9.25(c)

$$\begin{array}{l} A \rightarrow (\sim B) \\ \sim A \\ \hline B \end{array}$$

INVERSE REASONING

9.25 (e)

$$\begin{array}{l} A \rightarrow (\sim B) \\ (\sim B) \rightarrow T \\ \hline \therefore A \rightarrow T \end{array}$$

SYLLOGISM

9.26 (b) CONVERSE REASONING

IF SAM HAD TB, THEN SAM HAD A POSITIVE SKIN TEST
SAM HAD A POSITIVE SKIN TEST

∴ SAM HAD TB.

9.26 (d) MODUS TOLLENS

IF SAM HAD TB, THEN SAM HAD A POSITIVE SKIN TEST.
SAM DID NOT HAVE A POSITIVE SKIN TEST

∴ SAM DID NOT HAVE TB.

10.12 (b) ASSUME p', q', r' ARE STATEMENTS THAT REPLACE p, q, r, RESP.

$$\left.\begin{array}{ll} 1. & \sim p' \vee \sim q' \\ 2. & r \rightarrow (p' \wedge q') \end{array}\right\} \text{ASSUMED TRUE FOR DIRECT PROOF}$$

3. $\sim(p' \wedge q')$ — 1, DEMORGAN'S LAW
4. $\sim r'$ — 2, 3, MODUS TOLLENS

(LINE 3 COULD HAVE BEEN OMITTED AND $\sim r'$ CONCLUDED FROM 1, 2, MODUS TOLLENS)

10.12 (e) INVALID. DEMONSTRATE VIA TRUTH TABLE

10.12(i) VALID

0. ASSUME p', q', w', s' ARE STATEMENTS THAT REPLACE p, q, w, s, RESP.

$$\left.\begin{array}{l} 1.\ p' \leftrightarrow q' \\ 2.\ (p' \to q') \to w' \\ 3.\ (q' \to p') \to s' \end{array}\right\} \text{ASSUMED TRUE FOR DIRECT PROOF (SHOW } w' \wedge s'\text{)}$$

4.	$(p' \to q') \wedge (q' \to p')$	1, DEFINITION OF $\leftrightarrow$
5.	$p' \to q'$	4, TRUTH TABLE DEF. OF $\wedge$
6.	$q' \to p'$	4, TRUTH TABLE DEF. OF $\wedge$
7.	w'	2, 5, MODUS PONENS
8.	s'	3, 6, MODUS PONENS
9.	$w' \wedge s'$	7, 8, TRUTH TABLE DEF. OF $\wedge$

10.12(e) HINT: USE THE FACT THAT $\sim p \vee q$ IS EQUIVALENT TO $p \to q$ AT LEAST TWICE.

11.8 (a) 0. ASSUME p', q', r' ARE STATEMENTS THAT REPLACE p, q, r, RESP.

$$\left.\begin{array}{l} 1.\ p' \to q' \\ 2.\ \sim(p' \to r') \\ 3.\ q' \to r' \end{array}\right\} \text{ASSUMED TRUE FOR INDIRECT PROOF. GET \underline{ANY} CONTRADICTION}$$

$$\left.\begin{array}{ll} 4.\ p' \to r' & \text{1, 3, SYLLOGISM} \\ 5.\ \sim(p' \to r') & \end{array}\right\} \# \ 2$$

11.8 (c) 0. ASSUME p', q', r' ARE STATEMENTS THAT REPLACE p, q, r RESP.

1. $\sim p' \rightarrow [\sim q' \vee \sim r']$
2. $\sim q' \rightarrow \sim r'$
3. $\sim p' \wedge r'$

ASSUMED TRUE FOR INDIRECT PROOF
GET ANY CONTRADICTION

4. $\sim p'$ — 3, TT DEF. OF $\wedge$
5. r' — 3, TT DEF. OF $\wedge$
6. q' — 2, 5, MODUS TOLLENS
7. $\sim q' \vee \sim r'$ — 1, 4, MODUS PONENS
8. $q' \rightarrow \sim r'$ — 7, IMPLICATION EQUIVALENCE

THIS IS ENOUGH HINT FOR THIS PROBLEM. THE STUDENT CAN FINISH.

12.13 TRUE INSTANCE: 2 IS AN EVEN POSITIVE INTEGER AND 2 IS LESS THAN 10.
FALSE INSTANCE: 5 IS AN EVEN POSITIVE INTEGER AND 5 IS LESS THAN 10.

12.14 (a) YES. 1 IS A POSITIVE INTEGER IMPLIES 1 IS A REAL NUMBER

12.15 (b) 5, 7

12.16 (a) FALSE

12.16 (c) TRUE

12.16 (e) FALSE

12.16 (h) TRUE

13.7 (a) TRUE. THERE IS A REPLACEMENT FOR x FROM R THAT MAKES A TRUE INSTANCE OF "$x+7=10$".

13.8 (b) FALSE. EVERY REPLACEMENT FOR x FROM R MAKES A TRUE INSTANCE OF "$x \nless 5$".

13.9 (b) TRUE. THERE IS A REPLACEMENT FOR x FROM R THAT MAKES A TRUE INSTANCE OF "$x \notin N$".

13.13 (a) FALSE. EVERY REPLACEMENT FOR x FROM R MAKES A TRUE INSTANCE OF "$x \in R$ IMPLIES $x \in N$".

13.16 (a) TRUE. EVERY REPLACEMENT FOR x FROM THE UNDERSTOOD UNIVERSE FOR x, MAKES A TRUE INSTANCE OF "$x \in N$ IMPLIES $x \in R$".

13.18 (a) TRUE. EVERY REPLACEMENT FOR x FROM THE UNDERSTOOD UNIVERSE FOR x MAKES A TRUE INSTANCE OF "$x \in \phi$ IMPLIES $x \in N$".

13.20 (b) FALSE. THERE IS A REPLACEMENT FOR n FROM THE UNDERSTOOD UNIVERSE FOR n THAT MAKES A TRUE INSTANCE OF "($n \in E$ AND [$n \notin N$ OR 2 IS NOT A FACTOR OF n]) OR ([$n \in N$ AND 2 IS A FACTOR OF n] AND $n \notin E$)".

14.2 (a) TRUE

14.3 (a) TRUE

14.4 (a) TRUE

14.5 (c) TRUE

14.6 (b) FALSE

14.7 (b) TRUE

14.8 (c) FALSE

14.9 (a) TRUE

14.10 (b) TRUE

15.6 LET D = THE SET OF ALL DOGS

$\exists x \in D$, x HAS FLEAS

15.9 LET M = THE SET OF ALL MARRIED MEN.

LET P = THE SET OF ALL PEOPLE.

$\forall x \in M, \exists y \in P \ni y$ IS THE WIFE OF x

15.11 $\forall x \in N, \forall y \in N, \ x+y \in N.$

15.15 LET P = THE SET OF ALL PEOPLE

$\forall x \in P$, x DOES NOT LIKE MATH

15.17 LET P = THE SET OF ALL PEOPLE

$\forall x \in P, \exists y \in P \ni x$ LIKES y.

15.22 $\forall x \in R, \forall y \in R$, IF $x \neq 0$ AND $y \neq 0$, THEN $xy \neq 0$.

15.24 $\forall x \in R, \forall y \in R, \ x+y = y+x$

15.30 $\forall x \in R, \forall y \in R$, (IF $x<y$, THEN $\exists q \in Q \ni x<q<y$)

Q IS THE SET OF RATIONAL NUMBERS

16.17 $\exists x \ni x \in N$ AND $x \notin R$

16.20 $\forall x \in C, (x \notin A$ OR $x \notin B)$

16.24 $\exists x \in R, \exists y \in R, \exists z \in R \ni x(y+z) \neq xy + xz$

16.26 $\exists M$, (M IS A SUBSET OF N AND $M \neq \phi$) AND

$(\forall k \in M, \exists x \in M \ni k > x)$

(17, I, 1) PROOF START:

A, 1. ASSUME $e > 4$ (SHOW $3e - 5 > 2$)

2. $3e > 12$ 1, MULTIPLY BY 3

⋮

(17, I, 3) PROOF START:

A, 1. ASSUME ($K \in B$ AND $K < 15$). (SHOW $K \in W$)

2. $K \in B$ 1, TRUTH TABLE DEF. OF AND

3. $K \in R$ AND $K > 10$ 2, DEFINITION OF B

4. $K < 15$ 1, TRUTH TABLE DEF. OF AND

5. $15 < 21$ ARITHMETIC KNOWLEDGE

⋮ GET K IN M, THEN IN W TO COMPLETE THE PROOF

(18, H, 2) PROOF START:

A, 1. ASSUME K IS A REAL NUMBER AND $K > 5$ (SHOW $2K \in B$)

2. $K \in R$ 1, TRUTH TABLE DEFINITION OF AND

3. $K > 5$ 1, TRUTH TABLE DEFINITION OF AND

(18, H, 4) PROOF START:

A_1 1. ASSUME EVERY REAL NUMBER LESS THAN 30 IS IN E. (SHOW EVERY ELEMENT OF M IS IN E)

A_2 2. ASSUME $K \in M$ (SHOW K IS IN E)

3. $K \in R \wedge K < 21$ 2, DEFINITION OF M

⋮

(19, L, 1) PROOF START:

A_1 1. ASSUME $x' \in T$ (SHOW $3x'-2 < 25$)

2. $x' < 8$ — 1, DEFINITION OF T

...

(19, L, 3) IF $x' \in T$, THEN $x' \in W$

(19, L, 5) PROOF START:

A_1 1. ASSUME $(\forall x \in A, x > 5)$ AND $(\forall x \in B, x > 6)$

(SHOW $\forall g \in B, \forall z \in A, 2z + 3g > 28$)

A_2 2. ASSUME $g' \in B$ (SHOW $\forall z \in A, 2z + 3g' > 28$)

A_3 3. ASSUME $z' \in A$ (SHOW $2z' + 3g' > 28$)

4. $\forall x \in A, x > 5$ — 1, TRUTH TABLE DEF. OF AND

5. $z' > 5$ — 3, 4, INSTANCE OF 4

...

(20, D, 1) PROOF OUTLINE:

(SHOW $4b > 12 \rightarrow 15 < 5b$)

A_1 1. ASSUME $4b > 12$ (SHOW $15 < 5b$)

:

A_1^*

(SHOW $15 < 5b \rightarrow 4b > 12$)

A_2 6. ASSUME $15 < 5b$ (SHOW $4b > 12$)

:

A_2^*

11. $(4b > 12 \rightarrow 15 < 5b) \wedge (15 < 5b \rightarrow 4b > 12)$ — 5, 10, TT DEF $\wedge$

12. $4b > 12 \leftrightarrow 15 < 5b$ — 11, DEF $\leftrightarrow$

(20, D, 2) PROOF START: PROVE $\forall x \in R, (0 > 2x-4 \leftrightarrow x \in B)$

A_1 1. ASSUME $x' \in R$ (SHOW $0 > 2x'-4 \leftrightarrow x' \in B$)

(SHOW $0 > 2x'-4 \rightarrow x' \in B$)

A_2 2. ASSUME $0 > 2x'-4$ (SHOW $x' \in B$)

⋮

(21, E, 2) PROOF START:

A_1 1. ASSUME $B \subseteq E$ AND $E \subseteq G$ (SHOW $B \subseteq G$)

(SHOW $\forall x \in B, x \in G$)

A_2 2. ASSUME $x' \in B$ (SHOW $x' \in G$)

3. $B \subseteq E$ — 1, TRUTH TABLE DEF. OF AND

4. $\forall x \in B, x \in E$ — 3, DEFINITION OF $\subseteq$

⋮

(22, G, 2) PROOF START: (SHOW $\forall x \in E \cap T, x \in E \cup T$)

A_1 1. ASSUME $x' \in E \cap T$ (SHOW $x' \in E \cup T$)

2. $x' \in E$ AND $x' \in T$ — 1, DEFINITION OF $\cap$

⋮

(23, H, 2) PROOF START: (SHOW $\forall x \in H-(H-K), x \in K$)

A_1 1. ASSUME $x' \in H-(H-K)$ (SHOW $x' \in K$)

2. $x' \in H$ AND $x' \notin H-K$ — 1, DEFINITION OF $-$

A_2 3. NOW START AN INDIRECT PROOF OF ___?___

(24, G, 2) PROOF OUTLINE:

(SHOW $J \cap (H \cap K) \subseteq (J \cap H) \cap K$)

(SHOW $\forall x \in J \cap (H \cap K)$, $x \in (J \cap H) \cap K$)

A_1 1. ASSUME $x' \in J \cap (H \cap K)$ (SHOW $x' \in (J \cap H) \cap K$)

⋮

(SHOW $(J \cap H) \cap K \subseteq J \cap (H \cap K)$)

(SHOW $\forall x \in (J \cap H) \cap K$, $x \in J \cap (H \cap K)$)

(24, G, 4)

(SHOW $H \cap \phi \subseteq \phi$) (SHOW $\forall x \in H \cap \phi$, $x \in \phi$)

A_1 1. ASSUME $x' \in H \cap \phi$ (SHOW $x' \in \phi$)

⋮

(SHOW $\phi \subseteq H \cap \phi$) (SHOW $\forall x \in \phi$, $x \in H \cap \phi$)

ALTERNATE WAY: SHOW $H \cap \phi$ HAS NO ELEMENTS BY INDIRECT PROOF.

(25, D, 1) PROOF START:

A_1 1. ASSUME $A \subseteq E$ AND $B \subseteq F$ (SHOW $A \cup B \subseteq E \cup F$)

(SHOW $\forall x \in A \cup B$, $x \in E \cup F$)

A_2 2. ASSUME $x' \in A \cup B$ (SHOW $x' \in E \cup F$)

3. $\boxed{x' \in A \text{ OR } x' \in B}$

(SHOW $x' \in A \rightarrow x' \in E \cup F$)

A_3 4. <u>CASE 1</u> ASSUME $x' \in A$ (SHOW ________)

⋮

(25, G) $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ IS PROOF BY CASES BUT <u>NOT</u> PROOF BY CASES <u>WITHIN</u> PROOF BY CASES.

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ <u>IS</u> PROOF BY CASES WITHIN PROOF BY CASES.

YOUR MATHEMATICAL DEVELOPMENT MAY BE HINDERED IF YOU ARE GIVEN MORE HINT THAN THIS!

(26, G, 1) HINT: GET LINES IN THE PROOF THAT SAY $7 > 5$

$7 \in E \cup F$

THEN YOU CAN SAY NEXT: $\exists x \in E \cup F \ni x > 5$

(27, L, 2) PROOF START:

A, 1. ASSUME $(\exists x \in A, x \in E \cup F)$, $E \subseteq B$, AND $F \subseteq K$ (SHOW $\exists x \in A, x \in B \cup K$)

2. $\exists x \in A \ni x \in E \cup F$ 1, TRUTH TABLE DEF. OF AND

3. LET $x' \in A \ni x' \in E \cup F$ 2, LET RULE

4. $x' \in E \cup F$ 3

5. $\boxed{x' \in E \text{ OR } x' \in F}$

DO PROOF BY CASES

(28, F, 3) PROOF START

A, 1. ASSUME $A \neq \phi$ AND $A \subset B$ (SHOW $B-A \subset B$)

(SHOW $B-A \subseteq B$ AND $\exists x \in B \ni x \notin B-A$)

(SHOW $B-A \subseteq B$)

⋮

(SHOW $\exists x \in B \ni x \notin B-A$)

6. $A \neq \phi$ — 1, TRUTH TABLE DEF. OF AND

7. LET $x' \in A$ — 6, LET RULE

GET x' IN B AND NOT IN $B-A$

(29, G, 2)

1. ASSUME $A \subseteq B$ AND $E \subseteq F$ (SHOW $A \cup E \subseteq B \cup F$)

(SHOW $\forall x \in A \cup E, x \in B \cup F$)

2. ASSUME $x \in A \cup E$ (SHOW $x \in B \cup F$)

3. $x \in A$ OR $x \in E$ — 2, DEF. OF $\cup$

4. <u>CASE 1</u> $x \in A$

5. $x \in B$ — 1, SINCE $A \subseteq B$

6. $x \in B$ OR $x \in F$ — 5, TRUTH TABLE DEF. OF OR

7. $x \in B \cup F$ — 6, DEF OF $\cup$

8. <u>CASE 2</u> $x \in E$

(29, J, 1) PROOF START (SHOW $\forall x \in A, x \in B$)

1. ASSUME $z \in A$ (SHOW $z \in B$)

2. $\exists x \in [0,1] \ni z = 2x+8$ — 1, DEF. OF A

3. LET $x' \in [0,1] \ni z = 2x'+8$ — 2, LET RULE

(29, M, 1) PROOF START:

1. ASSUME $A \subset B$ (SHOW $A-B \subset A\cup B$)

   (SHOW $A-B \subseteq A\cup B \wedge \exists x \in A\cup B \ni x \notin A-B$)

   (SHOW $A-B \subseteq A\cup B$. SHOW $\forall x \in A-B, x \in A\cup B$)

   ⋮

   HAVE THESE LINES, WITH PROPER REASONS

   $A \subseteq B \wedge \exists x_1 \in B \ni x_1 \notin A$ ____________

   $x_1 \in B$ ____________

(30, K, 2)

A. SHOW $4(1)-3 = 1(2(1)-1)$

$$4(1)-3 = 4-3 = 1 = 1\cdot 1 = 1(2-1) = 1(2(1)-1)$$

B. ASSUME n IS A POSITIVE INTEGER AND $1+5+9+\cdots+(4n-3) = n(2n-1)$. (SHOW $1+5+9+\cdots+(4(n+1)-3) = (n+1)(2(n+1)-1)$

1.	$1+5+9+\cdots+(4n-3)+(4(n+1)-3)$	$=$	
2.	$n(2n-1)+4(n+1)-3$	$=$	B
3.	$2n^2-n+4n+4-3$	$=$	ALGEBRA
4.	$2n^2+3n+1$	$=$	"
5.	$(n+1)(2n+1)$	$=$	"
6.	$(n+1)(2n+2-1)$	$=$	"
7.	$(n+1)(2(n+1)-1)$		"

(31, G, 2) $2(8^3)+0(8^2)+3(8^1)+4(8^0)+5(8^{-1})+0(8^{-2})+6(8^{-3})$

(31, G, 3) NEXT NUMBER AFTER 5077_8 IS 5100_8
NEXT NUMBER AFTER $B9_{16}$ IS BA_{16}

(31, I, 1) $45\frac{5}{8} = 45.625$ (31, I, 3) 4003

(31, K, 2) $.01\overline{0011}$ (31, K, 5) $.C_{16}$

(31, M, 2) 353.16_8

(31, M, 4) 101000111011.01111111_2

(31, O, 2) 111100_2 (31, Q, 2) 32232_5

(31, Q, 4) 10011_2 (31, S) $A7_{16} \times 3B_{16} = 267D_{16}$

(31, U, 2) 1010101_2 WITH REMAINDER 100

(32, E, 1) $T \times E = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b),(4,a),(4,b)\}$

AN EXAMPLE OF r COULD BE

$r = \{(a,2),(a,3),(b,3)\}$

dom(r) = {a, b} = E. r COULD BE MANY OTHER THINGS.

(32, N, b) $|$ IS READ "IS A FACTOR OF"
REFLEXIVE, NOT SYMMETRIC, TRANSITIVE, NOT AN EQUIVALENCE RELATION. "IS A FACTOR OF" IS NOT ANTISYMMETRIC SO NOT A PARTIAL ORDER

$-1|1$ AND $1|-1$

(32, N, d) REFLEXIVE, SYMMETRIC, TRANSITIVE, EQUIVALENCE RELATION. r IS NOT ANTISYMMETRIC SO r IS NOT A PARTIAL ORDER

(32, N, f) NOT REFLEXIVE, $(3,3) \notin r$.
SYMMETRIC, NOT TRANSITIVE $(3,5), (5,3) \in r$ BUT $(3,3) \notin r$. NOT AN EQUIVALENCE RELATION
NOT ANTISYMMETRIC $(5,7) \in r$ AND $5 \neq 7$ BUT $(7,5) \in r$. SO NOT A PARTIAL ORDER

(32, S, 1, a) $[3] = \{3, 5, 6\}$
$P = \{\{1,2,4\}, \{3,5,6\}\}$

(32, S, 2, b) $r = \{(1,1), (1,5), (5,1), (5,5), (2,2), (3,3), (2,3), (3,2), (4,4), (6,6), (4,6), (6,4)\}$

(33, G, 2) (SHOW $\forall w, v \in (-\infty, -1)$, IF $f(w) = f(v)$, THEN $w = v$)

1. ASSUME $w, v \in (-\infty, -1)$ AND $f(w) = f(v)$ (SHOW $w = v$)
2. $-2w^2 + 3 = -2v^2 + 3$ — 1, DEF. OF f
3. $-2w^2 = -2v^2$ — 2, SUBTRACT 3
4. $w^2 = v^2$ — 3, DIVIDE BY 2
5. $|w| = |v|$ — 4, TAKE SQ. ROOT BOTH SIDES
6. $w < -1 < 0$ AND $v < -1 < 0$
7. $-w = -v$ — 6, 5, DEF. ABS. VALUE
8. $w = v$ — 7

(33,H,1) SHOW $\forall y \in (-2,\infty), \exists x \in (-\infty,0) \ni f(x) = y$

1. Assume $y \in (-2,\infty)$ (SHOW $\exists x \in (-\infty,0) \ni f(x) = y$)
2. $y > -2$ 1
3. $y+2>0$ 2
4. $\frac{y+2}{3} > 0$ 3
5. $\sqrt{\frac{y+2}{3}} > 0$ 4
6. $-\sqrt{\frac{y+2}{3}} < 0$ 5
7. Let $x = -\sqrt{\frac{y+2}{3}}$
8. $x \in (-\infty, 0)$ 6, 7
9. $f(x) = 3x^2 - 2 \overset{7}{=} 3\left(-\sqrt{\frac{y+2}{3}}\right)^2 - 2 = 3\left(\frac{y+2}{3}\right) - 2$
   $= y+2-2 = y$
10. $\exists x \in (-\infty,0) \ni f(x) = y$ 8, 9

(33, M, 2) LET $f(1) = 1$, $f(2) = 4$, $f(3) = 6$,
$6 = f(4) = f(5) = f(6) = f(7) = f(8) = \cdots$
$\text{dom}(f) = \{1,2,3,\ldots\} = N$ $\text{ran}(f) = \{1,4,6\}$

(33, P, 1) $P(A) = \{\phi, \{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

(33, P, 3) $P(A) = \{\phi, \{1\}\} = A_1$ $P(A_1) = \{\phi, \{\phi, \{1\}\},$
$\{\phi\}, \{\{1\}\}\}$, $\#A = 1$, $\# A_1 = 2$, $\# P(A_1) = 4$

(33, R, 2) ω^2

(33, R, 4) $\#P(A)$

(34, E, 2)

```
TASK FIVEToThe(n)
IF n = 0 THEN
   RETURN (1)
ELSE
   RETURN (5 · FIVEToThe (n-1))
```

(34, I, 1) FIND(2) = 5   FIND(7) = 17

(34, I, 3) START OF THE ANSWER

```
TASK POWER(m, p)
  IF p = 0 THEN
     RETURN (1)
  ELSE ...
```

(34, K, 1) START OF THE ANSWER

```
TASK IndxOfMax(A, n)
  IF n = 1 THEN
    RETURN (1)
  ELSE
    IF A(n) > A(IndxOfMax(A, n-1)) THEN
      RETURN (n)
    ELSE
      ...
```

(35, J, 2) $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$ (35, J, 4) $1 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

(35, J, 5) $25 \cdot 25 \cdot 25 \cdot 25 \cdot 25$ (35, J, 8) $365 \cdot 364 \cdot 363 \cdot 362 \cdot 361$

(35, N, 1) $P(5,5) = \frac{5!}{(5-5)!} = 5!$ ← (35, N, 4)

(35, N, 5) $P(26,6) = \frac{26!}{(26-6)!}$ NO REPITITIONS

26^6 WITH REPITITIONS

(35, N, 7) $P(8,5)$

(35, w, 1, b) $C(8+6,4) - C(8,4)$

(35, w, 1, d) $C(6,3) \cdot 8 + C(6,4)$

(35, w, 2, b) $10 \cdot 27$

(35, w, 2, d) $C(3,2) \cdot [C(20,4) - 2C(10,4)]$

(35, w, 3, a) $C(8,5)$

(36, E, 1) MAKE A CHART FOR ALL POSSIBLE CASES AND SHOW TRUE FOR EACH CASE. HAVE COLUMNS FOR $x, y, z, y \cdot z, x+(y \cdot z)$, $x+y$, $x+z$, AND $(x+y) \cdot (x+z)$

(36, I, 1, a) $\forall x \in B,\ x \cdot x = x$

(36, I, 1, c) $\forall x \in B,\ x \cdot x' = 0$

(36, I, 2, a) PROVE $\forall x \in B,\ x + x = x$

1. ASSUME $x \in B$. (SHOW $x+x=x$)
2. $x = x + 0$ IDENTITY, 5 DEF. BOOL. ALG.
3. $= x + (x \cdot x')$ 7 DEF. BOOL. ALG.
4. $= (x+x) \cdot (x+x')$ DISTRIBUTIVITY 4 DEF. BOOL. ALG.
5. $= (x+x) \cdot 1$ 7 DEF. BOOL. ALG.
6. $= x+x$ IDENTITY, 6 DEF. BOOL. ALG.

(36, J, 9, a) AN EXPRESSION D INVOLVING p AND q IS IN DISJUNCTIVE NORMAL FORM IFF $\exists$ A POSITIVE INTEGER K, $\exists$ MINTERMS $m_1, m_2, \ldots, m_K$ IN p AND q SUCH THAT $D = m_1 \vee m_2 \vee m_3 \vee \cdots \vee m_K$. (A MINTERM IN p AND q IS AN EXPRESSION OF THE FORM $x_1 \wedge x_2$ WHERE x_1 IS EITHER p OR $\sim p$ AND x_2 IS EITHER q OR $\sim q$.)

(36, J, 9, c) $(p \wedge q \wedge r) \vee (p \wedge q \wedge \overline{r}) \vee (\overline{p} \wedge q \wedge r)$
$\vee (\overline{p} \wedge q \wedge \overline{r}) \vee (p \wedge \overline{q} \wedge r)$

(36, J, 9, e) $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r)$
$\vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

(36, K, 8, a) A MAXTERM IN $P_1, P_2, P_3, \ldots, P_n$ IS AN EXPRESSION OF THE FORM

$$x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_n$$

WHERE, FOR EACH $i \in \{1, 2, \ldots, n\}$ x_i IS EITHER P_i OR $\sim P_i$.

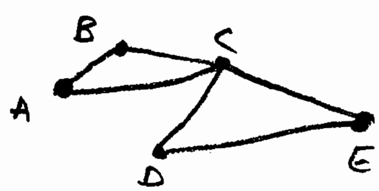
AN EXPRESSION D INVOLVING $P_1, P_2, \ldots, P_n$ IS IN CONJUNCTIVE NORMAL FORM IFF $\exists$ A POSITIVE INTEGER K, $\exists$ MAXTERMS $M_1, M_2, \ldots, M_K$ IN $P_1, P_2, P_3, \ldots, P_n$ SUCH THAT

$$D = M_1 \wedge M_2 \wedge M_3 \wedge \cdots \wedge M_K$$

(36, K, 8, b) $(\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee \sim r) \wedge (p \vee \sim q \vee \sim r)$

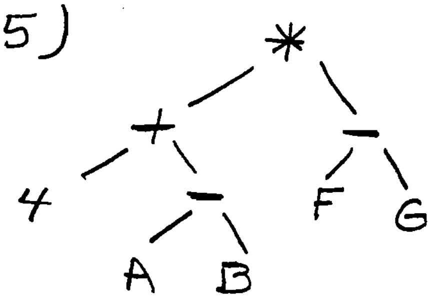
(37, T, 1)

(37, T, 3)



PATH
A − B − C − A − B − C − D − E − C − A
EDGE A, C REPEATED
(SINCE THERE IS NO CONFUSION OVER WHICH EDGES ARE TAKEN THEY ARE NOT NAMED)

(37, T, 5)

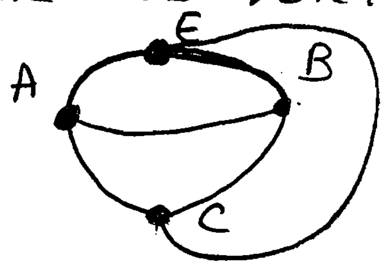


4 A B − + F G − *

(38, C, 1, G_2) YES A, e, B, l, C, g, D, h, A, i, D, k, B, f, C, j, A

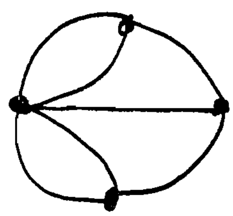
(38, C, 2) FOR AN EULER PATH WITH 2 ODD VERTICES, YOU CAN START AT EITHER OF THE ODD VERTICES AND YOUR EULER PATH MUST END AT THE OTHER ODD VERTEX.

(38, C, 3, H_2)



NO EULER PATH

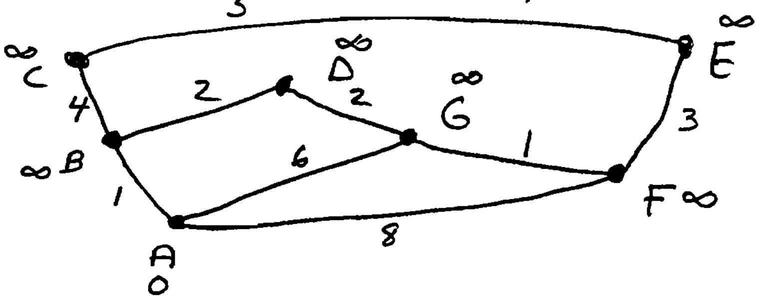
(38, C, 4)



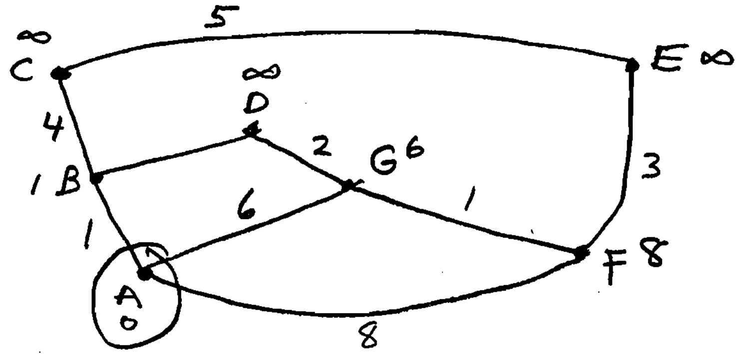
NO EULER PATH

(38, F, 1, G_6) A, j, E, k, B, g, C, h, D, i, A

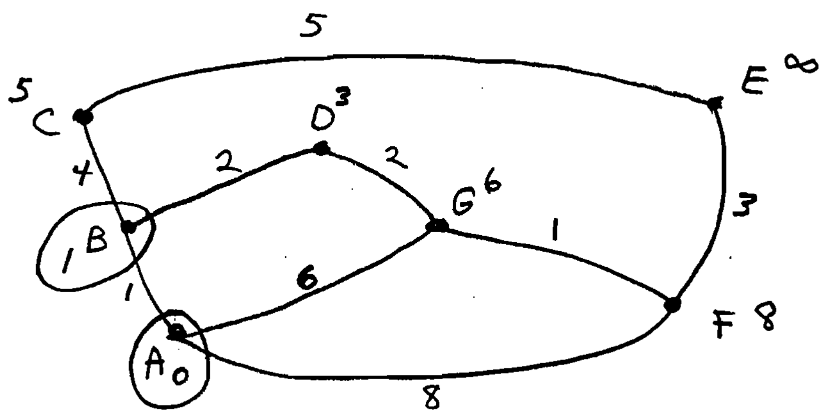
(38, J, 1)



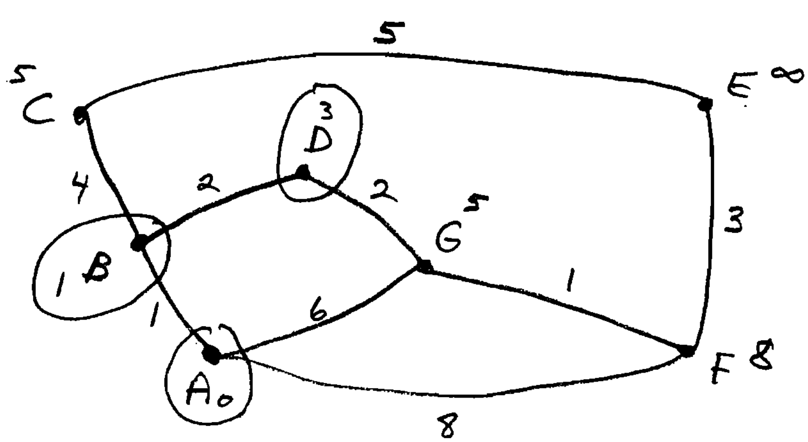
∞ C
5
D ∞
E ∞
4
2
2
G ∞
3
∞ B
6
1
1
F ∞
8
A
0



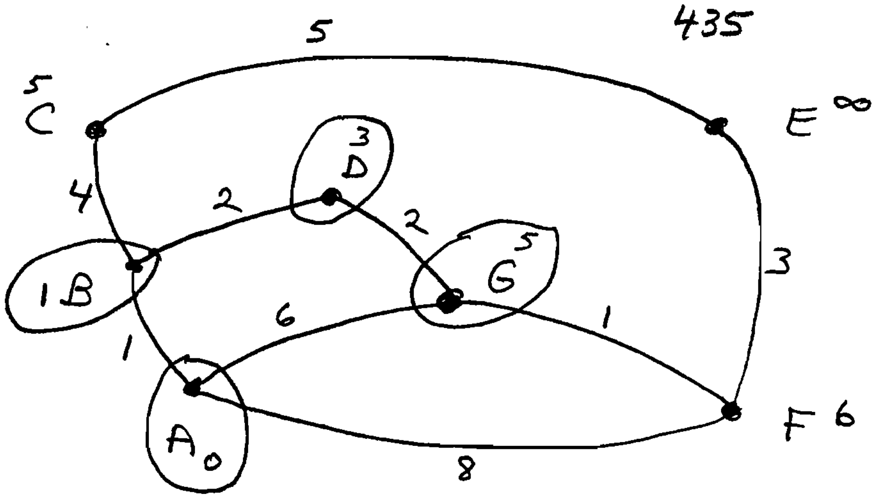
∞ C
5
E ∞
D ∞
4
2
G 6
3
1 B
6
1
1
F 8
A
0
8

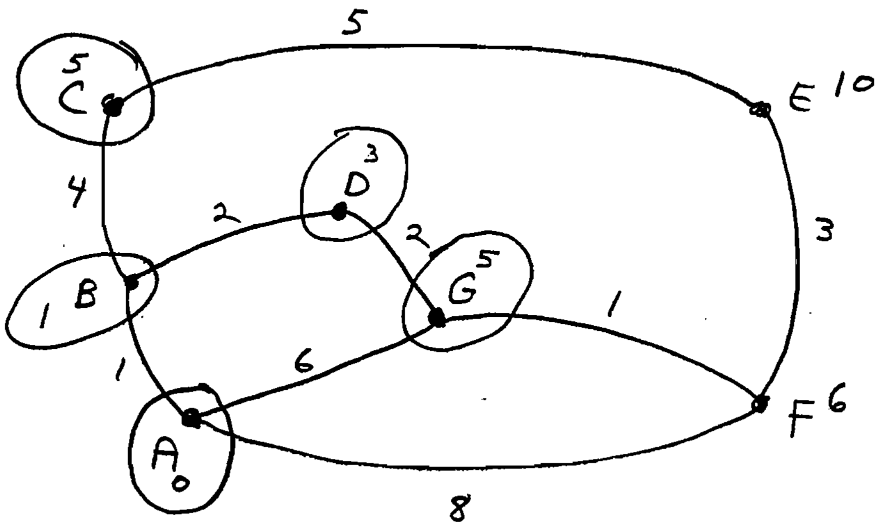


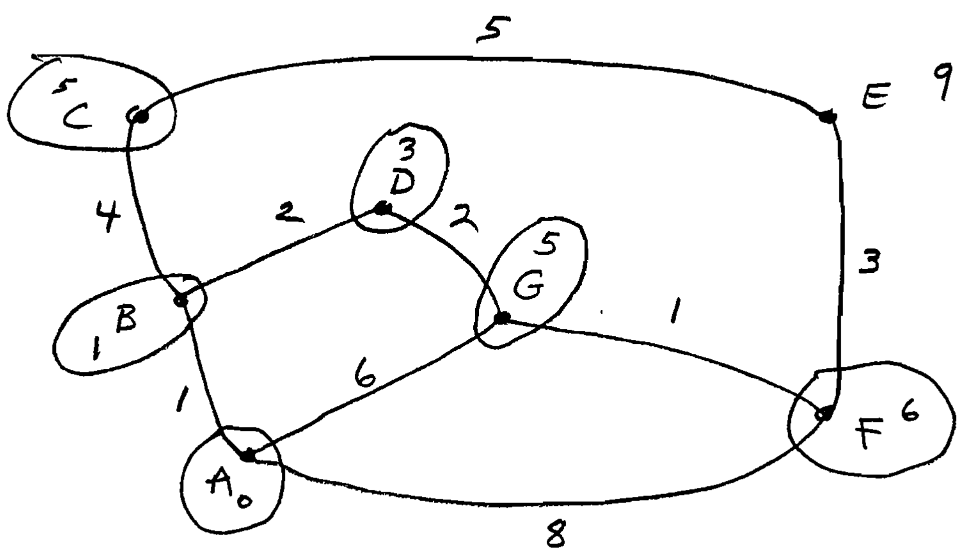
5
5 C
E ∞
D 3
4
2
2
G 6
1 B
1
3
6
1
F 8
A 0
8



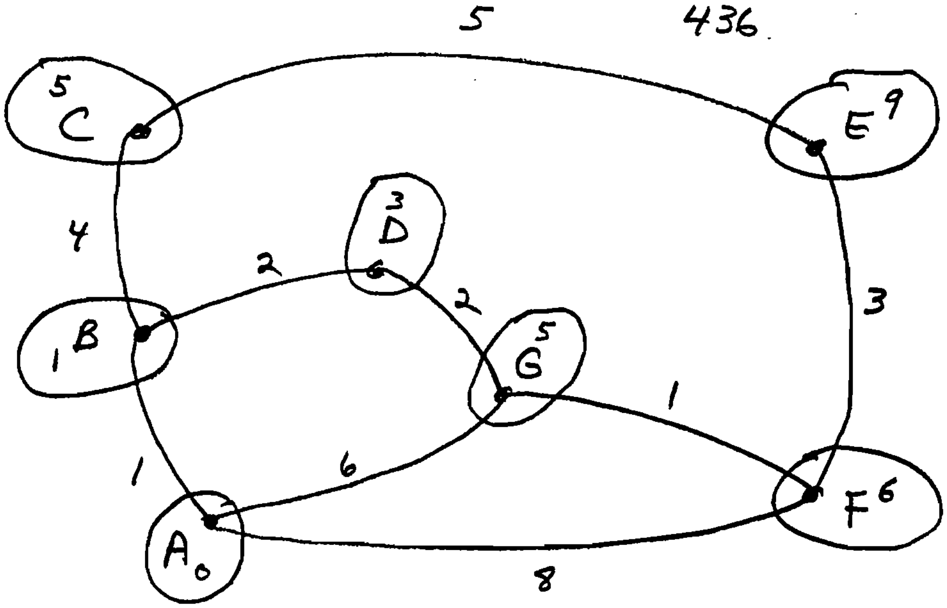
5
5 C
E ∞
3
D
4
2
2
G 5
3
1 B
1
1
6
F 8
A 0
8


5
5 C
E ∞
3 D
4
2
2
1 B
5 G
3
6
1
1
A 0
F 6
8


5
5 C
E 10
3 D
4
2
2
1 B
G 5
3
1
6
1
A 0
F 6
8

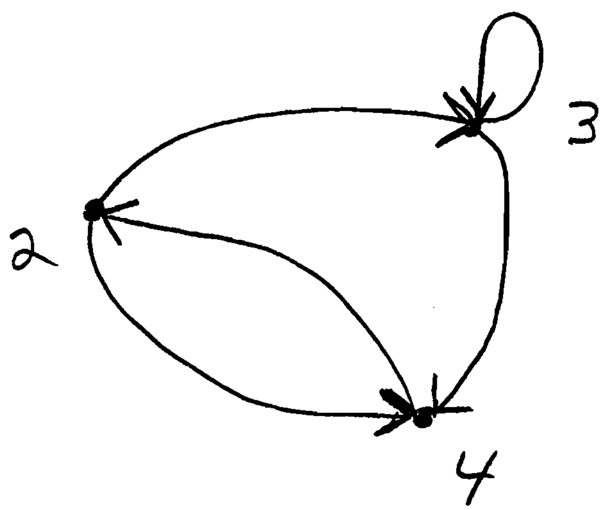

5
5 C
E 9
3 D
4
2
2
5 G
3
1 B
1
6
1
F 6
A 0
8

436.



PATH A - B - D - G - F - E

(38, m, 1)



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B

BASIC KNOWLEDGE QUESTIONS

(WITH ANSWERS)

BASIC KNOWLEDGE QUESTIONS WITH ANSWERS

CHAPTER 1

1. WHAT IS ANOTHER NAME FOR A STATEMENT?
   PROPOSITION
2. GIVE THE DEFINITION OF A STATEMENT.
   THE EXPRESSION IS 1) A SENTENCE, 2) A DECLARATIVE SENTENCE, AND 3) IT IS MEANINGFUL TO ASSIGN ONLY ONE OF THE VALUES TRUE OR FALSE.

CHAPTER 2

3. IS A STATEMENT PATTERN A STATEMENT? NO
4. IS AN INSTANCE OF A STATEMENT PATTERN A STATEMENT? YES
5. NAME EACH TYPE OF STATEMENT PATTERN BELOW:

$(\sim p) \rightarrow q$	IMPLICATION (OR CONDITIONAL)
$\sim (p \rightarrow q)$	NEGATION
$p \vee (\sim q)$	DISJUNCTION

6. WHAT IS A STATEMENT VARIABLE? A SYMBOL THAT CAN BE REPLACED BY ANY STATEMENT.
7. NAME THE ANTECEDENT OF $(p \vee q) \rightarrow (\sim w)$
   $p \vee q$
8. WHAT IS ANOTHER NAME FOR AN IMPLICATION?
   CONDITIONAL.
9. WHAT IS ANOTHER NAME FOR CONCLUSION?
   CONSEQUENT

BASIC KNOWLEDGE QUESTIONS CONTINUED

CHAPTER 3

10. TRUE OR FALSE : $1+1=2000$ IMPLIES $2<3$ (TRUE)

11. TRUE OR FALSE : $1<2$ OR $3<4$ (TRUE)

CHAPTER 4

12. $[p \vee (\sim q)] \rightarrow (\sim w)$

1	2	3	4	5	6	7

TO FILL IN COLUMN 5, LOOK IN COLUMNS __ AND __

2 AND 6

CHAPTER 5

13. WHAT IS A TAUTOLOGY? A STATEMENT PATTERN WHOSE EVERY INSTANCE IS TRUE.

14. GIVE THE DEFINITION OF $p \leftrightarrow q$.

$(p \rightarrow q) \wedge (q \rightarrow p)$

CHAPTER 6

15. AN IMPLICATION IS EQUIVALENT TO ITS:

CONVERSE INVERSE CONTRAPOSITIVE

16. THE INVERSE OF $(\sim A) \rightarrow B$ IS __________.

$A \rightarrow (\sim B)$

17. WHAT IS EQUIVALENT TO

	ANSWERS
$\sim[(\sim A) \vee B]$	$A \wedge (\sim B)$
$\sim[(\sim A) \rightarrow B]$	$(\sim A) \wedge (\sim B)$ OR $\sim(A \vee B)$
$(\sim A) \rightarrow B$	$(\sim B) \rightarrow A$ OR $A \vee B$

BASIC KNOWLEDGE QUESTIONS CONTINUED

18. NEGATE: $E \vee (\sim F)$ $\quad$ $(\sim E) \wedge F$

19. NEGATE: $T \to (\sim E)$ $\quad$ $T \wedge E$

CHAPTER 7

20. WRITE IN IF..., THEN... FORM
$1<2$ IS SUFFICIENT FOR $3<4$

IF $1<2$, THEN $3<4$

CHAPTER 8

21. WRITE IN EQUIVALENT FORM BY DISTRIBUTIVITY: $p \vee (q \wedge r)$

$(p \vee q) \wedge (p \vee r)$

CHAPTER 9

22. WHAT DO YOU MAKE A TRUTH TABLE OF TO SEE IF

$$\begin{array}{l} \sim p \vee q \\ \underline{\sim q} \\ \therefore p \end{array}$$

IS A VALID ARGUMENT PATTERN?

$[(\sim p \vee q) \wedge (\sim q)] \to p$

23. STATE THE MODUS PONENS ARGUMENT PATTERN IN TERMS OF THE VARIABLES p AND q.

$$\begin{array}{l} p \to q \\ \underline{p} \\ \therefore q \end{array}$$

BASIC KNOWLEDGE QUESTIONS CONTINUED

24. STATE THE MODUS TOLLENS ARGUMENT PATTERN IN TERMS OF THE VARIABLES p AND q.

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

CHAPTER 10

25. HOW DO YOU START A DIRECT PROOF OF

$$\begin{array}{l} p \to \sim q \\ \sim r \to q \\ p \\ \hline \therefore r \end{array}$$

0. ASSUME p', q', r' ARE STATEMENTS THAT REPLACE p, q, r RESPECTIVELY

$$\left.\begin{array}{ll} 1. & p' \to \sim q' \\ 2. & \sim r' \to q' \\ 3 & p' \end{array}\right\} \text{ASSUMED TRUE FOR DIRECT PROOF (SHOW } r' \text{ TRUE)}$$

CHAPTER 11

26. HOW DO YOU START AN INDIRECT PROOF OF

$$\begin{array}{l} p \to \sim q \\ \sim r \to q \\ p \\ \hline \therefore r \end{array}$$

ANSWER ON NEXT PAGE

BASIC KNOWLEDGE QUESTIONS CONTINUED

0. ASSUME p', q', r' ARE STATEMENTS THAT REPLACE p, q, r RESPECTIVELY

1. $p' \to \sim q'$
2. $\sim r' \to q'$
3. p'
4. $\sim r'$

ASSUMED TRUE FOR INDIRECT PROOF.

(GET ANY CONTRADICTION)

CHAPTER 12

27. GIVE AN EXAMPLE OF AN INSTANCE OF THE OPEN SENTENCE "$x+1 < 10$".

$7+1<10$

CHAPTER 13

28. HOW IS $\forall x \in I, x < 7$ READ?

FOR EVERY x IN I, x IS LESS THAN 7

29. HOW IS $\forall x \in I, x < 7$ DEFINED?

EVERY REPLACEMENT FOR x FROM I MAKES A TRUE INSTANCE OF "$x<7$".

30. HOW IS $\exists x \in I \ni x > 4$ READ?

THERE IS AN x IN I SUCH THAT $x > 4$

31. HOW IS $\exists x \in I \ni x > 4$ DEFINED?

THERE IS A REPLACEMENT FOR x FROM I THAT MAKES A TRUE INSTANCE OF "$x>4$".

Q6

CHAPTER 14

32. T OR F $\forall x \in R, \exists y \in R \ni xy = 1$ **FALSE**
DOES NOT WORK WHEN 0 REPLACES x

33. T OR F $\forall x \in R, \exists y \in R \ni x + y = 1$ **TRUE**
WHATEVER x IS REPLACED WITH, REPLACE y WITH THE NEGATIVE OF THAT VALUE PLUS ONE. (i.e. REPLACE x WITH 5, y WITH $-5+1$)

CHAPTER 15

P = SET OF ALL PEOPLE M = SET OF ALL MATH COURSES

34. TRANSLATE INTO A PROPERLY QUANTIFIED OPEN SENTENCE WITH NO HIDDEN QUANTIFIERS
a). THERE IS SOME MATH COURSE EVERYONE LIKES
ANSWER: $\exists x \in M \ni \forall y \in P$, y LIKES x
b). EVERYBODY LIKES SOME MATH COURSE
ANSWER: $\forall x \in P, \exists y \in M \ni$ x LIKES y

CHAPTER 16

35. NEGATE: $\forall x \in R$, (IF $x < 5$, THEN $x \in P$)
ANSWER: $\exists x \in R \ni (x < 5$ AND $x \notin P)$

36. NEGATE: $\forall x \in R, \exists y \in R \ni xy = 1$
ANSWER $\exists x \in R \ni \forall y \in R, xy \neq 1$

CHAPTER 17

37. HOW WERE YOU TAUGHT TO START A PROOF OF "IF $p \in Q$, THEN $p < 7$."
ANSWER: A, 1. ASSUME $p \in Q$. (SHOW $p < 7$)

Q 7

CHAPTER 18

38. HOW WERE YOU TAUGHT TO START A DIRECT PROOF OF "EVERY ELEMENT OF E IS AN ELEMENT OF F"

ANSWER: A, 1. ASSUME $p \in E$ (SHOW $p \in F$)

CHAPTER 19

39. HOW WERE YOU TAUGHT TO START A DIRECT PROOF OF "$\forall x \in Q, x > 3$"

ANSWER: A, 1. ASSUME $x' \in Q$ (SHOW $x' > 3$)

---

40. a) $x' \in A$ GIVEN

b) $\forall x \in A, (x \notin T \vee x \in M)$ GIVEN

NAME AN INSTANCE OF b)

ANSWER: $x' \notin T \vee x' \in M$

41. a) $x' \in A$ GIVEN

b) $\forall x \in A, \forall y \in B, x + y \in T$ GIVEN

NAME AN INSTANCE OF b)

ANSWER: $\forall y \in B, x' + y \in T$

42. ACCORDING TO WHAT YOU WERE TAUGHT, WHAT IS THE FIRST ASSUME AND SHOW TO PROVE

$$\forall x \in T, \forall y \in M, (x + y < 7 \rightarrow xy \in B)$$

ANSWER:

A, 1. ASSUME $x' \in T$

(SHOW $\forall y \in M, x' + y < 7 \rightarrow x'y \in B$)

Q 8

CHAPTER 20

43. WHAT ARE THE TWO MAIN THINGS THAT NEED TO BE PROVEN TO PROVE $p \in A \leftrightarrow p < 7$

ANSWER: 1. $p \in A \rightarrow p < 7$

2. $p < 7 \rightarrow p \in A$

CHAPTER 21

44. GIVE THE DEFINITION OF $H \subseteq K$

ANSWER: $\forall x \in H, x \in K$

CHAPTER 22

45. DEFINITION OF $A \cap B$. ANSWER: $\{x \mid x \in A \text{ AND } x \in B\}$

46. BY DEFINITION, WHAT IS THE NEXT LINE OF THE PROOF AFTER

a) $x' \in A \cap (B \cup C)$ GIVEN

ANSWER: $x' \in A$ AND $x' \in B \cup C$ a), DEF. $\cap$

47. BY DEFINITION, WHAT IS THE NEXT LINE OF THE PROOF AFTER

a) $x' \in P \cup (Q \cap W)$ GIVEN

ANSWER: $x' \in P$ OR $x' \in Q \cap W$ a), DEF. $\cup$

CHAPTER 23

48. BY DEFINITION, WHAT IS THE NEXT LINE OF THE PROOF AFTER

a) $x' \in H - (A \cup B)$ Given

ANSWER: $x' \in H$ AND $x' \notin (A \cup B)$

CHAPTER 24

49. TO PROVE $H \cap K = A \cup B$ BY THE AXIOM OF EXTENT, PROVE ________________.

ANSWER. $H \cap K \subseteq A \cup B$ AND $A \cup B \subseteq H \cap K$

CHAPTER 25

50. SUPPOSE YOU HAVE

a. $7 \varepsilon B$ OR $7 \varepsilon D$ GIVEN

b. $7 \varepsilon B \rightarrow 7 \varepsilon E \cup W$ GIVEN

c. $7 \varepsilon D \rightarrow 7 \varepsilon E \cup W$ GIVEN

WHAT CAN BE CONCLUDED BY PROOF BY CASES ?

ANSWER: $7 \varepsilon E \cup W$

CHAPTER 26

51. SUPPOSE YOU HAVE THESE LINES IN A PROOF:

a. $7 \varepsilon P \cap Q$ GIVEN

b. $8 \varepsilon A \cap B$ GIVEN

c. $7 \varepsilon H - K$ GIVEN

NAME A $\exists$ STATEMENT THAT THESE LINES PROVE.

ANSWER: $\exists x \varepsilon P \cap Q \ni x \varepsilon H-K$ OR $\exists x \varepsilon H-K \ni x \varepsilon P \cap Q$

52. SUPPOSE YOU HAVE THESE LINES IN A PROOF:

a. $x' \varepsilon A$ GIVEN

b. $y' \varepsilon B$ GIVEN

c. $x' + y' = 10$ GIVEN

NAME A $\exists$ STATEMENT THAT THESE LINES PROVE.

ANSWER: $\exists x \varepsilon A \ni x + y' = 10$ OR $\exists y \varepsilon B \ni x' + y = 10$

Q 10

CHAPTER 27

53. NAME A NEXT STATEMENT IN A PROOF, THAT FOLLOWS FROM: $\exists x \in A \ni x \in B$

<u>ANSWER</u>: LET $x' \in A \ni x' \in B$
(ALSO ACCEPTABLE → LET $x' \in A$)

54. NAME A NEXT STATEMENT IN A PROOF, THAT FOLLOWS FROM: $\exists x \in A, \exists y \in B \ni x+3y>7$

<u>ANSWER</u>: LET $x' \in A \ni \exists y \in B \ni x'+3y>7$

CHAPTER 28

55. GIVE THE DEFINITION OF $H \cup K \subset A-B$.

<u>ANSWER</u>: $H \cup K \subseteq A-B$ AND $\exists x \in A-B \ni x \notin H \cup K$

CHAPTER 29

56. $\{2x-3 \mid x \in [0,1]\} = \{\alpha \mid \underline{\qquad\qquad\qquad}\}$

<u>ANSWER</u>: $\{2x-3 \mid x \in [0,1]\} = \{\alpha \mid \underline{\exists x \in [0,1] \ni \alpha = 2x-3}\}$

CHAPTER 30

57. WHAT ARE THE STEPS TO PROVE:

$\forall$ positive integer n, $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION

a. PROVE $1 = \dfrac{1(1+1)}{2}$

b. ASSUME n IS A POSITIVE INTEGER AND $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

(Show $1+2+3+\cdots+n+1 = \dfrac{(n+1)((n+1)+1)}{2}$)

301 QUESTIONS

(CHAPTER 31)

58. NAME FOR BASE 16 NUMERATION SYSTEM?

59. NAME FOR BASE 2 NUMERATION SYSTEM?

60 213.42_5 — THIS IS CALLED THE ____ ____

61 NEXT INTEGER AFTER $9FF_{16}$

$$\begin{array}{r} 13_5 \\ -\ 4_5 \\ \hline \end{array}$$

62. $4_5 \cdot 3_5 = \_\_\_\__5$

(CHAPTER 32)

63. SUPPOSE r IS A RELATION ON SET B. GIVE THE DEFINITION OF

a. r IS REFLEXIVE ON B.
b. r IS SYMMETRIC ON B.
c. r IS TRANSITIVE ON B.
d. r IS ANTI-SYMMETRIC ON B.
e. r IS AN EQUIVALENCE RELATION ON B.
f. r IS A PARTIAL ORDER ON B

64 NAME A PARTITION OF $E = \{1, 2, 3\}$

(CHAPTER 33)

65 DEFINITION OF A FUNCTION

66 FOR $f : H \to W$

a. HOW READ
b. HOW DEFINED
c. DEF f IS 1-1
d. DEF, THM f IS ONTO W

301 QUESTIONS

67 NAME AN INFINITE COUNTABLE SET. $\{1,2,3,\ldots\} = N$

68 NAME AN INFINITE UNCOUNTABLE SET. Reals

69 WHAT IS THE FIRST INFINITE ORDINAL NUMBER? ω

70 WHAT IS THE FIRST UNCOUNTABLE ORDINAL NUMBER? Ω

71 NAME A SET WHOSE CARDINALITY IS GREATER THAN THE CARDINALITY OF THE SET OF REALS. $P(R)$

CHAPTER 34

72 FOR THE TASK BELOW WHAT IS WHAT(7)

```
TASK WHAT(n)
  IF n<5 THEN
    RETURN(8)
  ELSE
    RETURN (4 + WHAT(n-3))
```

CHAPTER 35

73 SAM HAS 5 SHIRTS AND 4 PANT. HOW MANY SHIRT-PANTS PAIRS ARE POSSIBLE.

74. FORMULA FOR $P(5,2)$ (i.e. ${}_5P_2$)? HOW READ?

75. FORMULA FOR $C(5,2)$ (i.e. ${}_5C_2$, $\binom{5}{2}$)? HOW READ

CHAPTER 36

76. FOR BOOLEAN ALGEBRA $\mathcal{B}$

a. $\forall x,y,z \in B$, $x+(y\cdot z) =$ ___

b. $\forall x \in B$, $x+x' =$ ___

c. $\forall x \in B$, $x \cdot x' =$ ___

d. $\forall x \in B$, $x+1 =$ ___

77. Give the dual of $\forall x,y \in B$, $x\cdot(x+y) = x+0$

301 QUESTIONS

78. GIVE AN EXAMPLE OF A MINTERM IN p, q, r

79. GIVE AN EXAMPLE OF AN EXPRESSION IN p, q IN DISJUNCTIVE NORMAL FORM. ________

80 AS TAUGHT IN CLASS, TO GET IN DISJUCTIVE NORMAL FORM IN p, q, r

$$(r \wedge p) \vee (r \wedge q) \equiv$$

$$((r \wedge p) \wedge (\_\_\_\_)) \vee ((r \wedge q) \wedge (\_\_\_\_))$$

80. GIVE AN EXAMPLE OF A MAXTERM IN p, q, r

81 GIVE AN EXAMPLE OF AN EXPRESSION IN p, q IN CONJUNCTIVE NORMAL FORM. ________

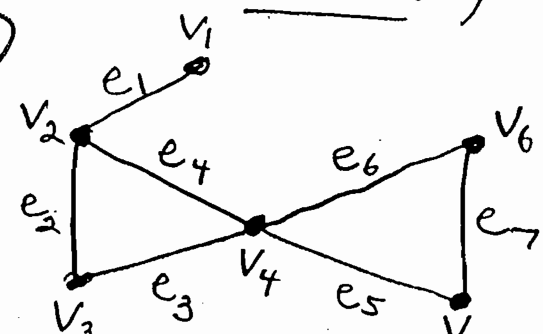
82 AS TAUGHT IN CLASS, TO GET IN CONJUNCTIVE NORMAL FORM IN p, q, r

$$(p \vee q) \wedge (\sim p \vee r) \equiv$$

$$((p \vee q) \vee (\_\_\_\_)) \wedge ((\sim p \vee r) \vee (\_\_\_\_))$$

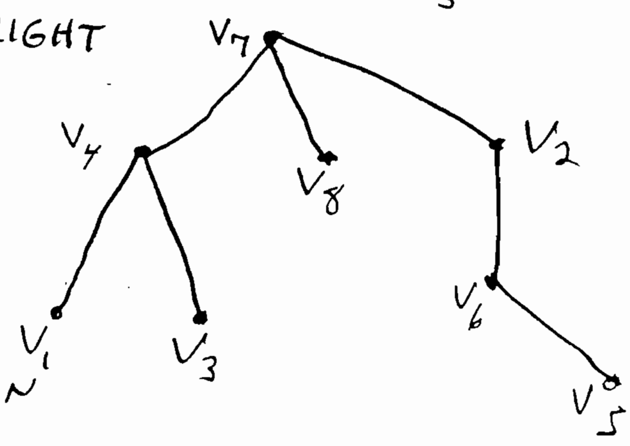
CHAPTER 37

83. FOR THE GRAPH AT THE RIGHT, NAME A CYCLE



84 FOR THE TREE AT THE RIGHT THE ROOT IS V_7.

1) NAME A SET OF SIBLINGS
2) NAME A CHILD OF V_2
3) NAME THE PARENT OF V_3



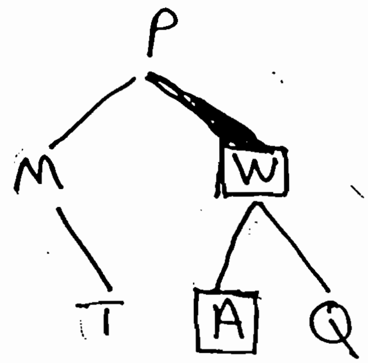
85 WHAT IS AN ACYCLIC GRAPH?

86 WHAT IS THE DEFINITION OF A TREE

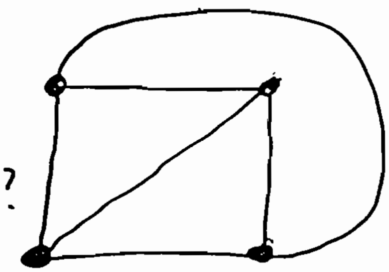
MAT 301 QUESTIONS

87. TRAVERSALS

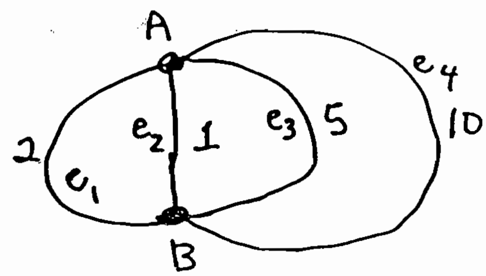
PREORDER
INORDER
POSTORDER



88. IS THERE AN EULER PATH FOR THE GRAPH AT THE RIGHT? WHY?



89. NAME THE HAMILTONIAN CYCLE FOR THE GRAPH AT THE RIGHT.



CHAPTER 38

90. AN EULER PATH FOR GRAPH G IS A PATH THAT ______________________________

91. FOR A CONNECTED GRAPH, AN EULER PATH HAS HOW MANY ODD VERTICES? ________

92. A HAMILTONIAN CYCLE FOR A GRAPH G IS A CYCLE THAT ____________________

TRUTH GEMS

THE ANSWER
IS IN
THE BACK OF
THE BOOK

TRUTH GEM

BE IN THE WILL OF GOD FOR WHAT YOU DO

A. ROM 15:32 ...that I may come to you with joy by the WILL OF GOD, and may be refreshed together with you.

B. I come to you in the WILL OF GOD with joy and we will have refreshing math.

C. Being in the WILL OF GOD taking this course and faithfully, wisely studying you will flourish.

D. HEB. 10:36 For you have need of endurance, so that after you have done the WILL OF GOD you may receive the promise.

TG-2

TRUTH GEM

WISDOM

A. PR 1:7 WISDOM IS THE PRINCIPAL THING; THEREFORE GET WISDOM. AND IN ALL YOUR GETTING, GET UNDERSTANDING.

B. <u>DEFINITIONS</u>:

1. <u>KNOWLEDGE</u>: FACTS, GAINED INFORMATION
2. <u>UNDERSTANDING</u>: WHY A FACT IS A FACT.
3. <u>WISDOM</u>: BEING LED BY THE SPIRIT. KNOWING WHAT TO DO AT ANY MOMENT.

C. PRAY FOR WISDOM <u>IN</u> <u>FAITH</u>: JAMES 1:5
IF ANY OF YOU LACKS WISDOM, LET HIM ASK OF GOD, WHO GIVES TO ALL LIBERALLY AND WITHOUT REPROACH, AND IT WILL BE GIVEN HIM.

TRUTH GEM

BEGIN

A. ACTS 1:1 "... of all that Jesus <u>BEGAN</u> both to do and teach."

B. MK 4:1 "And again He <u>BEGAN</u> to teach by the sea."

C. For a task to be accomplished, you must <u>BEGIN</u>.

D. <u>BEGINNINGS</u>:

1. <u>BEGIN</u> to s<u>ee</u> yourself as a faithful, good math student
2. <u>See</u> changes you need to make, and <u>BEGIN</u> on those changes.
3. <u>See</u> and learn <u>BEGINNINGS</u> of different problem types.

TRUTH GEM

PRESS ON

A. PLP 3:12 NOT THAT I HAVE ALREADY ATTAINED... BUT I <u>PRESS</u> <u>ON</u>, THAT I MAY LAY HOLD OF THAT FOR WHICH CHRIST JESUS HAS ALSO LAID HOLD OF ME.

B. LIKE A DISTANCE RUNNER OR CRUISE CONTROL

1. CONSTANCY, STEADFASTNESS
2. PATIENCE POWER

} GOOD WORDS

C. <u>PRESS</u> <u>ON</u> CYCLE DONE WITH A GOOD ATTITUDE (NOT AT LAST MINUTE)

→ HEAR THE NEW →

STUDY →

DO HOMEWORK →

REVIEW →

D. HEB 6:12 ... THAT YOU DO NOT BECOME SLUGGISH (LAZY), BUT IMITATE THOSE WHO THROUGH FAITH AND PATIENCE INHERIT THE PROMISES.

TRUTH GEM

COMPLETE IT

A. PLP 1:6 "... being confident of this very thing, that He who has BEGUN a good work in you will COMPLETE IT until the day of Jesus Christ.

B. THINGS GOD BEGAN, IN HIS WILL, GOD GIVES ABILITY AND PROVISION TO COMPLETE

1. SEE THESE THINGS THROUGH TO THE END.
2. VICTORY IS SWEET

C. THERE CAN BE BARRIERS TO BREAK THROUGH AT THE END.

1. LIKE A TAPE AT THE END OF A RACE.
2. LIKE THE SOUND BARRIER.

TRUTH GEM

BE ESTABLISHED

A. PS 90:17 And let the beauty of the Lord our God be upon us, and ESTABLISH the work of our hands for us;

B. Be established in the BEGIN - PRESS ON - COMPLETE IT cycle for working problems

C. Be established in knowing how to work certain problem types

D. Hebrew: Established = koon : things brought into incontrovertible existence like:

1. Your nature to faithfully study
2. Your ability to work certain problem types

E. To learn better how to be ESTABLISHED

1. Is 54:14a In righteousness you shall be ESTABLISHED
2. We will learn of righteousness

TRUTH GEM

RIGHTEOUSNESS

A. HEB 9:28a For He will finish the work and cut it short in RIGHTEOUSNESS;

B. RIGHTEOUSNESS = RIGHT STANDING WITH GOD BY FAITH.

PLP 3:9 and be found in Him, not having my own righteousness, which is from the law, but that which is through faith in Christ, the righteousness which is from God by faith.

C. BEING IN THE WILL OF GOD FOR WHAT YOU DO, IN RIGHT STANDING WITH GOD, THERE IS GREAT LIBERTY AND SPEEDUP IN WHAT YOU DO. (RIGHTEOUSNESS ENHANCED ACCELERATED LEARNING)

TG-8

TRUTH GEM

RIGHTEOUSNESS - SHALOM

A. IS 32:17 THE WORK OF RIGHTEOUSNESS (RIGHT STANDING) WILL BE PEACE (SHALOM: NOTHING MISSING, NOTHING BROKEN, COMPLETENESS, HEALTH, PROSPERITY, SAFETY, PEACE)

B. THE SHALOM LEARNING ENVIRONMENT: TEACHER & STUDENT AT PEACE AS THEY TEACH, STUDY, TAKE TESTS, & GRADE PAPERS.

C. "LET RIGHTEOUSNESS WORK FOR YOU" (ED TAYLOR)

D. SPEEDUP BY LEARNING IN A STATE OF SHALOM

TG-9

TRUTH GEM

RIGHTEOUSNESS - BOLD

A. PR. 28:1 THE WICKED FLEE WHEN NO ONE PURSUES, BUT THE RIGHTEOUS ARE AS BOLD AS A LION

B. PR. 30:30 A LION, WHICH IS MIGHTY AMONG BEASTS AND DOES NOT TURN AWAY FROM ANY;

C. LEARNING IS SLOWED DOWN BY WIMPILY, TIMIDLY TURNING AWAY FROM SOME MATH PROBLEMS.

D. LEARNING SPEEDUP BY BOLDLY (IN ACCORDANCE WITH YOUR RIGHTEOUS NATURE) TAKING ON THINGS ON YOUR PATH AND OVERCOMING.

E. NEXT, BOLDNESS AND HUMILITY, NOT CONTRADICTARY.

TRUTH GEM

RIGHTEOUSNESS: BOLD & HUMBLE

A. PLP 2:8a He (Jesus) HUMBLED Himself and became obedient.

B. James 4:6-7a "... God resists the proud, but gives grace to the HUMBLE." Therefore submit to God.

C. HUMILITY: Go God's way not your own way.

D. Some problems it takes boldness to solve. A false humility doctrine can cause someone to wimp out and be defeated.

E. There is a difference between boldness and aggression

TRUTH GEM

FAITH – NO FEAR

A. MK 5:36b "DO NOT BE AFRAID, ONLY BELIEVE." — HAVE FAITH

B. FEAR IS HAVING MORE FAITH IN THE POWER OF THE DEVIL TO DO HARM THAN THE POWER OF GOD TO DO GOOD. Philip Derber

C. FAITH IS LIKE PUTTING IT IN DRIVE,
DOUBT IS LIKE PUTTING IT IN NEUTRAL,
FEAR IS LIKE PUTTING IT IN REVERSE
UNBELIEF IS LIKE PUTTING IT IN PARK.
John Paul

D. ROM 12:21 "DO NOT BE OVERCOME BY EVIL, BUT OVERCOME EVIL WITH GOOD." GOOD OVERCOMES EVIL SO DO NOT FEAR.

E. FEAR ATTRACTS THE FEARED THING. FAITH IS THE SUPERNATURAL CONNECTION THAT RECEIVES THE OVERCOMING GOOD – THE THING BELIEVED FOR ... MATH UNDERSTANDING

TG-12

TRUTH GEM

GRACE

A. I Pet 4:10-11 part As each one has received a GIFT, minister it to one another as good stewards of the manifold GRACE of God... If anyone ministers let him do it as with the ABILITY which God supplies...

B. DEFINITION: GRACE - God's ability gift to live and function in the gifts and callings.

C. Grace is part of God's supernatural provision to do excellently what God has called us to do

D. 2 Cor 9:8 And God is able to make all GRACE abound toward you, that you, always having all sufficiency in all things, may have an abundance for every good work.

TRUTH GEM

GRACE - WORKS EFFECTIVELY

A. Parts of Gal 2:7-9 ...when they SAW that the gospel for the uncircumcised had been committed to me.. for He who... WORKED EFFECTIVELY in me toward the Gentiles, ... when James, Cephas, and John,... perceived the GRACE that had been given to me.

B. A GRACEFUL PERSON WORKS EFFECTIVELY

C. GRACE: GOD'S ABILITY GIFT TO LIVE AND FUNCTION IN THE GIFTS AND CALLINGS

D. BEING IN GOD'S WILL FOR TAKING THIS COURSE, THERE IS GRACE (SUPERNATURAL ABILITY TO WORK EFFECTIVELY) FOR YOU TO DO SO WELL IT CAN BE SEEN.

TRUTH GEM

IDOLATRY

A. COL. 3:5b ... PUT TO DEATH YOUR MEMBERS WHICH ARE ON THE EARTH: FORNICATION, UNCLEANNESS, PASSION, EVIL DESIRE, AND COVETEOUSNESS, WHICH IS IDOLATRY.

B. IDOLS: PEOPLE WORSHIPPED IMAGES OF THINGS THAT ARE NOTHING (I COR 8:4b ... WE KNOW THAT AN IDOL IS NOTHING ...)

C. MANY PEOPLE COVET THE IMAGE OF BEING EDUCATED, BUT THERE IS NO REALITY, NOTHING TO BACK UP THE IMAGE.

D. SOME WANT TO WORK IN GROUPS AND GET A GROUP GRADE WHEN IN TRUTH THEY DO NOT KNOW IT, OR WRITE A PAPER ON FEELINGS ABOUT MATH RATHER THAN DO MATH. THEY ARE CONTENT WITH THE GRADE, THE IMAGE (THE IDOL) EVEN THOUGH THEY DO NOT KNOW.

E. FROM I JN 5:21 ... KEEP YOURSELF FROM IDOLS. [DESIRE TO KNOW THE MATH, DESIRE TO BE ABLE TO OBJECTIVELY DEMONSTRATE THAT YOU KNOW.]

TG-15

TRUTH GEM

DISCIPLINE

A. 2 Tim 1:7 For God has not given us a spirit of timidity, but of power and love and DISCIPLINE.

B. The undisciplined have problems with math.

C. Regular steady study (in wisdom) prevails, not just a big flurry the night before the test.

D. You will have to keep retaking life's test on discipline until you pass it.

E. For the Christian, DISCIPLINE is a fruit of what we are given.
Gifts must be received.
Gifts must be cultivated.

TRUTH GEM

$\boxed{\text{MEDITATE}}$

A. PS 1:2 BUT HIS DELIGHT IS IN THE LAW OF THE LORD AND IN HIS LAW HE MEDITATES* DAY AND NIGHT.

*PONDERS BY TALKING TO HIMSELF

B. WHAT YOU ARE IN THE WILL OF GOD TO LEARN CAN BE MEDITATED.

C. PICK A DEFINITION, THEOREM, OR PROBLEM DERIVATION.

1. CHEW ON IT WORD FOR WORD SEEKING UNDERSTADING
2. SPEAK IT OUT SO YOU CAN HEAR IT
3. WRITE IT DOWN OVER AND OVER
4. REVIEW IT

TG-17

TRUTH GEM

MAGNIFY THE SOLUTION AND NOT THE PROBLEM

A. PS 34:3 OH, MAGNIFY THE LORD WITH ME AND LET US EXALT HIS NAME TOGETHER

B. FOCUS ON WHAT YOU SEE IS TRUE AND GOOD AND CAN BE DONE. DO THAT. THE PROBLEM SHRINKS. SEE SOMETHING ELSE THAT IS TRUE AND GOOD AND CAN BE DONE. DO THAT. THE PROBLEM SHRINKS. PRESS ON DOING THIS UNTIL THE PROBLEM IS GONE!

C. PR 3:27 DO NOT WITHHOLD GOOD FROM THOSE TO WHOM IT IS DUE, WHEN IT IS IN THE POWER OF YOUR HAND TO DO SO.

D. JAMES 4:17 THEREFORE TO HIM WHO KNOWS TO DO GOOD AND DOES NOT DO IT, TO HIM IT IS SIN.

TG-18

TRUTH GEM

LITTLE BY LITTLE

A. DEUT. 7:22 AND THE LORD YOUR GOD WILL DRIVE OUT THOSE NATIONS BEFORE YOU <u>LITTLE BY LITTLE</u>; YOU WILL BE UNABLE TO DESTROY THEM AT ONCE, (EX 23:29 LEST THE LAND BECOME DESOLATE AND) LEST THE BEASTS OF THE FIELD BECOME TOO NUMEROUS FOR YOU.

B. LEARNING MATHEMATICS IS LIKE POSSESSING NEW TERRITORY. YOU MUST DO IT <u>LITTLE BY LITTLE</u> AND GET ESTABLISHED IN, CULTIVATE, PATROL\GUARD THAT GAINED WISDOM\KNOWLEDGE\UNDERSTANDING

C. TO TRY TO LEARN ALL AT ONCE, YOUR MIND IS STRETCHED TOO THIN (LAND BECOMES DESOLATE) YOU ARE UNABLE TO PATROL\GUARD AGAINST THE BEASTS OF THE FIELD (CONFUSION ATTEMPTS, LACK OF REMEMBERANCE, FEAR OF A HERD OF QUESTIONS COMING AT YOU ON A TEST)

D. DO NOT WAIT UNTIL RIGHT BEFORE THE TEST TO TRY TO LEARN DAYS OF WORK. WISDOM: LITTLE BY LITTLE.

TRUTH GEM

BE STEADFAST IN HOMEWORK

A. I COR 15:58 THEREFORE, MY BELOVED BRETHREN, BE STEADFAST, IMMOVABLE, ALWAYS ABOUNDING IN THE WORK OF THE LORD, KNOWING THAT YOUR LABOR IS NOT IN VAIN IN THE LORD.

B. BEING IN THE WILL OF GOD BY TAKING THIS COURSE MEANS YOU ARE DOING THE WORK OF THE LORD, ... SO... BE STEADFAST

C. DISTRACTIONS, LOWER PRIORITY THINGS WILL TRY TO TAKE YOU AWAY FROM "ALWAYS ABOUNDING" IN STUDY... BE SET. DO NOT BE MOVED. HOLD FAST AND STUDY.

D. PS 16:8 I HAVE SET THE LORD ALWAYS BEFORE ME; BECAUSE HE IS AT MY RIGHT HAND I SHALL NOT BE MOVED.

TG-20

TRUTH GEM

PRIORITIZE

A. NEH 6:3 I AM DOING A GREAT WORK, SO THAT I CANNOT COME DOWN. WHY SHOULD THE WORK CEASE WHILE I LEAVE IT AND GO DOWN TO YOU?

B. HAVE GOD'S PRIORITIES FOR YOUR LIFE CLEAR, DECIDED, SET, ESTABLISHED... A DIVINE ORDER. WHEN SOMETHING COMES UP FOR A DECISION, DISCERN WHAT CATEGORY IT IS IN AND THE DECISION HAS ALREADY BEEN MADE.

C. BEING IN GOD'S WILL FOR TAKING THIS COURSE MEANS THIS COURSE IS A HIGH PRIORITY, SO REGULAR, NONDISTRACTED STUDY TIME IS A HIGH PRIORITY, SO... DO NOT LEAVE IT AND GO DOWN TO DO A LESSER PRIORITY.

TG-21

TRUTH GEM

OVERCOME

A. I JN 5:4 FOR WHATEVER IS BORN OF GOD OVERCOMES THE WORLD. AND THIS IS THE VICTORY THAT HAS OVERCOME THE WORLD— OUR FAITH.

B. BEING IN THE WILL OF GOD FOR TAKING THIS COURSE, AND HENCE DOING A GREAT WORK, YOU WILL BE COME AGAINST TO TRY TO STOP, HINDER, OR HARASS THE WORK OF REGULAR STUDY — OVERCOME IT WITH SUPERNATURAL HELP

C. WITH A BOLD, NONTIMID, STRONG AND COURAGEOUS INNER MAN — SAY NO AND RESIST AND OVERCOME THE OPPOSITION

D. EPH 3:16 ... THAT HE WOULD GRANT YOU, ACCORDING TO THE RICHES OF HIS GLORY, TO BE STRENGTHENED WITH MIGHT THROUGH HIS SPIRIT IN THE INNER MAN

TG-22

TRUTH GEM

DECISIONS VS. OTSGWFIADI

A. MK 14:36b ... NEVERTHELESS, NOT WHAT I WILL, BUT WHAT YOU WILL.

B. WHEN WHAT IS CALLED A DECISION IS PRESENTED TO YOU (LIKE TO START STUDYING ON DO SOMETHING ELSE - OR - TO KEEP STUDYING LONGER OR STOP). WHAT IS THE BASIS FOR YOUR DECISION?... YOUR OWN UNDERSTANDING? YOUR PLEASURES?

C. PR 3:5-6 TRUST IN THE LORD WITH ALL YOUR HEART AND LEAN NOT ON YOUR OWN UNDERSTANDING; IN ALL YOUR WAYS ACKNOWLEDGE HIM AND HE SHALL DIRECT YOUR PATHS

D. INSTEAD OF DECISION TIME ITS OTSGWFIADI TIME : Opportunity To Seek God's Will, Find It, And Do It.

E. PART OF MT 7:7 ... SEEK AND YOU WILL FIND...

TG-23

TRUTH GEM

DAVID AND GOLIATH

A. BEFORE GOLIATH'S DEFEAT: PART I SAMUEL 17:11

... THEY WERE DISMAYED AND GREATLY AFRAID.

B. PROPER BATTLE ATTITUDE: I SAM. 17:48b

... DAVID HURRIED AND RAN TOWARD THE ARMY TO MEET THE PHILISTINE.

C. AFTER GOLIATH'S DEFEAT: I SAM. 17:51b

... AND WHEN THE PHILISTINES SAW THAT THEIR CHAMPION WAS DEAD, THEY FLED.

D. THE LOSS OF ONE SOLDIER IN THE PHILISTINE ARMY CAUSED A MAJOR REVERSAL - FROM TAUNTING TO FLEEING ... A REASON: YOU DEFEAT AN ENEMY'S MOST POWERFUL WEAPON, THEIR STRENGTH IS GONE; WHAT THEY TRUSTED IN WAS GONE; THEY FLEE.

E. BY NOW YOU HAVE HAD VICTORY OVER SOME OF THE MOST POWERFUL PROBLEMS IN THE COURSE. SOME OF INTIMIDATION'S STRONGEST WEAPONS - FEAR OF HARD MATH PROBLEMS AND PAST EXPERIENCE OF DIFFICULTY WITH HARD MATH PROBLEMS, HAS BEEN OVERCOME. THIS ENEMY IS FLEEING... PURSUE ... PLUNDER WITH CONTINUED DISCIPLINE & CLARITY

F. PARTS OF I SAM 17:52,53 NOW THE MEN OF ISRAEL AND JUDAH ... PURSUED ... AND THEY PLUNDERED.

TRUTH GEM

JOY IN RESPONSIBILITY

A. MT 25:21 HIS LORD SAID TO HIM, "WELL DONE, GOOD AND FAITHFUL SERVANT; YOU WERE FAITHFUL OVER A FEW THINGS, I WILL MAKE YOU RULER OVER MANY THINGS. ENTER INTO THE JOY OF YOUR LORD.

B. IT TAKES RESPONSIBILITY TO BE GOOD IN MATH.

C. RESPONSIBILITY IS A GOOD WORD.

D. WHEN GOD PROMOTES YOU AFTER LONG FAITHFUL SERVICE TO MORE RESPONSIBILITY, GOD IS JOYFUL AT HAVING SOMEONE GOOD AND FAITHFUL OVER WHAT GOD CARES VERY MUCH FOR.

1. WE ARE TO ENTER INTO HIS JOY
2. IT IS COMMANDED.
3. GOD MAKES YOU RULER, SO THROUGH HIM YOU ARE ABLE.

E. MANY OF YOU HAVE BEEN GOOD AND FAITHFUL STUDENTS FOR YEARS; YOU HAVE BEEN PROMOTED TO THIS CLASS; ENTER INTO HIS JOY. THIS CLASS IS ALSO A PROVING GROUND FOR FURTHER PROMOTION BY GOD.

TRUTH GEM

HOLY SPIRIT BAPTISM IMPLIES BOLDNESS

A. ACTS 4:31b ... THEY WERE ALL FILLED WITH THE HOLY SPIRIT AND SPOKE THE WORD OF GOD WITH BOLDNESS

B. RECALL PR 28:1b .. THE RIGHTEOUS ARE AS BOLD AS A LION.

C. YOU NEED GREAT BOLDNESS TO WORK SOME MATH PROBLEMS.
RIGHTEOUSNESS BOLDNESS + HOLY SPIRIT POWER BOLDNESS = SOLVED PROBLEMS IN THIS COURSE.

D. JESUS TOLD HIS DISCIPLES TO WAIT UNTIL THE RECEIVED POWER FROM ON HIGH UNTIL THEY WENT OUT TO WITNESS

E. WHATEVER WE DO IN HIS WILL THIS BOLDNESS IS AVAILABLE.

F. LK 11:13b ... HOW MUCH MORE WILL YOUR HEAVENLY FATHER GIVE THE HOLY SPIRIT TO THOSE WHO ASK HIM.

TRUTH GEM

FOR EVERY PROBLEM, THERE IS A PROBLEM OBLITERATING REVELATION THAT OBLITERATES THE PROBLEM

A. ACTS 9:3 AS HE JOURNEYED HE CAME NEAR DAMASCUS, AND SUDDENLY A <u>LIGHT</u> <u>SHONE</u> AROUND HIM FROM HEAVEN.... GAL 1:23b HE WHO FORMERLY PERSECUTED US NOW PREACHES THE FAITH WHICH HE ONCE TRIED TO DESTROY.

B. PAUL HAD A GREAT PROBLEM. HE WANTED CHRISTIANS JAILED, EVEN HAD A PART IN A STONING. HE GOT A <u>REVELATION</u>. THAT PROBLEM WAS <u>OBLITERATED</u>.

C. FOR EVERY PROBLEM A PERSON HAS, THERE IS A PROMISE OF GOD THAT OBLITERATES THE PROBLEM.

D. II PET 1:4 ... BY WHICH HAVE BEEN GIVEN TO US EXCEEDINGLY GREAT AND PRECIOUS PROMISES, THAT THROUGH THESE YOU MAY BE PARTAKERS OF THE DIVINE NATURE, HAVING ESCAPED THE CORRUPTION THAT IS IN THE WORLD THROUGH LUST.

TRUTH GEM

LEARN FROM THE CLEAR CLEARLY DO

A. (JOSH. 11:15 a) AS THE LORD COMMANDED MOSES HIS SERVANT, SO MOSES COMMANDED JOSHUA, AND SO JOSHUA DID.

B. WHEN HIRED, THE FIRST THING DONE USUALLY IS <u>TRAINING</u>.

C. THERE IS A MAJOR NEED FOR PEOPLE TO BE <u>TAUGHT</u> <u>CLEARLY</u> FROM THOSE WHO <u>SEE</u> <u>CLEARLY</u> AND FOR THE TAUGHT ONES TO FAITHFULLY DO IT IN TUNE AND IN FOCUS.

D. (ROM 13:10 a) LOVE DOES NO HARM...
(HARM GENERALLY COMES FROM NOT DOING A JOB WELL, THE WAY YOU HAVE BEEN TRAINED)

E. LIFE IS NOT ALL SUBJECTIVE WORD PROBLEMS. MUCH OF IT IS LEARNING FROM THOSE WHO SEE CLEARLY AND DOING IT.

F. SEEK THE CLEAR ONES AND LEARN.

CALLING & DESTINY

A. EPH 1:18 ... the eyes of your understanding enlightened that you may know what is the hope (i.e. destiny) of His calling

B. (Jack Shoup) The calling is the office. The destiny is to be fulfilled in that office. Grace is the provision to fulfill your destiny within and by being in your calling

C. EXAMPLE: Part of my calling is to be a math teacher. Part of my destiny is to "make it plain". Since I have answered the call - there is grace to fulfill it.

Similar example: call: college student
destiny - x = major grade = y

D. You can answer the call and not fulfill your destiny (Jack Shoup)

E. PLP 3:14 I press toward the goal for the prize (destiny) of the upward call of God in Christ Jesus

TG-29

TRUTH GEM

ESTABLISHED 2

A. I PET 5:10 But may the God of all grace who called us to His eternal glory by Christ Jesus, after you have suffered a while, perfect, ESTABLISH, strengthen, and settle you.

B. Establish - brought into incontrovertible existence

C. Be established in ending things well

1. semesters - you have spent a semester getting math understanding - use it to make\redeem your grade.
2. projects
3. moving

D. We go through so much to get to the end - the goal. Do not lose it there (Do not throw away a letter grade just to leave a few minutes early going home at end of semester - you can pack after exams!)

E. 2 Jn :8 Look to yourselves, that we do not lose those things that we worked for, but that we may receive a full reward.

TRUTH GEM

Heb 13:9 ESTABLISHED[3]

A. DO NOT BE CARRIED ABOUT WITH VARIOUS AND STRANGE DOCTRINES. FOR IT IS GOOD THAT THE **HEART** BE ESTABLISHED BY **GRACE**, NOT WITH **FOODS** WHICH HAVE **NOT PROFITED** THOSE WHO HAVE BEEN OCCUPIED WITH THEM.

B. OUR HEARTS FEED ON TEACHINGS (DOCTRINES). SOME PROFIT. SOME DON'T.

C. WE ARE IN A CERTAIN SENSE, WALKING TEACHINGS

1. THOSE THAT HARM ARE LIKE PARASITES THAT DO NOT PROFIT -
2. THE ENTRANCE OF PROFITABLE TEACHINGS, CAN "DE-WORM" A ~~[illegible]~~ PERSON FROM PARASITE TEACHINGS + CAUSE YOU TO PROFIT.

TRUTH GEM TG-31

BE STRONG AND OF GOOD COURAGE X ≈ 4

A. "... be strong and of good courage" (Josh 1:6,9,18, ≈7)

B. There are things that some view as very hard to do, that with God being with you, you can do, but you have to be <u>STRONG</u> <u>AND</u> <u>OF</u> <u>GOOD</u> <u>COURAGE</u>!

C. You do not look at the opponent, but <u>begin</u>, doing the best you can do, <u>not relenting</u> on what you are in the will of God for.

D. What is <u>a</u> way to get strength & Courage? You can be commanded to have it!!!

So I command you: BE STRONG AND OF GOOD COURAGE!

NOT DIS-<u>COURAGED</u>
BUT EN-<u>COURAGED</u>

TG-32

DISCOURAGEMENT IS NOT AN OPTION

A. BE STRONG AND OF GOOD COURAGE Josh 1:6

"

"

BE STRONG AND VERY COURAGEOUS Josh 1:7

DIS-COURAGE: TO REMOVE, TAKE AWAY COURAGE
EN-COURAGE: TO INCREASE COURAGE Belinda French

B. IF YOU KNOW AND BELIEVE THE RIGHT THINGS AND DO NOT WAVER, THERE ARE NO GROUNDS FOR DISCOURAGEMENT!

C. WHEN YOU KNOW YOU HAVE

1. IRREVOCABLE ACCESS TO WISDOM
2. IRREVOCABLE ACCESS TO GRACE
3. IRREVOCABLE RIGHT STANDING WITH GOD
4. IRREVOCABLE FORGIVENESS ACCESS

THERE ARE NO GROUNDS FOR DISCOURAGEMENT!

D. DISCOURAGEMENT SLOWS DOWN LEARNING

E. NOW THANKS BE TO GOD
WHO ALWAYS LEADS US IN TRIUMPH
IN CHRIST 2 COR 2:14

F. IS 42:4A HE WILL NOT FAIL NOR BE DISCOURAGED
TILL HE HAS ESTABLISHED JUSTICE IN THE EARTH

TG-33

GOD-PACED LEARNING
NOT SELF-PACED LEARNING

A. (Mt 16:24) IF ANYONE DESIRES TO COME AFTER ME, LET HIM DENY HIMSELF, AND TAKE UP HIS CROSS AND FOLLOW ME.

B. SELF WILL WANT TO STOP AND HAVE PIZZA

C. SELF WILL DECEIVE ITSELF DUE TO HUNGER FOR SELF-ESTEEM

IN ONE STUDY THE U.S. CAME IN LAST IN MATH SCORES, BUT FIRST IN HOW THEY FELT ABOUT MATH

D. HUNGER TO BE GOD-ESTEEMED, GOD-PACED EMPOWERED BY GRACE.

GRACE-BOOSTED CARRYING YOUR OWN LOAD VS. PARALYZED DEPENDENCE

A. JN 5:7-8 Parts SIR, I HAVE NO ONE TO PUT ME IN THE POOL... JESUS SAID TO HIM, RISE TAKE UP YOUR BED AND WALK.

B. (K. HAGIN) SOME CONSIDER THEMSELVES HELPLESS DEPENDENT ON OTHER PEOPLE AND DO NOT REALIZE THERE IS TRANSFORMATIONAL WORD THAT CAN TRANSFORM THEM TO WHERE THEY CAN DO IT. THEY THINK THEY CANNOT LEARN WITHOUT AN INDIVIDUAL TUTOR OR GROUP WORK. WHAT IS NEEDED IS TO HEAR, BELIEVE AND FOLLOW THE RIGHT WISDOM WORD AND EMPOWERED BY GRACE THEY DO IT THEMSELF.

C. GAL. 6:2,5 BEAR ONE ANOTHER'S BURDENS... EACH ONE SHALL BEAR HIS OWN LOAD.

GIVE THE RIGHT WISDOM WORD THAT FREES OF HARMING DEPENDENCE, WHEN YOU ARE TO DO IT GRACE BOOSTED, NOT DEPENDENT ON OTHERS,

THEN THERE IS WORK YOU ARE MADE TO DO AS A SOURCE, NOT A DRAIN; DO IT HELPING OTHERS.

GUARD YOUR HEART TO KEEP BEING A GOOD STUDENT

A. PR. 4:23 "KEEP YOUR HEART WITH ALL DILIGENCE, FOR OUT OF IT SPRING THE ISSUES OF LIFE."

B. GUARD AGAINST ENVIRONMENTS OF

1. EXCESSIVE TALKING (JAMES 1:19)
2. NEGATIVE TALK
3. DISCOURAGING TALK

C. BE IN AND SET AN ENVIRONMENT OF

1. FAITHFULNESS
2. EXCELLENCE
3. WISE DILIGENCE
4. WORDS OF WISDOM AND LIFE

D. IT IS FROM YOUR HEART THAT THE FIRE TO DO AND EXCEL COMES FROM.

E. 2 JN :8 LOOK TO YOURSELVES, THAT WE DO NOT LOSE THOSE THINGS WE WORKED FOR BUT THAT WE MAY RECEIVE A FULL REWARD.'

TG-36

TRUTH GEM

HIGHER NATURE EDUCATION

A. JOHN 8:23 AND HE (JESUS) SAID TO THEM, "YOU ARE FROM BENEATH; I AM FROM ABOVE. YOU ARE OF THIS WORLD; I AM NOT OF THIS WORLD.

B. JESUS WAS NOT SPEAKING ABOUT DIRECTIONALLY UP OR DOWN, BUT ABOUT NATURES.

C. MANY EDUCATIONAL IDEAS COME FROM THE REALM OF MAN'S WISDOM AND APPEAL TO THE LOWER NATURE (EXCUSES, WHY YOU CAN'T LEARN, BLAMING OTHERS, LOWERED EXPECTATIONS)

D. TRUE HIGHER EDUCATION APPEALS TO THE HIGHER NATURE WAYS OF THE MOST HIGH. (FAITHFULNESS, DILLIGENCE, WISDOM, NO EXCUSES, SUCCESS, VICTORY, OVERCOMING, COURAGE, CHALLENGE, DOMINION OF MATERIAL TO BE LEARNED, WISDOM IN STUDY HABITS)

INMOST BEING COMPATIBILITY TEACHING AND LEARNING

A. ACTS 17:28A " FOR IN HIM WE LIVE AND MOVE AND HAVE OUR BEING.

B. SEEK AND FIND TEACHING\LEARNING METHODS IN ACCORDANCE TO THE WAY YOU ARE TO BE.

C. THINGS ARE TO BE RIGHT IN\WITH YOUR INMOST BEING.

D. MATT. 7:7 "... SEEK, AND YOU WILL FIND."

TRUTH GEM

"JUST AS" TEACHING\LEARNING

A. I JN 2:27 BUT THE ANOINTING WHICH YOU HAVE RECEIVED FROM HIM ABIDES IN YOU, AND YOU DO NOT NEED THAT ANYONE TEACH YOU; BUT AS THE SAME ANOINTING TEACHES YOU CONCERNING ALL THINGS, AND IS TRUE, AND IS NOT A LIE, AND JUST AS IT HAS TAUGHT YOU, YOU WILL ABIDE IN HIM.

B. AS (AT THE TIME) YOU ARE TAUGHT SOMETHING GOOD, YOU IMMEDIATELY DO IT JUST AS (IN LIKE MANNER) TO THE WAY YOU WERE TAUGHT. YOU WILL ABIDE (DWELL) IN THE FLOW OF THE TEACHER.

C. TO DELAY DOING HOMEWORK OR TO ATTEMPT DOING IT DIFFERENTLY THAN TAUGHT CAN KEEP YOU FROM ABIDING IN THE COURSE FLOW AND TRUTH.

TG-[redacted]39
TRUTH GEM

MAGNIFY THE SOLUTION AND NOT THE PROBLEM

A. PS 34:3 OH, MAGNIFY THE LORD WITH ME AND LET US EXALT HIS NAME TOGETHER

B. FOCUS ON WHAT YOU SEE IS TRUE AND GOOD AND CAN BE DONE. DO THAT. THE PROBLEM SHRINKS. SEE SOMETHING ELSE THAT IS TRUE AND GOOD AND CAN BE DONE. DO THAT. THE PROBLEM SHRINKS. PRESS ON DOING THIS UNTIL THE PROBLEM IS GONE!

C. PR 3:27 DO NOT WITHHOLD GOOD FROM THOSE TO WHOM IT IS DUE, WHEN IT IS IN THE POWER OF YOUR HAND TO DO SO.

D. JAMES 4:17 THEREFORE TO HIM WHO KNOWS TO DO GOOD AND DOES NOT DO IT, TO HIM IT IS SIN.

ONE
NOT DIS-INTEGRATED

A. I THESS 5:23 NOW MAY THE GOD OF PEACE HIMSELF SANCTIFY YOU COMPLETELY; AND MAY YOUR WHOLE SPIRIT, SOUL, AND BODY BE PRESERVED BLAMELESS AT THE COMING OF OUR LORD JESUS CHRIST.

B. MK 12:29 "... THE FIRST OF ALL THE COMMANDMENT IS: HEAR, O ISRAEL, THE LORD OUR GOD, THE LORD IS ONE.

C. YOU ARE ONE WHEN OUT OF LOVE OF GOD YOU FOCUS SPIRIT, SOUL, AND BODY ON THE GOD-LED TASK AT HAND ... THIS CLASS, IN THIS COURSE, NOW

- NOT: BODY HERE AND MIND ELSEWHERE
- NOT: BODY AND MIND HERE, BUT HEART NOT IN IT
- NOT: ABSENT IN THE BODY BUT WITH US IN SPIRIT

} DIS-INTEGRATED

D. DESIRE AND ADMIRE BEING ONE.

TRUTH GEM TG-41

RIGHT - SOAKED WORDS

A. PR. 8:6b,8 FROM THE OPENING OF MY LIPS WILL COME RIGHT THINGS ... ALL THE WORDS OF MY MOUTH ARE WITH RIGHTEOUSNESS, NOTHING CROOKED OR PERVERSE IS IN THEN.

B. WORDS ARE LIKE CONTAINERS, SPONGES; THEY CARRY THINGS IN THEM.

C. THE ENVIRONMENT IS TO BE PERMEATED WITH RIGHT - SPOKEN BY ONE WHO SAYS RIGHT WORDS SOAKED WITH UNDERSTANDING

D. TO LEARN YOU HEAR THE RIGHT WORDS FROM THE RIGHT PERSON, SPEAKING THEM RIGHTLY SOAKED WITH UNDERSTANDING

E. YOU KEEP REPEATING THE RIGHT WORDS UNTIL THEY GET SOAKED WITH UNDERSTANDING

THE ALTERNATIVE... LISTENING TO THE CONFUSED SAY CONFUSION SOAKED WORDS UNTIL UNDERSTANDING COMES & GREATLY SLOWS DOWN LEARNING.

F. TREAT CONFUSION FOR THE RATTLESNAKE IT IS.

G. GET UNDERSTANDING BY BEING PERMEATED WITH RIGHT SOAKED WORDS

TG-[redacted]42

TRUTH GEM

USE YOUR IRREVOCABLE GIFT

A. I PET 4:10 AS EACH ONE HAS RECEIVED A GIFT, MINISTER IT TO ONE ANOTHER AS GOOD STEWARDS OF THE MANIFOLD GRACE OF GOD

B. ROM 11:29 FOR THE GIFTS AND CALLING OF GOD ARE IRREVOCABLE.

C. EACH OF US HAS AN IRREVOCABLE GIFT

1. IT IS TO BE USED TO HELP US AND OTHERS
2. IT IS IRREVOCABLE SO THAT YOU CAN HAVE GREAT CONFIDENCE AT TIME OF NEED, THE GIFT WILL BE THERE TO BLESS

D. WHEN YOU DISCERN YOUR CALLING AND GIFTS

1. HONOR THE CALL AND GIFTS - DON'T DESIRE SOME OTHER.
2. USE EACH GIFT BY GRACE TO FULFILL THE CALL.

E. BEING IN THE WILL OF GOD FOR BEING HERE BE SURE THERE IS A GIFT YOU HAVE THAT GOD WILL GRACE FOR YOU TO USE TO SUCCEED WITH JOY.