ANALYSIS OF PRECALCULUS FOR TEACHERS

by

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THEOREMS TO BE STUDIED

(A Reference List)

"Do not be afraid, only believe*"
that these will be clear and
understandable in the season
they are to be taught, studied,
and understood.

Austin French

* Mark 5:36

Lemma 11-2 For every real R, if R>0, then for every pos. int. n, $R^n > 0$

Lemma 11A-1 For every positive integer n, for every $\varepsilon > 0$, $\sqrt[n]{\varepsilon} > 0$

Lemma 11A-2 For all reals L, R, if $0 < L < R$, then for every positive integer n, $L^n < R^n$

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THEOREM LIST

Thm A-1: For each real number x, $|x| \ge 0$.

Thm A-2: For each real number x, if $x \le 0$, then $|x| = -x$.

Thm A-3: For each real number x, $|x| < a$ iff $-a < x < a$.

Thm A-4: For each real number x, $|x| > a$ iff $x < -a$ or $x > a$.

Thm A-4a: For each real number x, if $x \ne 0$, then $|x| \ne 0$.

Thm A-5: For each real number x, $|ab| = |a||b|$.

Thm A-6: For each real number x, $|a/b| = |a|/|b|$.

Thm 9-1: For each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|1/x - 0| < \epsilon$.

Thm 9-2: For each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|-7/x - 0| < \epsilon$.

Thm 10-1: Given C is a real number. For each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|C/x - 0| < \epsilon$.

Lemma 11-1: For all real numbers L,R, if $0 < L < R$, then $L^2 < R^2$.

Thm 11-1: For each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|1/x^2 - 0| < \epsilon$.

Thm 11-2: For each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|1/x^n - 0| < \epsilon$.

Thm 12-1: Let $f(x) = (3x+1)/x$. For each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|f(x) - 3| < \epsilon$.

Thm 12-2: Let C be a constant real number and for all x, $f(x) = C$. For each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|f(x) - C| < \epsilon$.

Lemma 13-1: For all real numbers a,b, $|a+b| \le |a| + |b|$.

Thm 13-2: Given f(x) and g(x) are functions and L and M are reals. If (1) for each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|f(x) - L| < \epsilon$, and (2) for each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|g(x) - M| < \epsilon$, THEN for each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$,

$$|[f(x)+g(x)] - (L+M)| < \epsilon.$$

Thm 14-1: Given C and L are reals and f(x) is a function. If for each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|f(x) - L| < \epsilon$, THEN for each $\epsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|Cf(x) - CL| < \epsilon$.

VI

Thm 14-2 For every $\varepsilon>0$, there is an $N<0$ such that for all $x<N$, $\left|\frac{1}{x}-0\right|<\varepsilon$.

Thm 14-3 For every $\varepsilon>0$, there is an $N<0$ such that for all $x<N$, $\left|\frac{1}{x^2}-0\right|<\varepsilon$

Theorem 16-1 $\lim_{x\to\infty}\frac{1}{x}=0$ (See Thm 9-1)

Theorem 16-2 $\lim_{x\to\infty}\frac{-7}{x}=0$ (See Thm 9-2)

Theorem 16-3 Given c is a real no. $\lim_{x\to\infty}\frac{c}{x}=0$ (See Thm 10-1)

Theorem 16-4 $\lim_{x\to\infty}\frac{1}{x^2}=0$ (See Thm 11-1)

Theorem 16-5 Given n is a pos. int. $\lim_{x\to\infty}\frac{1}{x^n}=0$ (See Thm 11-2)

Theorem 16-6 $\lim_{x\to\infty}\frac{3x+1}{x}=3$ (See Thm 12-1)

Theorem 16-7 Given c is a real no. $\lim_{x\to\infty}c=c$ (See Thm 12-2)
(or For $f(x)=c$, $\lim_{x\to\infty}f(x)=c$)

Theorem 16-8 Given $f(x), g(x)$ are functions defined on the positive reals and L, M are reals. If $\lim_{x\to\infty}f(x)=L$ and $\lim_{x\to\infty}g(x)=M$, then $\lim_{x\to\infty}f(x)+g(x)=L+M$. (See Thm 13-2)

Theorem 16-9 Given c and L are real numbers and $f(x)$ is a function defined on the positive reals. If $\lim_{x\to\infty}f(x)=L$, then $\lim_{x\to\infty}c\,f(x)=cL$. (See Thm 14-1)

Theorem 16-10 $\lim_{x\to-\infty}\frac{1}{x}=0$ (See Thm 14-2)

Theorem 16-11 $\lim_{x\to-\infty}\frac{1}{x^2}=0$ (See Thm 14-3)

Lemma 17-1: For every real number L, $\frac{|L|}{|L|+1}<1$

Lemma 17-2: $|a-b|\geq|a|-|b|$

Lemma 17-3: $|x|=|-x|$ Note: $-(a-b)=b-a$

Thm 17-4: If $\lim_{x\to\infty}G(x)=M$ and $M\neq 0$, then there are $P>0$ and $D>0$ such that for all $x>P$, $|G(x)|>D$

Lemma 18-1 If $T \geq 0$ and $W > D > 0$, then $\frac{T}{W} \leq \frac{T}{D}$

Theorem 18-1 If $\lim_{x\to\infty} F(x) = T$, $\lim_{x\to\infty} G(x) = B$, and $B \neq 0$, then $\lim_{x\to\infty} \frac{F(x)}{G(x)} = \frac{T}{B}$

Theorem 19-1 $\lim_{x\to\infty} \frac{x^2-7}{3x^2+x} = \frac{1}{3}$

Corollary 19A-1 If $\lim_{x\to\infty} l(x) = L$, $\lim_{x\to\infty} r(x) = R$, and $R \neq 0$ then $\lim_{x\to\infty} l(x)\cdot r(x) = L\cdot R$

Theorem 19A-1 If $\lim_{x\to\infty} l(x) = L$ and $\lim_{x\to\infty} r(x) = 0$, then $\lim_{x\to\infty} l(x)\cdot r(x) = 0$

Theorem 19B If $\lim_{x\to\infty} l(x) = L$ and $\lim_{x\to\infty} r(x) = R$, then $\lim_{x\to\infty} l(x)\cdot r(x) = L\cdot R$

Theorem 20A For all reals T, B, c, d, if $B \neq 0$, then $\lim_{x\to\infty} \frac{Tx+c}{Bx+d} = \frac{T}{B}$.

Theorem 20B For all reals a, b, c, d, e, if $c \neq 0$, then $\lim_{x\to\infty} \frac{ax+b}{cx^2+dx+e} = 0$

Theorem 20C For all reals T, c, d, B, e, and f, if $B \neq 0$, then $\lim_{x\to\infty} \frac{Tx^2+cx+d}{Bx^2+ex+f} = \frac{T}{B}$

Theorem 21-1 For all $H > 0$, there is a $\delta > 0$ such that for all $0 < x < \delta$, $\frac{1}{x} > H$.

Theorem 21-2: For all $H > 0$, there is a $\delta > 0$ such that for all $3 < x < 3+\delta$, $\frac{1}{x-3} > H$.

Theorem 21-4 $\lim_{x\to 5^+} \frac{7}{x-5} = +\infty$

Theorem 22-1 For all reals c, a, if $c > 0$, then $\lim_{x\to a^+} \frac{c}{x-a} = +\infty$

Theorem 23-2 $\lim_{x\to 0^+} \frac{-3}{x} = -\infty$

Theorem 23-3 For all reals a, c, if $c < 0$, then $\lim_{x\to a^+} \frac{c}{x-a} = -\infty$

Theorem 24-1 Let $f(x) = ax+b$. For every real p, f is continuous at p.

Theorem 25-1. If f is continuous at p and $f(p) > 0$, then there are positive reals, δ, L, and U such that for all $x \in R$, if $|x-p| < \delta$, then $L < f(x) < U$

Theorem 25-2: If f is continuous at p and $f(p) < 0$, then there are a $\delta > 0$ and $L, U < 0$ such that for all $x \in R$, if $|x-p| < \delta$ then $L < f(x) < U$

Th^{m} 26-1 $\lim_{x \to p^+} x - p = 0^+$

Th^{m} 26-2 T is cont. at a, B is cont at a, and $\lim_{x \to a^+} D(x) = 0^+$

(a) if $T(a) > 0$ and $B(a) > 0$, then $\lim_{x \to a^+} \frac{T(x)}{B(x)D(x)} = +\infty$

(b) if $T(a) > 0$ and $B(a) < 0$, then $\lim_{x \to a^+} \frac{T(x)}{B(x)D(x)} = -\infty$

(c) if $T(a) < 0$ and $B(a) > 0$, then $\lim_{x \to a^+} \frac{T(x)}{B(x)D(x)} = -\infty$

(d) if $T(a) < 0$ and $B(a) < 0$, then $\lim_{x \to a^+} \frac{T(x)}{B(x)D(x)} = +\infty$

Th^{m} 28-1 $\lim_{x \to 5^-} \frac{-3}{x-5} = +\infty$

Th^{m} 28-4 $\lim_{x \to a^-} x - a = 0^-$

Th^{m} 28-5 T is cont. at a, B is cont at a, and $\lim_{x \to a^-} D(x) = 0^-$

(a) if $T(a) > 0$ and $B(a) > 0$, then $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = -\infty$

THEOREM 28C If f is continuous at p then kf is cont. at p

(b) if $T(a) > 0$ and $B(a) < 0$, then $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = +\infty$

(c) if $T(a) < 0$ and $B(a) > 0$, then $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = +\infty$

(d) if $T(a) < 0$ and $B(a) < 0$, then $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = -\infty$

Theorem 29-1 Let $f(x) = ax^2 + bx + c$. For all $p \in R$, f is continuous at p

Th^m 41-1: If $b(x)$ is continuous at p and $b(p) \neq 0$, then there are a $B>0$ and a $\delta>0$ such that for all $x \in R$, if $|x-p| < \delta$, then $|b(x)| > B$

Th^m 42: If $t(x)$ is continuous at p, $b(x)$ is continuous at p, and $b(p) \neq 0$, then $\frac{t(x)}{b(x)}$ is continuous at p.

Th^m 44-1: If $M(x) = \frac{ax^2+bx+c}{dx^2+ex+f}$ and $dp^2+ep+f \neq 0$, then M is continuous at p.

Th^m 45-1: Given M is continuous at p and $\lim_{x\to p^+} V(x) = +\infty$

a) If $M(p)>0$, then $\lim_{x\to p^+} M(x)V(x) = +\infty$.

b) If $M(p)<0$, then $\lim_{x\to p^+} M(x)V(x) = -\infty$

Th^m 46-1: Given M is continuous at p and $\lim_{x\to p^-} V(x) = -\infty$

a) If $M(p)>0$, then $\lim_{x\to p^-} M(x)V(x) = -\infty$

b) If $M(p)<0$, then $\lim_{x\to p^-} M(x)V(x) = +\infty$

Th^m 48-2 If $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to\infty} g(x) = +\infty$, then $\lim_{x\to\infty} f(x)\cdot g(x) = +\infty$.

Lemma 48-3 For every $T \in R$, there is a $B \in R$ such that $B>T$ and $B>0$.

Theorem 49-1 If $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to\infty} g(x) = K$, then $\lim_{x\to\infty} f(x)+g(x) = +\infty$.

Theorem 50 If $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to\infty} g(x) = +\infty$, then $\lim_{x\to\infty} f(x)+g(x) = +\infty$.

Theorem 52-1 $\lim_{x\to\infty} x = +\infty$.

Theorem 52-2 $\lim_{x\to\infty} 2x+9 = +\infty$

Theorem 53 $\lim_{x\to\infty} \frac{2x^2+7x+6}{x-1} = \lim_{x\to\infty} 2x+9+\frac{15}{x-1} = +\infty$

Possible other theorems involving functions of 2-variables

ABSOLUTE VALUE

I. INTUITIVELY: $|x|$ IS THE DISTANCE FROM 0.

$|5| = 5$. $|-5| = 5$

II. RIGOROUSLY: $|x| = x$ IF $x \geq 0$.
$= -x$ IF $x < 0$.

$|\underset{pos}{5}| = 5$ $|\underset{neg}{-5}| = -(-5) = 5$

IF THE THING INSIDE THE ABSOLUTE VALUE SIGNS IS POSITIVE, WRITE IT DOWN. IF THE THING INSIDE THE ABSOLUTE SIGNS IS NEGATIVE, WRITE IT DOWN AND PUT A "MINUS" IN FRONT.

A. SUPPOSE $a < -3$ AND $b > 7$.

$$|\underbrace{ab}_{\substack{neg \cdot pos \\ neg}}| = -ab \qquad |\underbrace{3a}_{\substack{pos \cdot neg \\ neg}}| = -3a$$

$$|\underbrace{-2ab}_{\substack{neg \cdot neg \cdot pos \\ pos}}| = -2ab$$

$$|\underbrace{7b}_{\substack{pos \cdot pos \\ pos}}| = 7b$$

$$|a-b| = |\underbrace{a+(-b)}_{\substack{neg + neg \\ neg}}| = -(a-b)$$
$$= -a+b$$
$$= b-a$$

B. THEOREM: $\sqrt{x^2} = |x|$

so $\sqrt{(x+1)^2} = |x+1|$

$$\sqrt{(-3-x^2)^2} = |-3-x^2| = |\underbrace{-(3+x^2)}_{\substack{\text{neg} \cdot \text{pos} \\ \text{neg}}}|$$

$$= -\left[-(3+x^2)\right] = 3+x^2.$$

C. THEOREM: $|x| < a$ IF AND ONLY IF $-a < x < a$.

1.) $|x| < 3$

$-3 < x < 3$

2.) $|3-2x| < 5$

$-5 < 3-2x < 5$

$-8 < -2x < 2$

$\frac{-8}{-2} > \frac{-2x}{-2} > \frac{2}{-2}$

$4 > x > -1$

$-1 < x < 4$

P. THEOREM: $|x| > a$ IF AND ONLY IF ($x > a$ OR $x < -a$)

1.) $|x| > 7$

$x > 7$ OR $x < -7$

2) $|7 - 2x| > 3$

$7 - 2x > 3$ OR $7 - 2x < -3$

$-2x > -4$ OR $-2x < -10$

$\frac{-2x}{-2} < \frac{-4}{-2}$ OR $\frac{-2x}{-2} > \frac{-10}{-2}$

$x < 2$ OR $x > 5$

IMPLICATIONS

I. IF p, THEN q.
p IMPLIES q.
$p \to q$.
} ALL MEAN THE SAME

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

<u>EXAMPLE</u>: "$1+1=3$ IMPLIES $2+2=4$" IS <u>TRUE</u>, SINCE IT IS FALSE $\to$ TRUE.

II. A WAY TO PROVE AN IMPLICATION TRUE (DIRECT PROOF).

A. DIRECT PROOF OF "IF p, THEN q."

ASSUME p TRUE. (SHOW q TRUE.)

THIS IS HOW YOU "<u>BEGIN</u>"

B. <u>EXAMPLE</u>: PROVE: IF $x<7$, THEN $-3x+5>-16$.

1. ASSUME $x<7$. (SHOW $-3x+5>-16$.)
2. $-3x>-21$ 1, MULT. BY -3.
3. $-3x+5>-21+5$ 2, ADD. 5.
4. $-3x+5>-16$ 3

C. HOMEWORK: PROVE EACH OF THE FOLLOWING.

1. IF $x<5$, THEN $2x-7<3$.
2. IF $x<-3$, THEN $-2x-3>3$.

* 3. IF $\varepsilon>0$ AND $x>\frac{1}{\varepsilon}$, THEN $\frac{1}{x}<\varepsilon$

* 4. IF $\varepsilon>0$ AND $x>0$, AND $x>\frac{7-\varepsilon}{\varepsilon}$, THEN $\frac{7}{x+1}<\varepsilon$

* WORKED ON NEXT PAGE.

IIC 3 PROOF: PROVE IF $\varepsilon > 0$ AND $x > \frac{1}{\varepsilon}$, THEN $\frac{1}{x} < \varepsilon$.

1. ASSUME $\varepsilon > 0$ AND $x > \frac{1}{\varepsilon}$ (SHOW $\frac{1}{x} < \varepsilon$)
2. $x\varepsilon > 1$ — 1, MULTIPLY BOTH SIDES BY $\varepsilon > 0$
3. $\frac{1}{\varepsilon} > 0$ — 1, SINCE $\varepsilon > 0$
4. $x > \frac{1}{\varepsilon} > 0$ — 1, 3
5. $\varepsilon > \frac{1}{x}$ — 2, 4, DIVIDE 2 BY $x > 0$
6. $\frac{1}{x} < \varepsilon$ — 5

IIC 4 PROOF: IF $\varepsilon > 0$, $x > 0$, AND $x > \frac{7-\varepsilon}{\varepsilon}$, THEN $\frac{7}{x+1} < \varepsilon$

1. ASSUME $\varepsilon > 0, x > 0$, AND $x > \frac{7-\varepsilon}{\varepsilon}$ (SHOW $\frac{7}{x+1} < \varepsilon$)
2. $\varepsilon x > 7 - \varepsilon$ — 1, MULTIPLY $x > \frac{7-\varepsilon}{\varepsilon}$ BY $\varepsilon > 0$
3. $\varepsilon x + \varepsilon > 7$ — 2, ADD ε
4. $\varepsilon(x+1) > 7$ — 3
5. $x + 1 > 0 + 1 = 1 > 0$ — 1, ADD 1 TO $x > 0$
6. $\varepsilon > \frac{7}{x+1}$ — 4, DIVIDE BY $x+1 > 0$, 5
7. $\frac{7}{x+1} < \varepsilon$ — 6

"EVERY" STATEMENTS ; "FOR EVERY" STATEMENTS

I Form: For each $x \in A$, $x < 7$
For any $x \in A$, $x < 7$
For all $x \in A$, $x < 7$
Every element of A is less than 7.

II A way to prove "Every" statements (Direct)
Assume $x \in A$ (Show $x < 7$)

III Illustrations of proof starts.

A. Prove: For each $x \in A$, $x < 23$
Assume $x \in A$ (Show $x < 23$)

B. Prove: Every element of A is an element of B.
Assume $x \in A$ (Show $x \in B$)

C. Prove: For each real number x, if $x < 7$, then $4x - 2 < 26$.
Assume x is a real number (Show if $x < 7$, then $4x - 2 < 26$)
Assume $x < 7$ (Show $4x - 2 < 26$)

IV COMPLETE PROOF. LET $H = \{x \mid x > -2\}$
Prove: For any $x \in H$, $-3x + 4 < 10$

1. Assume $x \in H$ (Show $-3x+4 < 10$)	
2. $x > -2$	1, def. of H
3 $-3x < 6$	2, multiply by -3
4. $-3x+4 < 10$	3, add 4

V. HOMEWORK: PROVE EACH OF THE FOLLOWING: $H = \{x \mid x > -2\}$

A. FOR EACH $x \in H$, $-7x+12<26$.

B. FOR EACH $y \in H$, $-2y<10$.

C. FOR EACH REAL NUMBER x, (IF $x>3$, THEN $-4x+10<-2$).

* D. FOR ANY REAL NUMBER x, (IF $x<-5$, THEN $-2x-12 \in H$).

* E. FOR EACH $\varepsilon>0$, (IF $x>\frac{1}{2\varepsilon}$, THEN $\frac{1}{2x}<\varepsilon$).

F. LET $J=\{x \mid 4x+8>0\}$.
PROVE: EVERY ELEMENT OF J IS AN ELEMENT OF H.

* WORKED ON NEXT PAGE

PROOF OF V, D: FOR EACH REAL NUMBER x, IF $x<-5$, THEN $-2x-12 \in H$ $\qquad H=\{x \mid x>-2\}$

1. ASSUME x IS A REAL NUMBER (SHOW IF $x<-5$, THEN $-2x-12 \in H$)
2. ASSUME $x<-5$ (SHOW $-2x-12 \in H$)
3. $-2x > (-2)(-5) = 10$ — 2, MULTIPLY BY -2
4. $-2x-12 > -2$ — 3, SUBTRACT 12
5. $-2x-12 \in H$ — 4, DEFINITION OF H

PROOF OF V, E: FOR EACH $\varepsilon>0$, IF $x>\frac{1}{2\varepsilon}$, THEN $\frac{1}{2x}<\varepsilon$.

1. ASSUME $\varepsilon>0$ (SHOW IF $x>\frac{1}{2\varepsilon}$, THEN $\frac{1}{2x}<\varepsilon$).
2. ASSUME $x>\frac{1}{2\varepsilon}$ (SHOW $\frac{1}{2x}<\varepsilon$)
3. $x\varepsilon > \frac{1}{2}$ — 2, MULTIPLY BY $\varepsilon>0$, 1
4. $\frac{1}{2\varepsilon}>0$ SINCE $\varepsilon>0$ — 1
5. $x > \frac{1}{2\varepsilon} > 0$ — 2, 4
6. $\varepsilon > \frac{1}{2x}$ — 3, DIVIDE BY $x>0$, 5
7. $\frac{1}{2x} < \varepsilon$ — 6

IF AND ONLY IF

I. p if and only if q
p iff q
$p \leftrightarrow q$

all mean the same thing

II. When you know "$p \leftrightarrow q$" is true that means p and q have a Common fate (i.e. when p is true, q is true. When q is true, p is true. When p is false, q is false. When q is false, p is false.).

III How to prove $p \leftrightarrow q$

Prove $p \rightarrow q$ and $q \rightarrow p$. That is, assume p true (show q true). Then, assume q true (show p true)

IV Examples of how to start "iff" proofs.

A. <u>Prove</u>: $|x| < 7$ iff $-7 < x < 7$

<u>Proof</u>: Part(1): Assume $|x| < 7$

⋮ (Show $-7 < x < 7$)

Part(2): Assume $-7 < x < 7$ (Show $|x| < 7$)

⋮

4A

IV B. Prove: For each real number x, x is even iff $x+2$ is even.

Proof: 1. Assume x is a real number (Show x is even iff $x+2$ is even)

Part(1) 2. Assume x is even (Show $x+2$ is even)

⋮

Part(2) 27. Assume $x+2$ is even (Show x is even)

⋮

$\boxed{4B}$

IV C. Prove: For each $\varepsilon > 0$, [if $x > 0$, then $\left(\frac{1}{x} < \frac{1}{\varepsilon} \text{ iff } x > \varepsilon\right)$]

<u>Proof:</u> 1. Assume $\varepsilon > 0$ [Show if $x > 0$, then $\left(\frac{1}{x} < \frac{1}{\varepsilon} \text{ iff } x > \varepsilon\right)$]

2. Assume $x > 0$ [Show $\frac{1}{x} < \frac{1}{\varepsilon}$ iff $x > \varepsilon$]

3. <u>Part (1)</u> Assume $\frac{1}{x} < \frac{1}{\varepsilon}$ (Show $x > \varepsilon$)

4.	$x\varepsilon > 0$	1, 2
5.	$\frac{1}{x}(x\varepsilon) < \frac{1}{\varepsilon}(x\varepsilon)$	3, 4
6.	$\varepsilon < x$	5
7	$x > \varepsilon$	6

8. <u>Part (2)</u> Assume $x > \varepsilon$ (Show $\frac{1}{x} < \frac{1}{\varepsilon}$)

9.	$x\varepsilon > 0$	1, 2
10.	$\frac{1}{x\varepsilon} > 0$	9
11.	$x\left(\frac{1}{x\varepsilon}\right) > \varepsilon\left(\frac{1}{x\varepsilon}\right)$	8, 10
12.	$\frac{1}{\varepsilon} > \frac{1}{x}$	11
13.	$\frac{1}{x} < \frac{1}{\varepsilon}$	12

4C

PROOF OF IV, D: FOR EVERY $\varepsilon > 0$, [IF $x > 0$, THEN $\frac{1}{x} < \varepsilon$ IFF $x > \frac{1}{\varepsilon}$]

1. ASSUME $\varepsilon > 0$ (SHOW IF $x > 0$, THEN $\frac{1}{x} < \varepsilon$ IFF $x > \frac{1}{\varepsilon}$)
2. ASSUME $x > 0$ (SHOW $\frac{1}{x} < \varepsilon$ IFF $x > \frac{1}{\varepsilon}$)
3. PART (1) ASSUME $\frac{1}{x} < \varepsilon$ (SHOW $x > \frac{1}{\varepsilon}$)
4. $1 < \varepsilon x$ — 3, MULTIPLY BY $x > 0$, 2
5. $\frac{1}{\varepsilon} < x$ — 4, DIVIDE BY $\varepsilon > 0$, 1
6. $x > \frac{1}{\varepsilon}$ — 5
7. PART (2) ASSUME $x > \frac{1}{\varepsilon}$ (SHOW $\frac{1}{x} < \varepsilon$)
8. $x\varepsilon > 1$ — 7, MULTIPLY BY $\varepsilon > 0$, 1
9. $\varepsilon > \frac{1}{x}$ — 8, DIVIDE BY $x > 0$, 2
10. $\frac{1}{x} < \varepsilon$ — 9

PROOF BY CASES

I. p OR q
$p \to r$
$q \to r$
$\therefore r$

EXAMPLE: 10. $x \in A$ OR $x \in B$
11. CASE 1 $x \in A$
⋮
17 $x \in C$
18. CASE 2 $x \in B$
⋮
25 $x \in C$
WE CAN THEREFORE CONCLUDE $x \in C$

II ANOTHER FORM OF PROOF BY CASES:

p OR q
$p \to r$
$q \to s$
$\therefore r$ OR s

EXAMPLE: 10. $x \in A$ OR $x \in B$
11. CASE 1: $x \in A$
⋮
17 $x \in C$
18. CASE 2: $x \in B$
⋮
25. $x \in D$

WE COULD THEREFORE CONCLUDE $x \in C$ OR $x \in D$

PROOFS CONCERNING ABSOLUTE VALUES USING PROOF BY CASES

THEOREM A-1: FOR EACH REAL NUMBER x, $|x| \geq 0$.

PROOF:

1. ASSUME x IS A REAL NUMBER. (SHOW $|x| \geq 0$)
2. $x \geq 0$ OR $x < 0$ — 1, TRICHOTOMY
3. CASE 1: $x \geq 0$. (SHOW $|x| \geq 0$)
4. $|x| = x$ — 3, DEF. OF ABS. VALUE
5. $|x| \geq 0$ — 3, 4, SUBSTITUTE ='s FOR ='s
6. CASE 2 $x < 0$. (SHOW $|x| > 0$)
7. $-x > 0$ — 6, MULT. BY -1 BOTH SIDES
8. $|x| = -x$ (neg) — 6, DEF. OF ABS. VALUE.
9. $|x| > 0$ — 7, 8, SUBSTITUTE ='s FOR ='s
10. $|x| \geq 0$ — 9

PROOFS CONCERNING ABSOLUTE VALUES (CONT.)
USING PROOF BY CASES.

THEOREM A-2: FOR EACH REAL NUMBER x, IF $x \leq 0$, THEN $|x| = -x$.

PROOF:

1. ASSUME x IS A REAL NUMBER. (SHOW IF $x \leq 0$, THEN $|x| = -x$.)
2. ASSUME $x \leq 0$. (SHOW $|x| = -x$)
3. $x < 0$ OR $x = 0$. 2
4. CASE 1 $x < 0$. (SHOW $|x| = -x$).
5. $|\underset{neg}{x}| = -x$. 4, DEF. OF ABS. VALUE.
6. CASE 2 $x = 0$. (SHOW $|x| = -x$).
7. $x > 0$ OR $x = 0$ 6
8. $x \geq 0$ 7
9. $|x| = x$ 8, DEF. OF ABS. VALUE.
10. $x \underset{6}{=} 0 = -x$
11. $|x| = -x$ 9, 10, SUBSTITUTE ='s FOR ='s

Theorem A-3 For each real x, $|x|<a$ iff $-a<x<a$

Proof: 1. Assume x is a real (Show $|x|<a$ iff $-a<x<a$)

2. $\boxed{\text{Part I}}$ Assume $|x|<a$ (Show $-a<x<a$)

3. $x \geq 0$ OR $x<0$ Trichotomy

4. Case 1 $x \geq 0$ (Show $-a<x<a$)

$4\frac{1}{2}$ $|x|=x$ 4, def. abs. val.

5. $x<a$ 2, $4\frac{1}{2}$, substitute = for ='s

6. $a>0$ 4, 5, transitive

7. $-a<0$ 6

8. $-a \overset{7}{<} 0 \overset{4}{\leq} x \overset{5}{<} a$

9. $-a<x<a$ 8

10. Case 2 $x<0$ (Show $-a<x<a$)

11 $|x|=-x$ 10, def. of abs. value

12. $-x<a$ 2, 11

13. $-x>0$ 10

14. $x>-a$ 12

15. $-a \overset{14}{<} x \overset{10}{<} 0 \overset{13}{<} -x \overset{11}{=} |x| \overset{2}{<} a$

16. $-a<x<a$

$\boxed{6'}$

17. $\boxed{\text{Part II}}$ Assume $-a<x<a$ (Show $|x|<a$)

18. $x \geq 0$ OR $x<0$

19. <u>Case 1</u> $x \geq 0$ (Show $|x|<a$)

20. $|x| = x$ 19, def of abs value

21. $|x|<a$ 17, 20

22. <u>Case 2</u> $x<0$ (Show $|x|<a$)

23. $|x| = -x$ 22, def. of abs. value

24 $a > -x$ 17, mult. by -1

25. $a > |x|$ 24, 23, subst. = for ='s

26. $|x| < a$ 25

6A Notes

$$|x| > a \longrightarrow (x < -a \text{ OR } x > a)$$

$|x| > a$

$x \geq 0$	$x < 0$
$\lvert x\rvert = x$	$\lvert x\rvert = -x$
$x > a$	$-x > a$
$x < -a$ OR $x > a$	$x < -a$
	$x < -a$ OR $x > a$

$$(x < -a \text{ OR } x > a) \longrightarrow |x| > a$$

$a \geq 0$

$x < -a$ or $x > a$

$x < -a$	$x > a$
$-x > a$	$x \geq 0$
$-x > 0$	$\lvert x\rvert = x$
$x < 0$	$\lvert x\rvert > a$
$\lvert x\rvert = -x$	
$\lvert x\rvert > a$	

$a < 0$

$|x| \geq 0$ Thm

$0 > a$

$|x| > a$

Theorem A-4 For each real number x, $[|x|>a \text{ iff } x<-a \text{ or } x>a]$

Proof: 1. Assume x is a real number
(Show $|x|>a$ iff $x<-a$ or $x>a$)

PART I SHOW $|x|>a \longrightarrow (x<-a \text{ OR } x>a)$

2. Assume $|x|>a$ (Show $x<-a$ OR $x>a$)
3. $x \geq 0$ or $x<0$ Trichotomy
4. Case 1: $x \geq 0$ (Show $x<-a$ OR $x>a$)
5. $|x|=x$ 4, definition of absolute value
6. $x>a$ 2,5, substitute ='s for ='s
7. $x<-a$ or $x>a$ 6, definition of OR
8. Case 2: $x<0$ (Show $x<-a$ OR $x>a$)
9. $|x|=-x$ 8, definition of absolute value
10. $-x>a$ 2,9, substitute ='s for ='s
11. $x<-a$ 10 multiply by -1
12. $x<-a$ or $x>a$ 11 definition of OR

PART II Show $(x<-a \text{ OR } x>a) \longrightarrow |x|>a$

13. Assume $(x<-a$ OR $x>a)$ (Show $|x|>a$)
14. $a \geq 0$ OR $a<0$ Trichotomy
15. Case 1: $a \geq 0$ (Show $|x|>a$)
16. Case 1A: $x<-a$ (Show $|x|>a$)
17. $-x>a$ 16, multiply by -1
18. $-x>0$ 17, 15, Transitive

19. $x<0$ 18, multiply by -1
20. $|x|=-x$ 19, definition of absolute value
21. $|x|>a$ 17, 20, substitute ='s for ='s
22. <u>Case 1B</u>: $x>a$ (Show $|x|>a$)
23. $x\geq 0$ 22, 15, transitive
24. $|x|=x$ 23, definition of absolute value.
25. $|x|>a$ 22, 24, substitute ='s for ='s.
26. <u>Case 2</u>: $a<0$ (Show $|x|>a$)
27. $|x|\geq 0$ Theorem A-1
28. $0>a$ 26
29. $|x|>a$ 27, 28, Transitive

<u>Theorem A-4a</u> For every real number x, if $x\neq 0$, then $|x|\neq 0$

Theorem A-5 For all real numbers a, b , $|ab| = |a||b|$

Proof: 1. Assume each of a and b is a real number (Show $|ab| = |a||b|$)

2. Case 1 $a \geq 0$ and $b \geq 0$

3.	$ab \geq 0$	2
4.	$\lvert ab \rvert = ab$	
5	$\lvert a \rvert = a$	2
6	$\lvert b \rvert = b$	2
7.	$\lvert a \rvert \lvert b \rvert = ab$	5, 6
8	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	7, 4

9. Case 2 $a \geq 0$ and $b < 0$

⋮

Case 3 $a < 0$ and $b \geq 0$

⋮

Case 4 $a < 0$ and $b < 0$

⋮

Theorem A-6 For all real numbers a, b, if $b \neq 0$, then $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Proof is similar to above

THERE EXIST STATEMENTS

I. There exists $x \in A$ such that $x \in B$.
There is $x \in A$ such that $x \in B$.
For some $x \in A$, $x \in B$.
→ THESE

ALL MEAN THE SAME THING WHICH IS: There is at least one element in A that is in the set B.

II A way to prove a "THERE EXISTS" statement is to exhibit one element with the desired property.

$$\text{Let } T = \{x \mid 5x - 3 < 8\}$$

EXAMPLE 1: PROVE THERE IS A POSITIVE INTEGER THAT IS AN ELEMENT OF T

1.	2 IS A POSITIVE INTEGER.	
2.	$5(2) - 3 = 7$	
3.	$5(2) - 3 < 8$	2
4.	$2 \in T$	3, DEFINITION OF T
5	THERE IS A POSITIVE INTEGER THAT IS AN ELEMENT OF T. (namely 2, 1, 4)	

EXAMPLE 2: THERE IS A NEGATIVE INTEGER n SUCH THAT $n \in T$

1.	LET $n = -1$.	
2.	n IS A NEGATIVE INTEGER.	1
3.	$5n - 3 = 5(-1) - 3 = -8$	1
4.	$5n - 3 < 8$	3
5	$n \in T$	4, DEFINITION OF T
6.	THERE IS A NEGATIVE INTEGER n SUCH THAT $n \in T$	2,5

<u>EXAMPLE 3</u>: PROVE FOR ALL REAL NUMBERS x, y IF $x < y$, THEN THERE IS A REAL NUMBER b SUCH THAT $x < b < y$.

1. ASSUME x AND y ARE REAL NUMBERS (SHOW IF $x < y$, THEN THERE IS A REAL NUMBER b SUCH THAT $x < b < y$)
2. ASSUME $x < y$ (SHOW THERE IS A REAL NUMBER b SUCH THAT $x < b < y$)
3. $x + x < x + y$ 2, ADD x
4. $2x < x + y$ 3
5. $x + y < y + y$ 2, ADD y
6. $x + y < 2y$ 5
7. $x < \frac{x+y}{2}$ 4, DIVIDE BY 2
8. $\frac{x+y}{2} < y$ 6, DIVIDE BY 2
9. $x < \frac{x+y}{2} < y$ 7, 8
10. $\frac{x+y}{2}$ IS A REAL NUMBER   1, CLOSURE
11. THERE IS A REAL NUMBER b SUCH THAT $x < b < y$ (NAMELY $b = \frac{x+y}{2}$, 9, 10)

Recall $|a-b|$ is the distance between a and b.

THEOREM 9-1 For each $\varepsilon > 0$, there is a positive real number P such that for all $x > P$, $\left|\frac{1}{x} - 0\right| < \varepsilon$

Proof: 1. Assume $\varepsilon > 0$ (Show there is a positive real number P such that for all $x > P$, $\left|\frac{1}{x} - 0\right| < \varepsilon$)

2. Let $P = \frac{1}{\varepsilon}$

3. P is a positive real number   1, 2
   (Show for all $x > P$, $\left|\frac{1}{x} - 0\right| < \varepsilon$)

4. Assume $x > P$ (Show $\left|\frac{1}{x} - 0\right| < \varepsilon$)

5. $x > 0$ 2, 3, Since $x > P$ and P is positive.

6. $x > \frac{1}{\varepsilon}$ 4, 2

7. $\frac{\varepsilon}{x} > 0$ 1, 5

8. $\frac{\varepsilon}{x} \cdot x > \frac{\varepsilon}{x} \cdot \frac{1}{\varepsilon}$ 6, 7, mult. both sides by same pos. no.

9. $\varepsilon > \frac{1}{x}$ 8

10 $\frac{1}{x} < \varepsilon$ 9

11. $\frac{1}{x}$ IS POSITIVE   5

12. $\left|\frac{1}{x}\right| = \frac{1}{x}$ 11, Def^n. of absolute value

13. $\left|\frac{1}{x}\right| < \varepsilon$ 12, 10

14. $\left|\frac{1}{x} - 0\right| < \varepsilon$ 13

THEOREM 9-2 For each $\varepsilon > 0$, there is a positive real number P such that for all $x > P$,
$$\left|\frac{-7}{x} - 0\right| < \varepsilon$$

Proof:

1. Assume $\varepsilon > 0$ (Show there is a positive real number P such that for all $x > P$, $\left|\frac{-7}{x} - 0\right| < \varepsilon$)
2. Let $P = \frac{7}{\varepsilon}$
3. P is a positive real number 1, 2
   (Show for all $x > P$, $\left|-\frac{7}{x} - 0\right| < \varepsilon$)
4. Assume $x > P$ (Show $\left|-\frac{7}{x} - 0\right| < \varepsilon$)

5.	x is positive	4, 3
6.	$x > \frac{7}{\varepsilon}$	4, 2
7.	$x\varepsilon > 7$	6, mult. by <u>positive</u> ε
8.	$\varepsilon > \frac{7}{x}$	7, 5, Divide by <u>positive</u> x
9.	$\lvert x\rvert = x$	5, Def^n of absolute value
10.	$\lvert -7\rvert = 7$	Def of absolute value
11.	$\varepsilon > \frac{\lvert -7\rvert}{\lvert x\rvert}$	8, 9, 10
12.	$\left\lvert\frac{-7}{x}\right\rvert = \frac{\lvert -7\rvert}{\lvert x\rvert}$	Theorem A-6
13.	$\varepsilon > \left\lvert\frac{-7}{x}\right\rvert$	11, 12
14.	$\left\lvert-\frac{7}{x}\right\rvert < \varepsilon$	13
15	$\left\lvert-\frac{7}{x} - 0\right\rvert < \varepsilon$	14

<u>THEOREM 10-1</u>: Given c is a real number.
For each $\varepsilon > 0$, there is a positive real number P such that for each $x > P$,

$$\left|\frac{c}{x} - 0\right| < \varepsilon$$

Proof of Theorem 10-1

1. $c \neq 0$ or $c = 0$
2. <u>Case 1</u>: $c \neq 0$ (Show For every $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $\left|\frac{c}{x} - 0\right| < \varepsilon$)
3. Assume $\varepsilon > 0$ (Show there is a $P > 0$ such that for all $x > P$, $\left|\frac{c}{x} - 0\right| < \varepsilon$)
4. Let $P = \frac{|c|}{\varepsilon}$
5. $|c| \neq 0$ — 2, Theorem A-4a
6. $|c| \geq 0$ — Theorem A-1
7. $|c| > 0$ — 5, 6
8. $P > 0$ — 3, 7, 4

(Show For all $x > P$, $\left|\frac{c}{x} - 0\right| < \varepsilon$)

9. Assume $x > P$ (Show $\left|\frac{c}{x} - 0\right| < \varepsilon$)
10. $x > \frac{|c|}{\varepsilon}$ — 9, 4, Substitution
11. $x\varepsilon > |c|$ — 10, 3, multiply by ε
12. $x > 0$ — 9, 8, transitivity
13. $\varepsilon > \frac{|c|}{x}$ — 11, 12, divide by x
14. $|x| = x$ — 12, definition of abs. value
15. $\varepsilon > \frac{|c|}{|x|}$ — 13, 14, substitution
16. $\varepsilon > \left|\frac{c}{x}\right|$ — 15, Theorem A-6
17. $\left|\frac{c}{x}\right| < \varepsilon$ — 16
18. $\left|\frac{c}{x} - 0\right| < \varepsilon$ — 17
19. <u>Case 2</u>: $c = 0$ (Show For every $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $\left|\frac{c}{x} - 0\right| < \varepsilon$)
20. Assume $\varepsilon > 0$ (Show there is a $P > 0$ such that for all $x > P$, $\left|\frac{c}{x} - 0\right| < \varepsilon$)
21. Let $P = 42 > 0$ (Show for all $x > P$, $\left|\frac{c}{x} - 0\right| < \varepsilon$)
22. $0 \overset{\text{def}}{=} |0| \overset{19}{=} \left|\frac{c}{x}\right| = \left|\frac{c}{x} - 0\right|$
23. $\varepsilon \overset{20}{>} 0 \overset{22}{=} \left|\frac{c}{x} - 0\right|$
24. $\left|\frac{c}{x} - 0\right| < \varepsilon$ — 23

21.5 Assume $x > P$
(Show $\left|\frac{c}{x} - 0\right| < \varepsilon$)

Lemma 11-1 For all real numbers L, R, if $0<L<R$, then $L^2 < R^2$

Proof: ① Assume L, R are reals (Show [redacted] if $0<L<R$, then $L^2<R^2$)

② Assume $0<L<R$ (Show $L^2<R^2$)

③ $L^2 < LR$ (2, mult both sides of $L<R$ by pos L)

④ $LR < R^2$ (2, " $L<R$ " pos R)

⑤ $L^2 < R^2$ (3,4)

Theorem 11-1 For each $\varepsilon>0$, there is a positive real number P such that for each real number $x > P$, $\left|\frac{1}{x^2} - 0\right| < \varepsilon$.

Proof: ① Assume $\varepsilon>0$ (Show there is a positive real number P such that for each real number $x>P$, $\left|\frac{1}{x^2}-0\right|<\varepsilon$)

② Let $P = \frac{1}{\sqrt{\varepsilon}}$; P is a pos. real (Show for each real number $x>P$, $\left|\frac{1}{x^2}-0\right|<\varepsilon$)

③ Assume x is a real number $>P$ (Show $\left|\frac{1}{x^2}-0\right|<\varepsilon$)

④ $x > \frac{1}{\sqrt{\varepsilon}}$ (2,3)

⑤ $x\sqrt{\varepsilon} > 1$ 4

⑥ $x>0$ 4, 2

⑦ $\sqrt{\varepsilon} > \frac{1}{x}$ 5, 6

⑧ $0<\frac{1}{x}<\sqrt{\varepsilon}$ 7, 6

⑨ $\frac{1}{x^2} < \varepsilon$ 8, Lemma 11-1

⑩ $\frac{1}{x^2} \geq 0$

⑪ $\left|\frac{1}{x^2}\right|$ [redacted] $\frac{1}{x^2}$ 10

⑫ $\left|\frac{1}{x^2}\right| < \varepsilon$

⑬ $\left|\frac{1}{x^2} - 0\right| < \varepsilon$

Another theorem that could be proven is: Given n is a pos. int

Theorem 11-2 For each $\varepsilon>0$, there is a positive real number P such that for each real number $x > P$, $\left|\frac{1}{x^n} - 0\right| < \varepsilon$

Lemma 11-2: For every real number R, if $R>0$, then for every positive integer n, $R^n > 0$

<u>Proof by induction</u>: To prove for each positive integer n, $P(n)$ true by induction do the following

Prove $P(1)$ true.

Assume n is a positive integer and $P(n)$ is true

Show $P(n+1)$ is true.

<u>Definition 11A-1</u>: For $x \geq 0$, for all positive <u>even</u> integers n, $\sqrt[n]{x}$ is the nonnegative number y such that $y^n = x$. For any $x \in R$, for all odd positive integers n, $\sqrt[n]{x}$ is the number y such that $y^n = x$.

<u>Lemma 11A-1</u>: For every positive integer n, for every $\varepsilon > 0$, $\sqrt[n]{\varepsilon} > 0$.

<u>Lemma 11A-2</u> For all reals L, R, if $0 < L < R$ then for every positive integer n, $L^n < R^n$..

Proof: (Mallow)

1. Assume L and R are real numbers. (Show if $0<L<R$, then for every positive integer n, $L^n < R^n$)
2. Assume $0 < L < R$ (Show by mathematical induction - for every positive integer n, $L^n < R^n$)
3. $L^1 < R^1$ — 2
4. Assume n is a positive integer and $L^n < R^n$ (Show $L^{n+1} < R^{n+1}$)
5. $L^{n+1} = L^n \cdot L < L \cdot R^n$ — 4, mult. by L, 2
6. $L \cdot R^n < R \cdot R^n = R^{n+1}$ — mult $L<R$ by $R^n > 0$ Lemma 11-2
7. $L^{n+1} < R^{n+1}$ — 5, 6, Transitive

11 B

Proof of Theorem 11-2 (Mallow)

Prove: For each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $\left|\frac{1}{x^n} - 0\right| < \varepsilon$

1. Assume $\varepsilon > 0$ (Show there is a $P > 0$ such that for all $x > P$, $\left|\frac{1}{x^n} - 0\right| < \varepsilon$)
2. Let $P = \frac{1}{\sqrt[n]{\varepsilon}} > 0$ — 1, Lemma 11A-1

   (Show for all $x > P$, $\left|\frac{1}{x^n} - 0\right| < \varepsilon$)
3. Assume $x > P$ (Show $\left|\frac{1}{x^n} - 0\right| < \varepsilon$)
4. $x > 0$ — 3, 2, Transitivity
5. $x > \frac{1}{\sqrt[n]{\varepsilon}}$ — 3, 2 Substitution
6. $x\sqrt[n]{\varepsilon} > 1$ — 5, multiply by $\sqrt[n]{\varepsilon} > 0$, 1, Lemma 11A-1
7. $\sqrt[n]{\varepsilon} > \frac{1}{x} \overset{4}{>} 0$ — 6, 4, divide by x
8. $\varepsilon > \frac{1}{x^n}$ — 7, Lemma 11A-2
9. $\frac{1}{x^n} > 0$ — 4, Lemma 11-2
10. $\left|\frac{1}{x^n}\right| = \frac{1}{x^n}$ — 9, definition of absolute value
11. $\varepsilon > \left|\frac{1}{x^n}\right|$ — 8, 10, Substitution
12. $\left|\frac{1}{x^n}\right| < \varepsilon$ — 11
13. $\left|\frac{1}{x^n} - 0\right| < \varepsilon$ — 12.

Let $f(x) = \frac{3x+1}{x}$

Theorem 12-1 For each $\varepsilon > 0$, there is a positive real number P such that for all real numbers $x > P$, $|f(x) - 3| < \varepsilon$.

Proof: ① Assume $\varepsilon > 0$ (Show there is a positive real number P such that for all real numbers $x > P$, $|f(x) - 3| < \varepsilon$)

② Let $P = \frac{1}{\varepsilon}$; P is a positive real. (Show for all real numbers $x > P$, $|f(x) - 3| < \varepsilon$)

③ Assume x is a real number $> P$. (Show $|f(x) - 3| < \varepsilon$)

④	$x > 1/\varepsilon > 0$	2,3
⑤	$x \geq 0$	4
⑥	$\lvert x \rvert = x$	5
⑦	$\lvert x \rvert > 1/\varepsilon$	4,6
⑧	$\varepsilon \lvert x \rvert > 1$	7
⑨	$\varepsilon > \frac{1}{\lvert x \rvert}$	8
⑩	$\frac{1}{\lvert x \rvert} < \varepsilon$	9
⑪	$\frac{\lvert 1 \rvert}{\lvert x \rvert} < \varepsilon$	10, Def$^{\underline{n}}$ of abs. value
⑫	$\left\lvert \frac{1}{x} \right\rvert < \varepsilon$	11, Th$^{\underline{m}}$ A-6
⑬	$\left\lvert \left(3 + \frac{1}{x}\right) - 3 \right\rvert < \varepsilon$	12
⑭	$\left\lvert \frac{3x+1}{x} - 3 \right\rvert < \varepsilon$	13
⑮	$\lvert f(x) - 3 \rvert < \varepsilon$	14

Theorem 12-2. Given c is a real number and for each $x \in \mathbb{R}$, $f(x) = c$. For each $\varepsilon > 0$, there is a positive [redacted] real number P such that for all real numbers $x > P$, $|f(x) - c| < \varepsilon$

Proof: ① Assume $\varepsilon > 0$ (Show there is a positive real number P such that for all real numbers $x > P$, $|f(x) - c| < \varepsilon$)

② Let $P = 37$; P is a positive real. (Show for all real numbers $x > P$, $|f(x) - c| < \varepsilon$.

Assume $x > P$ (Show $|f(x) - c| < \varepsilon$)

③	$0 < \varepsilon$	(1)
④	$\lvert 0 \rvert < \varepsilon$	3
⑤	$\lvert c - c \rvert < \varepsilon$	4
⑥	$\lvert f(x) - c \rvert < \varepsilon$	

Lemma 13-1 $|a+b| \le |a| + |b|$

Theorem 13-2 **If** ① for each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|f(x) - L| < \varepsilon$ **AND** ② for each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|g(x) - M| < \varepsilon$ **THEN** for each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|(f(x)+g(x)) - (L+M)| < \varepsilon$.

Proof: 1. Assume ① for each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|f(x) - L| < \varepsilon$ AND ② for each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|g(x) - M| < \varepsilon$ (Show for each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|(f(x)+g(x)) - (L+M)| < \varepsilon$)

2. Assume $\varepsilon' > 0$ (Show there is a $P > 0$ such that for all $x > P$, $|(f(x)+g(x)) - (L+M)| < \varepsilon'$)

3. $\frac{\varepsilon'}{2} > 0$ 2

4. There is a $P_1 > 0$ such that for all $x > P_1$, $|f(x) - L| < \frac{\varepsilon'}{2}$ 1,3

5. There is a $P_2 > 0$ such that for all $x > P_2$, $|g(x) - M| < \frac{\varepsilon'}{2}$ 1,3

6. Let $P = \max\{P_1, P_2\}$, $P \geq P_1 > 0$, $P \geq P_2 > 0$ so $P > 0$ 4,5

(Show for all $x > P$ $|(f(x)+g(x)) - (L+M)| < \varepsilon'$)

7. Assume $x > P$ (Show $|(f(x)+g(x)) - (L+M)| < \varepsilon'$)

8. $x > P \overset{6}{\geq} P_1$ so $x > P_1$

9. $|f(x) - L| < \frac{\varepsilon'}{2}$ 8,4

10. $x > P \overset{6}{\geq} P_2$ so $x > P_2$

11. $|g(x) - M| < \frac{\varepsilon'}{2}$ 10,5

12. $|(f(x)+g(x)) - (L+M)| = |f(x)+g(x) - L - M|$

$= |(f(x) - L) + (g(x) - M)|$

$\overset{\text{Lemma 13.1}}{\leq} |f(x) - L| + |g(x) - M| \overset{9,11}{<} \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'$

Theorem 14-1 [C, L constants] **IF** for each $\varepsilon > 0$, there is a $P > 0$, such that for all $x > P$,

$$|f(x) - L| < \varepsilon$$

THEN for each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|C f(x) - CL| < \varepsilon$

Proof: 1. Assume for each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|f(x) - L| < \varepsilon$ (Show for each $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$ $|c f(x) - cL| < \varepsilon$

2. Assume $\varepsilon > 0$ (Show there is a $P > 0$ such that for all $x > P$, $|c f(x) - cL| < \varepsilon$)

3. $|c| \geq 0$ $\quad$ Th^m A-1

4. $1 > 0$

5. $|c| + 1 > |c| + 0 = |c| \geq 0$ $\quad$ 4, Add $|c|$, 3

6. $|c| + 1 > 0$ $\quad$ 5

7. $\frac{\varepsilon}{|c|+1} > 0$ $\quad$ 2, 7

8. There is a $P_1 > 0$ such that for all $x > P_1$, $|f(x) - L| < \frac{\varepsilon}{|c|+1}$ $\quad$ 1, 7

9. Assume $x > P_1$ (Show $|c f(x) - cL| < \varepsilon$)

10. $|f(x) - L| < \frac{\varepsilon}{|c|+1}$ $\quad$ 8, 9

11. $|c| < |c| + 1$ $\quad$ Add $|c|$ to both sides of $0 < 1$

12. $\frac{|c|}{|c|+1} < 1$ $\quad$ 6, 11, divide 11 by $|c|+1$

13. $|c f(x) - cL| = |c(f(x) - L)| \overset{\text{Th}^m \text{A-5}}{=} |c|\,|f(x) - L|$

$$\underset{10}{\leq} |c| \frac{\varepsilon}{|c|+1} = \frac{|c|}{|c|+1}\varepsilon \underset{12}{<} 1 \cdot \varepsilon = \varepsilon$$

Theorem 14-2: For each $\varepsilon > 0$, there is an $N < 0$ such that for all $x < N$, $\left|\frac{1}{x} - 0\right| < \varepsilon$.

Proof: 1. Assume $\varepsilon > 0$ (Show there is an $N < 0$ such that for all $x < N$, $\left|\frac{1}{x} - 0\right| < \varepsilon$)

2. Let $N = \frac{-1}{\varepsilon}$; $N < 0$ 1

(Show for all $x < N$, $\left|\frac{1}{x} - 0\right| < \varepsilon$)

3. Assume $x < N$ (Show $\left|\frac{1}{x} - 0\right| < \varepsilon$)

4.	$x < \frac{-1}{\varepsilon}$	2, 3
5.	$x < 0$	4, 2
6.	$x\varepsilon < -1$	4
7.	$\varepsilon > -\frac{1}{x}$	6, 5, divide by x negative
8.	$\frac{1}{x} < 0$	5
9.	$\left\|\frac{1}{x}\right\| = -\frac{1}{x}$	8, def^n abs. val.
10.	$\varepsilon > \left\|\frac{1}{x}\right\|$	7, 9
11.	$\varepsilon > \left\|\frac{1}{x} - 0\right\|$	10
12.	$\left\|\frac{1}{x} - 0\right\| < \varepsilon$	11

Theorem 14-3 For each $\varepsilon > 0$ there is an $N < 0$ such that for all $x < N$, $\left|\frac{1}{x^2} - 0\right| < \varepsilon$

Proof: 1. Assume $\varepsilon > 0$ (Show there is an $N < 0$ such that for all $x < N$, $\left|\frac{1}{x^2} - 0\right| < \varepsilon$)

2. Let $N = \frac{-1}{\sqrt{\varepsilon}}$; $N < 0$ 1, Lemma 11A-1

(Show for all $x < N$, $\left|\frac{1}{x^2} - 0\right| < \varepsilon$)

3. Assume x is a real number $< N$ (Show $\left|\frac{1}{x^2} - 0\right| < \varepsilon$)

4.	$x < \frac{-1}{\sqrt{\varepsilon}}$	2, 3
5.	$-x > \frac{1}{\sqrt{\varepsilon}}$	4, multiply by -1
6.	$x < 0$	2, 3
7.	$\lvert x\rvert = -x$	6, Definition of absolute value
8.	$\lvert x\rvert > \frac{1}{\sqrt{\varepsilon}} > 0$	7, 5, 1
9.	$\lvert x\rvert^2 = \lvert x\rvert\cdot\lvert x\rvert = \lvert x\cdot x\rvert = \lvert x^2\rvert \overset{x^2 \geq 0}{=\!=} x^2$	Th^m A-5
10.	$x^2 = \lvert x\rvert^2 > \frac{1}{\varepsilon}$	8, 9, Lemma 11-1
11.	$\varepsilon x^2 > 1$	10, multiply by $\varepsilon > 0$, 1
12.	$\varepsilon > \frac{1}{x^2}$	11, Divide by $x^2 > 0$, 6
13.	$\frac{1}{x^2} \geq 0$	6, Algebra Knowledge
14.	$\left\lvert\frac{1}{x^2}\right\rvert = \frac{1}{x^2}$	13, Def. of abs. value
15.	$\varepsilon > \left\lvert\frac{1}{x^2}\right\rvert$	12, 14
16.	$\varepsilon > \left\lvert\frac{1}{x^2} - 0\right\rvert$	15
17.	$\left\lvert\frac{1}{x^2} - 0\right\rvert < \varepsilon$	16

Definition 15-1 $\lim_{x\to\infty} f(x) = L$ IFF for each $\varepsilon > 0$, there is a positive real number P such that for all real numbers $x > P$, $|f(x) - L| < \varepsilon$.

Definition 15-2 $\lim_{x\to-\infty} f(x) = L$ IFF for each $\varepsilon > 0$, there is a negative real number N such that for all real numbers $x < N$, $|f(x) - L| < \varepsilon$

Definition 15-3 $y = L$ is a horizontal asymptote for $f(x)$ IFF either $\lim_{x\to\infty} f(x) = L$ OR $\lim_{x\to-\infty} f(x) = L$

THEOREMS RESTATED USING LIMIT LANGUAGE

Theorem 16-1 $\lim_{x \to \infty} \frac{1}{x} = 0$ (See Th^m 9-1)

Theorem 16-2 $\lim_{x \to \infty} \frac{-7}{x} = 0$ (See Th^m 9-2)

Theorem 16-3 Given c is a real no. $\lim_{x \to \infty} \frac{c}{x} = 0$ (See Th^m 10-1)

Theorem 16-4 $\lim_{x \to \infty} \frac{1}{x^2} = 0$ (See Th^m 11-1)

Theorem 16-5 Given n is a pos. int. $\lim_{x \to \infty} \frac{1}{x^n} = 0$ (See Th^m 11-2)

Theorem 16-6 $\lim_{x \to \infty} \frac{3x+1}{x} = 3$ (See Th^m 12-1)

Theorem 16-7 Given c is a real no. $\lim_{x \to \infty} c = c$ (See Th^m 12-2)
(or For $f(x) = c$, $\lim_{x \to \infty} f(x) = c$)

Theorem 16-8 Given $f(x), g(x)$ are functions defined on the positive reals and L, M are reals. If $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to \infty} g(x) = M$, then $\lim_{x \to \infty} f(x) + g(x) = L + M$. (See Th^m 13-2)

Theorem 16-9 Given c and L are real numbers and $f(x)$ is a function defined on the positive reals. If $\lim_{x \to \infty} f(x) = L$, then $\lim_{x \to \infty} c\,f(x) = cL$. (See Th^m 14-1)

Theorem 16-10 $\lim_{x \to -\infty} \frac{1}{x} = 0$ (See Th^m 14-2)

Theorem 16-11 $\lim_{x \to -\infty} \frac{1}{x^2} = 0$ (See Th^m 14-3)

Lemma 17-1 For every real number L, $\frac{|L|}{|L|+1} < 1$

Proof: 1. Assume L is a real number. (Show $\frac{|L|}{|L|+1} < 1$)

2. $0 < 1$

3. $0 \le |L| < |L| + 1$ — 2, Add $|L|$ to both sides

4. $\frac{|L|}{|L|+1} < 1$ — 3, Divide both sides by the positive number $|L|+1$

Lemma 17-2 For all real numbers a, b, $|a-b| \ge |a| - |b|$

Proof: 1. Assume a and b are real numbers. (Show $|a-b| \ge |a| - |b|$)

2. $|a| = |(a-b)+b| \le |a-b| + |b|$ — Lemma 13-1

3. $|a| - |b| \le |a-b|$ — 2, Subtract $|b|$ from both sides

4. $|a-b| \ge |a| - |b|$

Lemma 17-3 For every real number x, $|x| = |-x|$

Proof: 1. Assume x is a real number. (Show $|x| = |-x|$)

2. $|-x| = |(-1)x| = |-1||x| = 1|x| = |x|$

Note: For all real numbers a, b $\quad -(a-b) = b-a$

Theorem 17-4 Given: $G(x)$ is a function defined on the positive reals and M is a real number. IF $\lim_{x\to\infty} G(x) = M$ and $M \ne 0$ THEN there are positive reals D and P such that for all real numbers $x > P$, $|G(x)| > D$

"Bounded away from 0"

Proof: 1. Assume $\lim_{x\to\infty} G(x) = M$ and $M \ne 0$ (Show there are positive reals D and P such that for all real numbers $x > P$, $|G(x)| > D$)

2. For every $\varepsilon > 0$, there is a positive real number P such that for all real numbers $x > P$, $|G(x) - M| < \varepsilon$. (1, def of limit)

3. $|M| > 0$ — 1, Since $M \ne 0$

4. $\frac{|M|}{2} > 0$

5. Let $D = \frac{|M|}{2}$

6. There is a positive real number P such that for all real numbers $x > P$, $|G(x) - M| < \frac{|M|}{2}$ (2, 4)

(Show for all real $x > P$, $|G(x)| > D$)

7. Assume x is a real number $> P$ (Show $|G(x)| > D$)

8. $|M| - |G(x)| \underset{17-2}{\le} |M - G(x)| = |G(x) - M| \underset{6,7}{<} \frac{|M|}{2}$

9. $D \underset{5}{=} \frac{|M|}{2} = |M| - \frac{|M|}{2} \underset{8}{<} |G(x)|$

Lemma 18-1 Given T, D, W are reals. IF $T \geq 0$ and $W > D > 0$, then $\frac{T}{W} \leq \frac{T}{D}$. Proof: 1. Assume $T \geq 0$ and $W > D > 0$ (Show $\frac{T}{W} \leq \frac{T}{D}$)

2. $D < W$ (1)
3. $\frac{1}{WD} > 0$ (1)
4. $\left(\frac{1}{WD}\right)D < \left(\frac{1}{WD}\right)W$ (2, mult both sides by $\frac{1}{WD} > 0$)
5. $\frac{1}{W} < \frac{1}{D}$ (4)
6. $\frac{T}{W} \leq \frac{T}{D}$ (5, 1) mult. both sides by $T \geq 0$

THEOREM 18-1 Given: $F(x)$ and $G(x)$ are functions defined on the positive reals and T, B are reals.
IF $\lim_{x \to \infty} F(x) = T$, $\lim_{x \to \infty} G(x) = B$, and $B \neq 0$, THEN $\lim_{x \to \infty} \frac{F(x)}{G(x)} = \frac{T}{B}$.

Proof: 1. Assume $\lim_{x \to \infty} F(x) = T$, $\lim_{x \to \infty} G(x) = B$ and $B \neq 0$ (Show $\lim_{x \to \infty} \frac{F(x)}{G(x)} = \frac{T}{B}$) (Show for every $\varepsilon > 0$, there is a positive real number P such that for all real $x > P$, $\left|\frac{F(x)}{G(x)} - \frac{T}{B}\right| < \varepsilon$.)

2. Assume $\varepsilon > 0$ (Show there is a positive real number P such that for all real $x > P$, $\left|\frac{F(x)}{G(x)} - \frac{T}{B}\right| < \varepsilon$.)

3. There are positive reals D and P_1 such that for all real $x > P_1$, $|G(x)| > D$. (1, Thm 17-4)

4. For each $\varepsilon > 0$, there is a positive real number P such that for all real $x > P$, $|F(x) - T| < \varepsilon$. (1, def)

5. For each $\varepsilon > 0$, there is a positive real number P such that for all real $x > P$, $|G(x) - B| < \varepsilon$.

6. $\frac{\varepsilon D}{2} > 0$ 2, 3

7. There is a positive real number P_2 such that for all real $x > P_2$, $|F(x) - T| < \frac{\varepsilon D}{2}$. (4,6)

8. $\frac{\varepsilon D|B|}{2(|T|+1)} > 0$ (2,1,3)

9. There is a positive real number P_3 such that for all real $x > P_3$, $|G(x) - B| < \frac{\varepsilon D|B|}{2(|T|+1)}$ (8,5)

10. Let P be the maximum of P_1, P_2, P_3. So P is positive and $P \geq P_1$, $P \geq P_2$, $P \geq P_3$. (Show for all real $x > P$, $\left|\frac{F(x)}{G(x)} - \frac{T}{B}\right| < \varepsilon$)

11. Assume x is a real number $> P$ (Show $\left|\frac{F(x)}{G(x)} - \frac{T}{B}\right| < \varepsilon$)
A long string of equalities and inequalities will now happen.

12. $\left|\frac{F(x)}{G(x)} - \frac{T}{B}\right| = \left|\frac{BF(x) - TG(x)}{G(x)B}\right| = \left|\frac{(BF(x) - TB) + (TB - TG(x))}{G(x)B}\right| =$

$$\frac{|B(F(x)-T) + T(B-G(x))|}{|G(x)||B|} \underset{3,\,L18\text{-}1}{\leq} \frac{|B(F(x)-T)+T(B-G(x))|}{D|B|} \underset{L13\text{-}1}{\leq}$$

$$\frac{|B(F(x)-T)| + |T(B-G(x))|}{D|B|} = \frac{|B||F(x)-T| + |T||B-G(x)|}{D|B|} =$$

$$\frac{|B||F(x)-T|}{D|B|} + \frac{|T||G(x)-B|}{D|B|} = \frac{|F(x)-T|}{D} + \frac{|T|}{D|B|}|G(x)-B| \underset{11,7,9}{<}$$

$$\frac{1}{D}\left(\frac{\varepsilon D}{2}\right) + \frac{|T|}{D|B|}\frac{\varepsilon D|B|}{2(|T|+1)} = \frac{\varepsilon}{2} + \frac{|T|}{|T|+1}\frac{\varepsilon}{2} \underset{L17\text{-}1}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\lim_{x\to\infty} \frac{x^2-7}{3x^2+x} = \frac{1}{3}$$

$\boxed{\text{THEOREM 19-1}}$

1. For each $x \neq 0$ $\frac{x^2-7}{3x^2+x} = \frac{\frac{1}{x^2}(x^2-7)}{\frac{1}{x^2}(3x^2+x)} = \frac{1 - \frac{7}{x^2}}{3 + \frac{1}{x}}$

$\boxed{\text{COROLLARY 19A-1}}$ If $\lim_{x\to\infty} l(x) = L$ and $\lim_{x\to\infty} r(x) = R$ and $R \neq 0$, then $\lim_{x\to\infty} l(x)r(x) = LR$.

Proof: 1. Assume $\lim_{x\to\infty} l(x) = L$, $\lim_{x\to\infty} r(x) = R$, and $R \neq 0$

(Show $\lim_{x\to\infty} l(x)r(x) = L\cdot R$)

2. $\lim_{x\to\infty} 1 = 1$ Thm 16.7

3. $\lim_{x\to\infty} \frac{1}{r(x)} = \frac{1}{R}$ 2, 1, Theorem 18-1

4. $\frac{1}{R} \neq 0$ 1

5. $\lim_{x\to\infty} l(x)r(x) = \lim_{x\to\infty} \frac{l(x)}{\frac{1}{r(x)}} \overset{4}{\underset{T18-1}{=}} \frac{L}{\frac{1}{R}} = LR$

THEOREM 19A-1 If $\lim_{x\to\infty} l(x) = L$ and $\lim_{x\to\infty} r(x) = 0$, then $\lim_{x\to\infty} l(x)\cdot r(x) = 0$.

Proof: 1. Assume $\lim_{x\to\infty} l(x) = L$ and $\lim_{x\to\infty} r(x) = 0$

(Show $\lim_{x\to\infty} l(x)\cdot r(x) = 0$) (Show for every $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|l(x)r(x) - 0| < \varepsilon$)

2. Assume $\varepsilon_1 > 0$ (Show there is a $P > 0$ such that for all $x > P$, $|l(x)\cdot r(x) - 0| < \varepsilon_1$)

3. For every $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|l(x) - L| < \varepsilon$. 1, def of $\lim_{x\to\infty} f(x) = L$

4\. For every $\varepsilon > 0$, there is a $P > 0$ such that for all $x > P$, $|r(x) - 0| < \varepsilon$. 1, def of $\lim_{x\to\infty} r(x) = 0$

5\. $\frac{\varepsilon_1}{|L|+1} > 0$ 2, Theorem A-1

6\. There is a $P_1 > 0$ such that for all $x > P_1$, $|r(x) - 0| < \frac{\varepsilon_1}{|L|+1}$ 4, 5

7\. $1 > 0$

8\. There is a $P_2 > 0$ such that for all $x > P_2$ $|l(x) - L| < 1$ 7, 3

9 Let $P = \max\{P_1, P_2\}$. Thus $P > 0$, $P \geq P_1$, $P \geq P_2$

(Show for all $x > P$, $|l(x) \cdot r(x) - 0| < \varepsilon_1$)

10\. Assume $x > P$ (Show $|l(x) \cdot r(x) - 0| < \varepsilon_1$)

11\. $|l(x)| - |L| \overset{L17\text{-}2}{\leq} |l(x) - L| < 1$ 10, 8, since $x > P \geq P_2$, 9

12\. $|l(x)| < |L| + 1$ 11, Add $|L|$ to both sides

13\. $|l(x) \cdot r(x) - 0| = |l(x) \cdot r(x)| = |l(x)||r(x)|$

$\overset{12}{\leq} (|L|+1)|r(x)| = (|L|+1)|r(x) - 0| \overset{6,\, x > P \geq P_1,\, 10}{<} (|L|+1)\frac{\varepsilon_1}{|L|+1} = \varepsilon_1$

<u>THEOREM 19B</u> If $\lim_{x\to\infty} l(x) = L$ and $\lim_{x\to\infty} r(x) = R$, then $\lim_{x\to\infty} l(x) \cdot r(x) = L \cdot R$.

Proof: 1. Assume $\lim_{x\to\infty} l(x) = L$ and $\lim_{x\to\infty} r(x) = R$ (Show $\lim_{x\to\infty} l(x) \cdot r(x) = L \cdot R$)

2\. $R \neq 0$ or $R = 0$

3\. <u>Case 1</u> $R \neq 0$

4\. $\lim_{x\to\infty} l(x) \cdot r(x) = L \cdot R$ 1, 3, Corollary 19A-1

5 <u>Case 2</u> $R = 0$

6 $\lim_{x\to\infty} l(x) \cdot r(x) = 0 \overset{5}{=} L \cdot R$ Theorem 19A-1

2. $\lim_{x\to\infty} 1 = 1$ Thm 16-7

3. $\lim_{x\to\infty} \frac{1}{x^2} = 0$ Thm 16-4

4. $\lim_{x\to\infty} \frac{-7}{x^2} = \lim_{x\to\infty} (-7)\frac{1}{x^2} = -7(0) = 0$ Thm 16-9

5. $\lim_{x\to\infty} 1 + \frac{-7}{x^2} = \lim_{x\to\infty} 1 + \lim_{x\to\infty} -\frac{7}{x^2} = 1 + 0 = 1$ Thm 16-8, (2,4)

6. $\lim_{x\to\infty} 1 + \frac{-7}{x^2} = \lim_{x\to\infty} 1 - \frac{7}{x^2} = 1$ (5)

7. $\lim_{x\to\infty} 3 = 3$ Thm 16-7

8. $\lim_{x\to\infty} \frac{1}{x} = 0$ Thm 16-1

9. $\lim_{x\to\infty} 3 + \frac{1}{x} = \lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{1}{x} = 3 + 0 = 3$ Thm 16-8, (7,8)

10. $\lim_{x\to\infty} 3 + \frac{1}{x} = 3 \neq 0$ 9

11. $\lim_{x\to\infty} \frac{1 - \frac{7}{x^2}}{3 + \frac{1}{x}} = \frac{1}{3}$ 6,10 Thm 18-1

12. $\lim_{x\to\infty} \frac{x^2 - 7}{3x^2 + x} = \frac{1}{3}$ 1,11

PROBLEM 20-1 PROVE

$$\lim_{x\to\infty} \frac{5x^2 + 6x}{-2x^2 + 3} = -\frac{5}{2}$$

THEOREM 20A: For all real numbers T, B, c, d, if $B \neq 0$, then $\lim_{x \to \infty} \frac{Tx+c}{Bx+d} = \frac{T}{B}$

Proof:

0\. Assume T, B, c, d are real numbers and $B \neq 0$

1\. For all $x \neq 0$, $\frac{Tx+c}{Bx+d} = \frac{\frac{1}{x}(Tx+c)}{\frac{1}{x}(Bx+d)} = \frac{T+\frac{c}{x}}{B+\frac{d}{x}}$

2\. $\lim_{x \to \infty} T = T$ — Theorem 16-7

3\. $\lim_{x \to \infty} \frac{1}{x} = 0$ — Theorem 16-1

4\. $\lim_{x \to \infty} \frac{c}{x} = \lim_{x \to \infty} c \cdot \frac{1}{x} = c \cdot 0 = 0$ — Theorem 16-9, 3

5\. $\lim_{x \to \infty} T + \frac{c}{x} = T + 0 = T$ — 2, 4, Theorem 16-8

6\. $\lim_{x \to \infty} B = B$ — Theorem 16-7

7\. $\lim_{x \to \infty} \frac{d}{x} = \lim_{x \to \infty} d \cdot \frac{1}{x} = d \cdot 0 = 0$ — 3, Theorem 16-9

8\. $\lim_{x \to \infty} B + \frac{d}{x} = B + 0 = B \neq 0$ — 6, 7, Theorem 16-8

9\. $\lim_{x \to \infty} \frac{T+\frac{c}{x}}{B+\frac{d}{x}} = \frac{T}{B}$ — 5, 8, Theorem 18-1

10\. $\lim_{x \to \infty} \frac{Tx+c}{Bx+d} \overset{1}{=} \lim_{x \to \infty} \frac{T+\frac{c}{x}}{B+d/x} \overset{9}{=} \frac{T}{B}$

THEOREM 20B: For all real numbers $T, c, d, B, e,$ and f, if $B \neq 0$ then $\lim\limits_{x \to +\infty} \dfrac{Tx^2+cx+d}{Bx^2+ex+f} = \dfrac{T}{B}$

Proof: 1. Assume $T, c, d, B, e,$ and f are real numbers and $B \neq 0$

2. For all $x \neq 0$, $\dfrac{Tx^2+cx+d}{Bx^2+ex+f} = \dfrac{\frac{1}{x^2}(Tx^2+cx+d)}{\frac{1}{x^2}(Bx^2+ex+f)} = \dfrac{T+\frac{c}{x}+\frac{d}{x^2}}{B+\frac{e}{x}+\frac{f}{x^2}}$

3. $\lim\limits_{x \to +\infty} T = T$ Theorem 16-7

4. $\lim\limits_{x \to +\infty} \frac{c}{x} = 0$ Theorem 16-3

5. $\lim\limits_{x \to +\infty} \frac{1}{x^2} = 0$ Theorem 16-4

6. $\lim\limits_{x \to +\infty} \frac{d}{x^2} = \lim\limits_{x \to +\infty} d \cdot \frac{1}{x^2} = d \cdot 0 = 0$ 5, Theorem 16-9

7. $\lim\limits_{x \to +\infty} T+\frac{c}{x} = T+0 = T$ 3, 4, Theorem 16-8

8. $\lim\limits_{x \to +\infty} T+\frac{c}{x}+\frac{d}{x^2} = T+0 = T$ 7, 6, Theorem 16-8

9. $\lim\limits_{x \to +\infty} B = B$ Theorem 16-7

10. $\lim\limits_{x \to +\infty} \frac{e}{x} = 0$ Theorem 16-3

11 $\lim\limits_{x \to +\infty} \frac{f}{x^2} = \lim\limits_{x \to +\infty} f \cdot \frac{1}{x^2} = f \cdot 0 = 0$ 5, Theorem 16-9

12. $\lim\limits_{x \to +\infty} B+\frac{e}{x} = B+0 = B$ 9, 10, Theorem 16-8

13. $\lim\limits_{x \to +\infty} B+\frac{e}{x}+\frac{f}{x^2} = B+0 = B$ 12, 11 Theorem 16-8

14. $\lim\limits_{x \to +\infty} \dfrac{T+\frac{c}{x}+\frac{d}{x^2}}{B+\frac{e}{x}+\frac{f}{x^2}} = \dfrac{T}{B}$ 1, 8, 13, Theorem 18-1

15. $\lim\limits_{x \to +\infty} \dfrac{Tx^2+cx+d}{Bx^2+ex+f} \overset{2}{=} \lim\limits_{x \to +\infty} \dfrac{T+\frac{c}{x}+\frac{d}{x^2}}{B+\frac{e}{x}+\frac{f}{x^2}} \overset{14}{=} \dfrac{T}{B}$

APPLICATIONS OF THEOREM 20A

$$\lim_{x\to+\infty} \frac{5x-7}{3x-2} = \frac{5}{3} \qquad \begin{matrix} T=5 & c=-7 \\ B=3 & d=-2 \end{matrix}$$

$$\lim_{x\to+\infty} \frac{4}{7x+3} = \lim_{x\to+\infty} \frac{0x+4}{7x+3} = \frac{0}{7} = 0 \qquad \begin{matrix} T=0 & c=4 \\ B=7 & d=-2 \end{matrix}$$

APPLICATION OF THEOREM 20 B

$$\lim_{x\to+\infty} \frac{4x-3}{5x^2+7x+6} = 0 \qquad \begin{matrix} c=4 & d=-3 & T=0 \\ B=5 & e=7 & f=6 \end{matrix}$$

Note: they are the same

APPLICATIONS OF THEOREM 20B

$$\lim_{x\to+\infty} \frac{5x^2-7x+6}{-3x^2+4x+2} = \frac{5}{-3} \qquad \begin{matrix} T=5 & c=-7 & d=6 \\ B=-3 & e=4 & f=2 \end{matrix}$$

$$\lim_{x\to+\infty} \frac{4x-3}{5x^2+7x+6} = \lim_{x\to+\infty} \frac{0x^2+4x-3}{5x^2+7x+6} = \frac{0}{5} = 0$$

$$\begin{matrix} T=0 & c=4 & d=-3 \\ B=5 & e=7 & f=6 \end{matrix}$$

Theorem 20B is really a corollary of Theorem 20C

"H as in height"

Theorem 21-1 For every $H>0$, there is an $\delta>0$ such that for all $0<x<\delta$, $\frac{1}{x}>H$.

Proof: 1. Assume $H>0$ (Show there is an $\delta>0$ such that for all $0<x<\delta$, $\frac{1}{x}>H$.)

2. Let $\delta=\frac{1}{H}$; $\delta>0$ (Show for all $0<x<\delta$, $\frac{1}{x}>H$)

3. Assume $0<x<\delta$ (Show $\frac{1}{x}>H$)

4. $x<\frac{1}{H}$ — 2,3

5. $xH<1$ — 1,4

6. $H<\frac{1}{x}$ — 3,5

7. $\frac{1}{x}>H$

Theorem 21-2 For every $H>0$, there is a $\delta>0$ such that for all $3<x<3+\delta$, $\frac{1}{x-3}>H$.

Proof: 1. Assume $H>0$ (Show there is a $\delta>0$ such that for all $3<x<3+\delta$, $\frac{1}{x-3}>H$)

2. Let $\delta=\frac{1}{H}$ (Note $\delta>0$) (Show for all $3<x<3+\delta$, $\frac{1}{x-3}>H$)

3. Assume $3<x<3+\delta$ (Show $\frac{1}{x-3}>H$)

4. $0<x-3<\delta$ — (3)

5. $0<x-3<\frac{1}{H}$ — (4,2)

6. $H(x-3)<1$ — (5,1)

7. $H<\frac{1}{x-3}$ — (6, div both sides by $x-3$, 4)

8. $\frac{1}{x-3}>H$

DEFINITION 21-3 Suppose the function $f(x)$ is defined to the right of the real number a (i.e. for $x>a$). $\lim_{x\to a^+} f(x)=+\infty$ iff for every $H>0$ there is a $\delta>0$ such that for all $a<x<a+\delta$, $f(x)>H$.

Theorem 21-4 $\lim_{x\to 5^+}\frac{7}{x-5}=+\infty$

(Show for every $H>0$, there is a $\delta>0$ such that for all $5<x<5+\delta$, $\frac{7}{x-5}>H$)

Proof: 1. Assume $H>0$ (Show there is a $\delta>0$ such that for all $5<x<5+\delta$, $\frac{7}{x-5}>H$)

2. Let $\delta=\frac{7}{H}$; note $\delta>0$ (Show for all $5<x<5+\delta$, $\frac{7}{x-5}>H$)

3. Assume $5<x<5+\delta$ (Show $\frac{7}{x-5}>H$)

4. $0<x-5<\delta$ (3)

5. $0<x-5<\frac{7}{H}$ (2,4)

6. $H(x-5)<7$ (5,1)

7. $H<\frac{7}{x-5}$ (5,6)

8. $\frac{7}{x-5}>H$ (7)

NO COROLLARY OF THEOREM 22-1

PROBLEM 22-1

H, δ, proof of $\lim_{x\to -2^+}\frac{8}{x+2}=+\infty$

Theorem 22-1 For all reals c, a, if $c>0$, then $\lim_{x\to a^+}\frac{c}{x-a}=+\infty$

Proof: 1. Assume c, a are reals (Show if $c>0$, then $\lim_{x\to a^+}\frac{c}{x-a}=+\infty$)

2. Assume $c>0$ (Show $\lim_{x\to a^+}\frac{c}{x-a}=+\infty$)
(Show for every $H>0$, there is a $\delta>0$ such that for all $a<x<a+\delta$, $\frac{c}{x-a}>H$)

3. Assume $H>0$ (Show there is a $\delta>0$ such that for all $a<x<a+\delta$, $\frac{c}{x-a}>H$)

4. Let $\delta=\frac{c}{H}$; note $\delta>0$ (Show for all $a<x<a+\delta$, $\frac{c}{x-a}>H$)

5. Assume $a<x<a+\delta$ (Show $\frac{c}{x-a}>H$)

6. $0<x-a<\delta$ 5

7. $0<x-a<c/H$ (6,4)

8. $H(x-a)<c$ (7,3)

9. $H<\frac{c}{x-a}$ (8,7)

10 $\frac{c}{x-a}>H$ 9

Example : $\lim_{x\to -3^+}\frac{5}{x+3}=+\infty$ since $5>0$ and $\frac{5}{x-(-3)}=\frac{5}{x+3}$

Definition 23-1 Suppose $f(x)$ is defined for $x > a$. $\lim_{x \to a^+} f(x) = -\infty$ iff for each $D < 0$ there is a $\delta > 0$ such that for all $a < x < a+\delta$, $f(x) < D$.

Theorem 23-2 $\lim_{x \to 0^+} \frac{-3}{x} = -\infty$ (Show for every $D<0$ there is a $\delta>0$ such that for all $0 < x < 0+\delta$, $\frac{-3}{x} < D$)

1. Assume $D<0$ (Show there is a $\delta>0$ such that for all $0<x<0+\delta$, $-\frac{3}{x}<D$)
2. Let $\delta = \frac{-3}{D}$; note $\delta>0$. (Show that for all $0<x<0+\delta$, $-\frac{3}{x}<D$)
3. Assume $0<x<0+\delta$ (Show $-\frac{3}{x}<D$)
4. $x < -3/D$ 3,2
5 $Dx > -3$ 4,1
6. $D > -3/x$ 5,3
7 $-\frac{3}{x} < D$

Theorem 23-3 For all reals a, c, if $c<0$, then $\lim_{x\to a^+} \frac{c}{x-a} = -\infty$.

1. Assume a, c are reals (Show if $c<0$, then $\lim_{x\to a^+}\frac{c}{x-a} = -\infty$)
2. Assume $c<0$ (Show $\lim_{x\to a^+}\frac{c}{x-a}=-\infty$) (Show for every $D<0$ there is a $\delta>0$ such that for all $a<x<a+\delta$, $\frac{c}{x-a}<D$)
3. Assume $D<0$ (Show there is a $\delta>0$ such that for all $a<x<a+\delta$, $\frac{c}{x-a}<D$)
4. Let $\delta=\frac{c}{D}$; note $\delta>0$ (Show for all $a<x<a+\delta$, $\frac{c}{x-a}<D$)
5. Assume $a<x<a+\delta$ (Show $\frac{c}{x-a}<D$)
6. $0<x-a<\delta$ (5)
7. $0<x-a<c/D$ (4,6)
8. $D(x-a)>c$ (7,3)
9. $D>\frac{c}{x-a}$ (8,7)
10 $\frac{c}{x-a}<D$ (9)

Example $\lim_{x\to 4^+}\frac{-7}{x-4} = -\infty$

Be able to present all on pages 21, 22, 23 (You will be asked to prove 2)

Let R denote the set of real numbers.

Definition 24-1 Suppose f is defined on all the reals. For every $p \in R$, f is continuous at p iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all real numbers x, if $|x-p| < \delta$, then $|f(x) - f(p)| < \varepsilon$

Theorem 24-1. Suppose $a, b \in R$ and $f(x) = ax+b$. For every $p \in R$, f is continuous at p.

1. Assume $\varepsilon > 0$ (Show there is a $\delta > 0$ such that for all real numbers x, if $|x-p| < \delta$, then $|f(x) - f(p)| < \varepsilon$)
2. Let $\delta = \frac{\varepsilon}{|a|+1}$ (Show for all real numbers x, if $|x-p| < \delta$, then $|f(x) - f(p)| < \varepsilon$)
3. Assume x is a real (Show if $|x-p| < \delta$, then $|f(x) - f(p)| < \varepsilon$)
4. Assume $|x-p| < \delta$ (Show $|f(x) - f(p)| < \varepsilon$)
5. $|x-p| < \frac{\varepsilon}{|a|+1}$ — 4, 2, substitution
6. $|a||x-p| \leq \frac{|a|\varepsilon}{|a|+1} = \frac{|a|}{|a|+1}\varepsilon < \varepsilon$ — 5, mult. by $|a|$, Th^{m} A-1, Lemma 17-1
7. $|a(x-p)| < \varepsilon$ — 6, Th^{m} A-5
8. $|ax - ap| < \varepsilon$ — 7
9. $|ax+b-ap-b| < \varepsilon$ — 8
10. $|(ax+b)-(ap+b)| < \varepsilon$ — 9
11. $|f(x) - f(p)| < \varepsilon$ — 10, def^{n} of $f(x)$

Thm 25-1 Suppose $f(x)$ is defined for all reals x and $p \in R$. If f is continuous at p and $f(p) > 0$, then there are positive reals δ, L, and U such that $\forall x \in R$, if $|x-p| < \delta$, then $L < f(x) < U$

Proof: 1. Assume f is continuous at p and $f(p) > 0$ (Show there are positive reals δ, L, and U such that $\forall x \in R$, if $|x-p| < \delta$, then $L < f(x) < U$)

2. $\frac{f(p)}{2} > 0$ — 1.

3. $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in R$, if $|x-p| < \delta$, then $|f(x) - f(p)| < \varepsilon$ — 1, def of cont at p.

4. $\exists \delta > 0$ such that $\forall x \in R$, if $|x-p| < \delta$ then $|f(x) - f(p)| < \frac{f(p)}{2}$ — 2, 3

5. Let $L = \frac{f(p)}{2}$ and $U = \frac{3f(p)}{2}$; Note $L, U > 0$ — 1
(Show $\forall x \in R$, if $|x-p| < \delta$, then $L < f(x) < U$)

6. Assume $x \in R$ (Show if $|x-p| < \delta$, then $L < f(x) < U$)

7. Assume $|x-p| < \delta$ (Show $L < f(x) < U$)

8. $|f(x) - f(p)| < \frac{f(p)}{2}$ — 4, 6, 7

9. $-\frac{f(p)}{2} < f(x) - f(p) < \frac{f(p)}{2}$ — 8, Thm A-3

10. $f(p) - \frac{f(p)}{2} < f(x) < f(p) + \frac{f(p)}{2}$ — 9, Add $f(p)$

11. $L = \frac{f(p)}{2} < f(x) < \frac{3f(p)}{2} = U$ — 10, 5

$U = \frac{f(p)}{2}$ $\quad L = \frac{3f(p)}{2}$

Remember

Thm 25-2: Suppose $f(x)$ is defined for all reals x and $p \in R$. If f is continuous at p and $f(p) < 0$, then there are a $\delta > 0$ and $L, U < 0$, such that for all $x \in R$, if $|x-p| < \delta$, then $L < f(x) < U$. 7-9-93

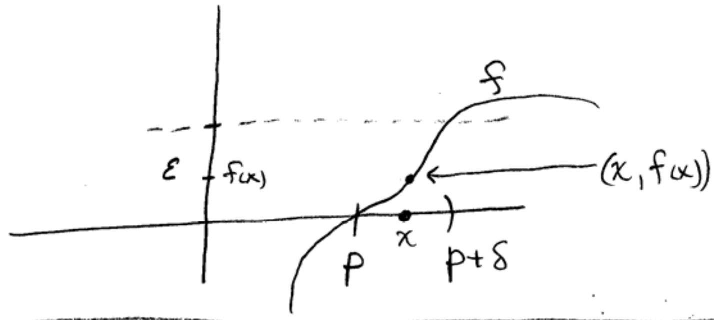
1. Assume f is continuous at p and $f(p) < 0$.

(Show there are a $\delta > 0$, $L, U < 0$ ∋ $\forall x \in R$, if $|x-p| < \delta$, then $L < f(x) < U$.)

2. $\frac{-f(p)}{2} > 0$ — 1
3. $\forall \varepsilon > 0, \exists \delta > 0$ ∋ $\forall x \in R$, if $|x-p| < \delta$, then $|f(x) - f(p)| < \varepsilon$ — 1, def. of Cont
4. There is a $\delta > 0$ ∋ $\forall x \in R$, if $|x-p| < \delta$, then $|f(x) - f(p)| < \frac{-f(p)}{2}$ — 2,3
5. Let $L = \frac{3f(p)}{2}$, $U = \frac{f(p)}{2}$, note $L, U < 0$ — 1.

(Show $\forall x \in R$, if $|x-p| < \delta$, then $L < f(x) < U$)

6. Assume $x \in R$ (Show if $|x-p| < \delta$, then $L < f(x) < U$)
7. Assume $|x-p| < \delta$ (Show $L < f(x) < U$)
8. $|f(x) - f(p)| < \frac{-f(p)}{2}$ — 4, 6, 7
9. $\frac{f(p)}{2} < f(x) - f(p) < \frac{-f(p)}{2}$ — 8, Thm A-3
10. $\frac{f(p)}{2} + f(p) < f(x) < f(p) - \frac{f(p)}{2}$ — 9, add $f(p)$
11. $L = \frac{3f(p)}{2} < f(x) < \frac{f(p)}{2} = U$ — 10, H.S., 5
12. $L < f(x) < U$ — 11, H.S.



Def 25-3 $\lim_{x \to p^+} f(x) = 0^+$ iff $\forall \varepsilon > 0, \exists \delta > 0$ such that for all $p < x < p+\delta$, $0 < f(x) < \varepsilon$

Thm 26-1 $\lim_{x \to p^+} x - p = 0^+$

(Show $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall p < x < p+\delta, 0 < x-p < \varepsilon$)

1. Assume $\varepsilon > 0$ (Show $\exists \delta > 0$ such that for all $p < x < p+\delta$, $0 < x-p < \varepsilon$)
2. Let $\delta = \varepsilon$; Note $\delta > 0$ 1
   (Show for all $p < x < p+\delta$, $0 < x-p < \varepsilon$)
3. Assume $p < x < p+\delta$ (Show $0 < x-p < \varepsilon$)
4. $0 < x-p < \delta$ 3, Subtract p
5. $0 < x-p < \varepsilon$ 4, 2.

Theorem 26-2. Given $T(x)$, $B(x)$, $D(x)$ are functions, $a \in \mathbb{R}$, T is continuous at a, B is continuous at a and

$$\lim_{x \to a^+} D(x) = 0^+$$

(a) if $T(a) > 0$ and $B(a) > 0$, then

$$\lim_{x \to a^+} \frac{T(x)}{B(x)\,D(x)} = +\infty$$

(b) if $T(a) > 0$ and $B(a) < 0$, then

$$\lim_{x \to a^+} \frac{T(x)}{B(x)\,D(x)} = -\infty$$

(c) if $T(a) < 0$ and $B(a) > 0$, then

$$\lim_{x \to a^+} \frac{T(x)}{B(x)\,D(x)} = -\infty$$

(d) if $T(a) < 0$ and $B(a) < 0$, then

$$\lim_{x \to a^+} \frac{T(x)}{B(x)\,D(x)} = +\infty$$

Th$^{\underline{m}}$ 26-2 <u>Given</u> T and B are continuous at a and $\lim\limits_{x \to a^+} D(x) = 0^+$

(a) if $T(a) > 0$ and $B(a) > 0$, then
$$\lim_{x \to a^+} \frac{T(x)}{B(x)D(x)} = +\infty$$

<u>Proof</u>: 1. Assume $T(a) > 0$ and $B(a) > 0$

(Show $\lim\limits_{x \to a^+} \frac{T(x)}{B(x)D(x)} = +\infty$)

(Show $\forall H > 0, \exists \delta > 0$ such that for all $a < x < a + \delta$, $\frac{T(x)}{B(x)D(x)} > H$)

1.5. Assume $H > 0$ (Show $\exists \delta > 0$ such that for all $a < x < a + \delta$, $\frac{T(x)}{B(x)D(x)} > H$)

2. T is cont. at a, B is cont. at a GIVEN
3. There are $\delta_1 > 0$, $L_1 > 0$, $U_1 > 0$ such that $\forall x \in R$, if $|x - a| < \delta_1$, then $L_1 < T(x) < U_1$ (2, 1 T25.1)
4. There are $\delta_2 > 0$, $L_2 > 0$, $U_2 > 0$ such that $\forall x \in R$, if $|x - a| < \delta_2$, then $L_2 < B(x) < U_2$

(2, 1, Th^m 25-1)

$\boxed{26.C}$

$\boxed{\ni \equiv \text{such that}}$

5. $\dfrac{L_1}{U_2 H} > 0$ 3, 4, 1.5

6. $\lim\limits_{x\to a^+} D(x) = 0^+$ Given

7. $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall a < x < a+\delta, 0 < D(x) < \varepsilon$ 6, Def

8. $\exists \delta_3 > 0 \ni \forall a < x < a+\delta_3, 0 < D(x) < \dfrac{L_1}{U_2 H}$ 5, 7

9. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. $\delta > 0$. $\delta \le \delta_1, \delta \le \delta_2, \delta \le \delta_3$
(Show $\forall a < x < a+\delta, \dfrac{T(x)}{B(x)D(x)} > H$) 3, 4, 8

10. Assume $a < x < a+\delta$ (Show $\dfrac{T(x)}{B(x)D(x)} > H$)

11. $0 < x-a < \delta$ 10

12. $|x-a| = x-a$ 11

13. $|x-a| < \delta$ 11, 12

14. $L_1 < T(x) < U_1$ 13, 3, since $|x-a| < \delta \le \delta_1$

15. $L_2 < B(x) < U_2$ 13, 4, since $|x-a| < \delta \le \delta_2$

16. $a < x < a+\delta_3$ 10 since $\delta \le \delta_3$

17. $0 < D(x) < \dfrac{L_1}{U_2 H}$ 8, 16

18. $\dfrac{T(x)}{B(x)D(x)} \overset{14}{>} \dfrac{L_1}{B(x)D(x)} \overset{15}{>} \dfrac{L_1}{U_2 D(x)} \overset{17}{>} \dfrac{L_1}{U_2\left(\frac{L_1}{U_2 H}\right)} = \dfrac{L_1 U_2 H}{U_2 L_1} = H$

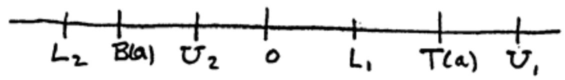
Given: T and B are continuous at a and $\lim\limits_{x\to a^+} D(x) = 0^+$ 26 D [redacted]

Proof of Theorem 26-2(b)

1. Assume $T(a)>0$ and $B(a)<0$ (Show $\lim\limits_{x\to a^+} \frac{T(x)}{B(x)D(x)} = -\infty$)

(Show for all $\mathbb{D}<0$, there is a $\delta>0$ such that for all $a<x<a+\delta$, $\frac{T(x)}{B(x)D(x)} < \mathbb{D}$)

2. T is continuous at a and B is continuous at a. GIVEN



3. There are positive reals δ_1, L_1, and U_1 such that for all $x\in R$, if $|x-a|<\delta_1$, then $0<L_1<T(x)<U_1$ Th^{m}25-1
4. There are a $\delta_2>0$ and $L_2, U_2<0$ such that for all $x\in R$, if $|x-a|<\delta_2$, then $L_2<B(x)<U_2<0$ Th^{m}25-2, 1,2

1.5 Assume $\mathbb{D}<0$ (Show there is a $\delta>0$ such that for all $a<x<a+\delta$, $\frac{T(x)}{B(x)D(x)}<\mathbb{D}$ $\left[\left|\frac{T(x)}{B(x)D(x)}\right|>|\mathbb{D}|\right]$)

5. $\frac{L_1}{|L_2||\mathbb{D}|}>0$ 3, 4, 1.5
6. $\lim\limits_{x\to a^+} D(x) = 0^+$ GIVEN
7. For every $\varepsilon>0$, there is a $\delta>0$ such that for all $a<x<a+\delta$, $0<D(x)<\varepsilon$. 6, Def^n
8. There is a $\delta_3>0$ such that for all $a<x<a+\delta_3$,
$$0<D(x)<\frac{L_1}{|L_2||\mathbb{D}|} \qquad 5,7$$
9. Let $\delta=\min\{\delta_1,\delta_2,\delta_3\}$. So $\delta>0$ and $\delta\le\delta_1$, $\delta\le\delta_2$, and $\delta\le\delta_3$. (Show for all $a<x<a+\delta$, $\frac{T(x)}{B(x)D(x)}<\mathbb{D}$)

26E [redacted]

10. Assume $a < x < a+\delta$ $\left(\text{Show } \frac{T(x)}{B(x)D(x)} < D,\ \left|\frac{T(x)}{B(x)D(x)}\right| > |D|\right)$

11. $0 < x-a < \delta$ — 10, subtract a

12. $|x-a| = x-a$ — 11, definition of absolute value

13. $|x-a| < \delta$ — 11, 12

14. $0 < L_1 < T(x) < U_1$ — 3, 13, since $|x-a| < \delta \overset{9}{\le} \delta_1$

15. $L_2 < B(x) < U_2 < 0$ — 4, 13, since $|x-a| < \delta \overset{9}{\le} \delta_2$

16. $a < x < a+\delta_3$ — 10, since $\delta \overset{9}{\le} \delta_3$

17. $0 < D(x) < \dfrac{L_1}{|L_2||D|}$ — 16, 8

18. $$\left|\frac{T(x)}{B(x)D(x)}\right| \overset{\substack{A\text{-}5\\A\text{-}6}}{=} \frac{|T(x)|}{|B(x)||D(x)|} \overset{\substack{|T(x)|=T(x)\\14}}{>} \frac{L_1}{|B(x)||D(x)|} \overset{\substack{-L_2 > -B(x) > 0\\15,\ |L_2| > |B(x)|}}{>}$$
$$\frac{L_1}{|L_2||D(x)|} \overset{17}{>} \frac{L_1}{|L_2|\frac{L_1}{|L_2||D|}} = |D|$$

19. $\dfrac{T(x)}{B(x)D(x)} < 0$ — 14, 15, 17

20. $\dfrac{-T(x)}{B(x)D(x)} > -D$ — 18, def^n abs. value, substitution, 15

21. $\dfrac{T(x)}{B(x)D(x)} < D$ — 20, multiply by -1

26F [redacted]

Proof of 26-2(d)

1. Assume $T(a) < 0$ and $B(a) < 0$ (Show $\lim\limits_{x \to a^+} \frac{T(x)}{B(x)D(x)} = +\infty$)
(Show for all $H > 0$, there is a $\delta > 0$ such that for all $a < x < a+\delta$, $\frac{T(x)}{B(x)D(x)} > H$)

1.5 Assume $H > 0$ (Show there is a $\delta > 0$ such that for all $a < x < a+\delta$, $\frac{T(x)}{B(x)D(x)} > H$ $\left[\left|\frac{T(x)}{B(x)D(x)}\right| > H\right]$)

2. T is cont. at a and B is cont. at a. GIVEN

3. There are a $\delta_1 > 0$ and $L_1, U_1 < 0$ such that for all $x \in R$, if $|x-a| < \delta_1$, then $L_1 < T(x) < U_1 < 0$. 1,2, Thm 25-2

4. There are a $\delta_2 > 0$ and $L_2, U_2 < 0$ such that for all $x \in R$, if $|x-a| < \delta_2$, then $L_2 < B(x) < U_2 < 0$. 1,2, Thm 25-2

5. $\frac{|U_1|}{|L_2| H} > 0$ 1.5, 3, 4

6. $\lim\limits_{x \to a^+} D(x) = 0^+$ GIVEN

7. For every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $a < x < a+\delta$, $0 < D(x) < \varepsilon$ 6, Definition

8. There is a $\delta_3 > 0$ such that for all $a < x < a+\delta_3$,
$0 < D(x) < \frac{|U_1|}{|L_2| H}$ 5, 7

9. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. So $\delta > 0$ and $\delta \le \delta_1$, $\delta \le \delta_2$, and $\delta \le \delta_3$. (Show for all $a < x < a+\delta$, $\frac{T(x)}{B(x)D(x)} > H$)

10. Assume $a < x < a+\delta$ $\left(\text{Show } \frac{T(x)}{B(x)D(x)} > H, \left|\frac{T(x)}{B(x)D(x)}\right| > H\right)$

11. $0 < x-a < \delta$ — 10, Subtract a

12. $|x-a| = x-a$ — 11, definition of absolute value

13. $|x-a| < \delta$ — 12, 11

14. $L_1 < T(x) < U_1 < 0$ — 3, 13, since $|x-a| < \delta \overset{9}{\le} \delta_1$

15. $L_2 < B(x) < U_2 < 0$ — 4, 13, since $|x-a| < \delta \overset{9}{\le} \delta_2$

16. $a < x < a+\delta_3$ — 10 since $\delta \overset{9}{\le} \delta_3$

17. $0 < D(x) < \frac{|U_1|}{|L_2|H}$ — 8, 16

$|T(x)| = -T(x) > -U_1 = |U_1|$ $\qquad$ $|L_2| = -L_2 > -B(x) = |B(x)|$

18. $$\left|\frac{T(x)}{B(x)\,D(x)}\right| \overset{A-5,\ A-6}{=} \frac{|T(x)|}{|B(x)|\,|D(x)|} \overset{14}{>} \frac{|U_1|}{|B(x)|\,|D(x)|} \overset{15}{>} \frac{|U_1|}{|L_2|\,|D(x)|} \overset{17}{=} \frac{|U_1|}{|L_2|\,D(x)} \overset{17}{>} \frac{|U_1|}{|L_2|\left(\frac{|U_1|}{|L_2|H}\right)} = H$$

19. $\frac{T(x)}{B(x)\,D(x)} > 0$ — 14, 15, 17

20. $\frac{T(x)}{B(x)\,D(x)} > H$ — 18, 19, definition of abs. value

Def$^{\underline{n}}$ 27-1 $\lim_{x \to a^-} f(x) = +\infty$ IFF
$\forall H > 0, \exists \delta > 0 \ni \forall a - \delta < x < a, f(x) > H$

Theorem 28-1 $\lim_{x \to 5^-} \frac{-3}{x-5} = +\infty$

(Show $\forall H > 0, \exists \delta > 0 \ni \forall 5 - \delta < x < 5, \frac{-3}{x-5} > H$)

1. Assume $H > 0$ (Show $\exists \delta > 0 \ni \forall 5 - \delta < x < 5, \frac{-3}{x-5} > H$)
2. Let $\delta = \frac{3}{H}$; Note $\delta > 0$ (Show $\forall 5 - \delta < x < 5, \frac{-3}{x-5} > H$)
3. Assume $5 - \delta < x < 5$ (Show $\frac{-3}{x-5} > H$)
4. $5 - \frac{3}{H} < x < 5$ 3, 2
5. $-\frac{3}{H} < x - 5 < 0$ 4
6. $-3 < H(x-5)$ 5
7. $\frac{-3}{x-5} > H$ 6, 5, Divide by $x - 5 < 0$

Def 28-2: $\lim_{x \to a^-} f(x) = -\infty$ IFF
$\forall D < 0, \exists \delta > 0 \ni \forall a - \delta < x < a, f(x) < D$

Def 28-3: $\lim_{x \to a^-} f(x) = 0^-$ IFF
$\forall \varepsilon > 0, \exists \delta > 0 \ni \forall a - \delta < x < a, -\varepsilon < f(x) < 0$

Thm 28-4: $\lim_{x \to a^-} x - a = 0^-$

(Show $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall a - \delta < x < a, \; -\varepsilon < x - a < 0$)

1. Assume $\varepsilon > 0$ (Show $\exists \delta > 0 \ni \forall a - \delta < x < a$, $-\varepsilon < x - a < 0$)
2. Let $\delta = \varepsilon$; Note $\delta > 0$ (Show $\forall a - \delta < x < a, -\varepsilon < x - a < 0$)
3. Assume $a - \delta < x < a$ (Show $-\varepsilon < x - a < 0$)
4. $a - \varepsilon < x < a$ 3, 2
5. $-\varepsilon < x - a < 0$ 4

Thm 28-5: Given T is continuous at a, B is continuous at a, and $\lim_{x \to a^-} D(x) = 0^-$.

(a) if $T(a) > 0$ and $B(a) > 0$, then $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = -\infty$

(b) if $T(a) > 0$ and $B(a) < 0$, then $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = +\infty$

(c) if $T(a) < 0$ and $B(a) > 0$, then $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = +\infty$

(d) if $T(a) < 0$ and $B(a) < 0$, then $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = -\infty$

Given: T, B are cont. at a and $\lim_{x \to a^-} D(x) = 0^-$ $\boxed{28A}$

Proof of Theorem 28-5 (a)

1. Assume $T(a) > 0$ and $B(a) > 0$ (Show $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = -\infty$)
(Show for all $\mathbb{D} < 0$, there is a $\delta > 0$ such that for all $a-\delta < x < a$, $\frac{T(x)}{B(x)D(x)} < \mathbb{D}$)

1.5 Assume $\mathbb{D} < 0$ (Show there is a $\delta > 0$ such that for all $a-\delta < x < a$, $\frac{T(x)}{B(x)D(x)} < \mathbb{D}$; $\left|\frac{T(x)}{B(x)D(x)}\right| > |\mathbb{D}|$)

2. T is cont. at a and B is cont. at a. Given

3. There are a $\delta_1 > 0$, $L_1 > 0$, $U_1 > 0$ such that for all $x \in R$, if $|x-a| < \delta_1$, $0 < L_1 < T(x) < U_1$ 1, 2, Th^m 25-1

4. There are a $\delta_2 > 0$, $L_2 > 0$, $U_2 > 0$ such that for all $x \in R$, if $|x-a| < \delta_2$, then $0 < L_2 < B(x) < U_2$ 1, 2, Th^m 25-1

5. $\frac{L_1}{U_2|\mathbb{D}|} > 0$ 3, 4, 1.5, Th^m A-1, Th^m A-4a

6. $\lim_{x \to a^-} D(x) = 0^-$ Given

7. For every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $a-\delta < x < a$, $-\varepsilon < D(x) < 0$. 6, Def^n

8. There is a $\delta_3 > 0$ such that for all $a-\delta_3 < x < a$, $-\left(\frac{L_1}{U_2|\mathbb{D}|}\right) < D(x) < 0$. 7, 5

9. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. So $\delta > 0$, $\delta \le \delta_1$, $\delta \le \delta_2$, $\delta \le \delta_3$ 3, 4, 8
(Show for all $a-\delta < x < a$, $\frac{T(x)}{B(x)D(x)} < \mathbb{D}$

10. See 11

28 B

11. Assume $a-\delta < x < a$ $\left(\text{Show } \frac{T(x)}{B(x)D(x)} < D;\ \left|\frac{T(x)}{B(x)D(x)}\right| > |D|\right)$

12. $-\delta < x-a < 0 \overset{9}{<} \delta$ — 11, subtract a

13 $|x-a| < \delta$ — 12, Thm A-3

14. $0 < L_1 < T(x) < U_1$ — 3, 13, $|x-a| < \delta \overset{9}{\le} \delta_1$

15. $0 < L_2 < B(x) < U_2$ — 4, 13, $|x-a| < \delta \overset{9}{\le} \delta_2$

16. $a-\delta_3 \overset{9}{\le} a-\delta \overset{11}{<} x < a$ — $-\delta_3 \le -\delta$ since $\delta \overset{9}{\le} \delta_3$

17 $-\left(\frac{L_1}{U_2|D|}\right) < D(x) < 0$ — 8, 16

17.5 $\frac{L_1}{U_2|D|} > -D(x) = |D(x)| > 0$ — mult 17 by -1; def^n of abs. val.

18. $$\left|\frac{T(x)}{B(x)D(x)}\right| \overset{\text{A-5, A-6}}{=} \frac{|T(x)|}{|B(x)||D(x)|} \overset{\text{def q.u. 14,15}}{=} \frac{T(x)}{B(x)|D(x)|} \overset{14}{>} \frac{L_1}{B(x)|D(x)|}$$

$$\overset{15}{>} \frac{L_1}{U_2|D(x)|} \overset{17.5}{>} \frac{L_1}{U_2\left(\frac{L_1}{U_2|D|}\right)} = \frac{L_1 \cdot U_2}{U_2 L_1}|D| = |D|$$

19. $\frac{T(x)}{B(x)D(x)} < 0$ — 14, 15, 17

20. $-\frac{T(x)}{B(x)D(x)} > -D$ — 18, definition of absolute value

21 $\frac{T(x)}{B(x)D(x)} < D$ — 20 multiply by -1

$\boxed{28C}$

<u>Theorem 28C</u> Let $k \in R$. If f is continuous at p, then $k \cdot f$ is continuous at p.

<u>Proof</u>: 1. Assume f is continuous at p (Show $k \cdot f$ is continuous at p) (Show for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $|kf(x) - kf(p)| < \varepsilon$.)

2. Assume $\varepsilon > 0$ (Show there is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $|kf(x) - kf(p)| < \varepsilon$.)

3. $\frac{\varepsilon}{|k|+1} > 0$ 2, Thm A-1

4. For every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $|f(x) - f(p)| < \varepsilon$ 1, def^n of cont.

5. There is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $|f(x) - f(p)| < \frac{\varepsilon}{|k|+1}$ 3, 4

(Show for all $x \in R$, if $|x-p| < \delta$, then $|kf(x) - kf(p)| < \varepsilon$)

6. Assume $x \in R$ and $|x-p| < \delta$ (Show $|kf(x) - kf(p)| < \varepsilon$)

7. $|f(x) - f(p)| < \frac{\varepsilon}{|k|+1}$ 5, 6

8. $|k|\,|f(x) - f(p)| \overset{7}{\leq} |k|\left(\frac{\varepsilon}{|k|+1}\right) = \frac{|k|}{|k|+1}\varepsilon \overset{L12\text{-}1}{<} (1)\varepsilon = \varepsilon$

9. $|kf(x) - kf(p)| = |k(f(x) - f(p))| \overset{A\text{-}5}{=} |k| \cdot |f(x) - f(p)| \overset{8}{<} \varepsilon$

Given: T, B are cont. at a and $\lim\limits_{x\to a^-} D(x) = 0^-$ 28 D

Proof of Theorem 28–5 (b)

1. Assume $T(a) > 0$ and $B(a) < 0$ (Show $\lim\limits_{x\to a^-} \frac{T(x)}{B(x)D(x)} = +\infty$)
2. T is continuous at a and B is continuous at a. Given
3. $-1 \cdot B = -B$ is continuous at a. 2, Th^m 28C, $k=-1$
4. $-B(a) > 0$ 1
5. $\lim\limits_{x\to a^-} \frac{T(x)}{-B(x)D(x)} = -\infty$ 1, 2, 3, 4, Th^m 28–5a
6. For all $D<0$, there is a $\delta>0$ such that for all $a-\delta < x < a$, $\frac{T(x)}{-B(x)D(x)} < D$. 5, def^n

(Show For all $H>0$, there is a $\delta>0$ such that for all $a-\delta<x<a$, $\frac{T(x)}{B(x)D(x)} > H$)

7. Assume $H>0$ (Show there is a $\delta>0$ such that for all $a-\delta<x<a$, $\frac{T(x)}{B(x)D(x)} > H$)
8. $-H<0$ 7, mult by -1
9. There is a $\delta>0$ such that for all $a-\delta<x<a$, $\frac{T(x)}{-B(x)D(x)} < -H$ (show for all $a-\delta<x<a$, $\frac{T(x)}{B(x)D(x)} > H$) 6, 8
10. Assume $a-\delta<x<a$ (Show $\frac{T(x)}{B(x)D(x)} > H$)
11. $\frac{T(x)}{-B(x)D(x)} < -H$ 9, 10
12. $\frac{T(x)}{B(x)D(x)} > H$ 11, multiply by -1

28E

Given: T, B are continuous at a and $\lim_{x \to a^-} D(x) = 0^-$

Th^m 28-5d

If $T(a) < 0$ and $B(a) < 0$, then $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = -\infty$

1.	Assume $T(a) < 0$ and $B(a) < 0$ (Show $\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} = -\infty$)	
2.	T is continuous at a and B is continuous at a	given
3.	$^-1 \cdot T = {}^-T$ is continuous at a	2, Th^m 28C, $K = {}^-1$
4.	$^-1 \cdot B = {}^-B$ is continuous at a	2, Th^m 28C, $K = {}^-1$
5.	$^-T(a) > 0$ and $^-B(a) > 0$	1, mult. by $^-1$
6.	$\lim_{x \to a^-} \frac{T(x)}{B(x)D(x)} \overset{H.S.}{=} \lim_{x \to a^-} \frac{^-T(x)}{^-B(x)D(x)} = -\infty$	3, 4, 5, given $\lim_{x \to a^-} D(x) = 0^-$ Th^m 28-5a

Theorem 29-1 Given $a, b, c \in R$ and $f(x) = ax^2+bx+c$
For all $p \in R$, f is continuous at p.

1. Assume $p \in R$ (Show f is continuous at p)
2. case 1 $a=0$
3. f is continuous at p Th^{m} 24-1
4. case 2 $a \neq 0$
(Show for every $\varepsilon>0$, there is a $\delta>0$ such that for all $x \in R$, if $|x-p|<\delta$, then $|f(x)-f(p)|<\varepsilon$.)
5. Assume $\varepsilon>0$ (Show there is a $\delta>0$ such that for all $x \in R$, if $|x-p|<\delta$, then $|f(x)-f(p)|<\varepsilon$)
6. $\dfrac{\varepsilon}{2|a||p|+|a|+|b|}>0$ since $a \neq 0$, hence $|a| \neq 0$
7. Let δ be the minimum of 1 and $\dfrac{\varepsilon}{2|a||p|+|a|+|b|}$; note $\delta>0$
(Show for all $x \in R$, if $|x-p|<\delta$, then $|f(x)-f(p)|<\varepsilon$)
8. Assume $x \in R$ (Show if $|x-p|<\delta$, then $|f(x)-f(p)|<\varepsilon$)
9. Assume $|x-p|<\delta$ (Show $|f(x)-f(p)|<\varepsilon$)
10. case 2-A $|ax+ap+b|=0$
11. $|x-p||ax+ap+b|=0<\varepsilon$
12. $|(x-p)(ax+ap+b)|<\varepsilon$
13. $|(x-p)(ax+ap+b)| = |(x-p)(a(x+p)+b)| = |a(x-p)(x+p)+b(x-p)|$
$= |a(x^2-p^2)+bx-bp| = |ax^2+bx+c-ap^2-bp-c|$
$= |ax^2+bx+c-(ap^2+bp+c)| = |f(x)-f(p)|$
14. $\boxed{|f(x)-f(p)|<\varepsilon}$
15. case 2-B $|ax+ap+b| \neq 0$
16. $|x-p|<\delta \leq 1$ 9, 7
17. $|x|-|p| \leq |x-p|<1$ Th^{m}, 16 $|x|$ gets bounded.
18. $|x|<|p|+1$
19. $0<|ax+ap+b| \leq |ax|+|ap+b| \leq |ax|+|ap|+|b| = |a||x|+|a||p|+|b|$
$\underset{18}{\leq} |a|(|p|+1)+|a||p|+|b|$

$= |a||p| + |a| + |a||p| + |b| = 2|a||p| + |a| + |b|$

20 $\quad 0 < |ax+ap+b| \le 2|a||p| + |a| + |b|$

21 $\quad \dfrac{1}{2|a||p|+|a|+|b|} \le \dfrac{1}{|ax+ap+b|}$ $\quad$ 20

22. $\quad \dfrac{\varepsilon}{2|a||p|+|a|+|b|} < \dfrac{\varepsilon}{|ax+ap+b|}$ $\quad$ 21

23. $\quad |x-p| \underset{9}{<} \delta \underset{7}{\le} \dfrac{\varepsilon}{2|a||p|+|a|+|b|} \underset{22}{\le} \dfrac{\varepsilon}{|ax+ap+b|}$

24 $\quad |x-p| < \dfrac{\varepsilon}{|ax+ap+b|}$

25 $\quad |x-p||ax+ap+b| < \varepsilon$ $\quad$ 24, 20

26 $\quad |x-p||ax+ap+b| = |(x-p)(ax+ap+b)| = |f(x)-f(p)|$ $\quad$ by the same algebra as 13

27. $\quad \boxed{|f(x)-f(p)| < \varepsilon}$

So in both cases 2-A and 2-B we have $|f(x)-f(p)| < \varepsilon$
so we have $|f(x)-f(p)| < \varepsilon$

28. So in case 2 we have proven: for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x-p| < \delta$, then $|f(x)-f(p)| < \varepsilon$

29. f is continuous at p

30. In both cases 1 + 2 we proved f is continuous at p
So f is continuous at p.

Definition 30-1 $\;x = a$ is a vertical asymptote for $f(x)$ iff at least one of the following happens:
$\lim_{x\to a^+} f(x) = +\infty$ or $\lim_{x\to a^+} f(x) = -\infty$ or $\lim_{x\to a^-} f(x) = +\infty$ or $\lim_{x\to a^-} f(x) = -\infty$

Horizontal and vertical asymptote summary

Horizontal asymptotes: if either of $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ has a limit of a real number k, then $y = k$ is a horizontal asymptote.

Vertical asymptotes: For each a such that $\lim_{x\to a^\pm} f(x) = \pm\infty$, $x = a$ will be a vertical asymptote. (To find these – try all values a that make the denominator 0)

For $f(x) = \frac{x^2+4}{x^2-x-6}$, find all horizontal and vertical asymptotes, plot, analyze, sketch (RIGOROUSLY PROVE ALL RESULTS)

I HORIZONTAL ASYMPTOTES

1. $\lim_{x\to\infty} \frac{x^2+4}{x^2-x-6} = \frac{1}{1} = 1$ Thm 20C

2. $y=1$ is a horizontal asymptote

   def^n of horizontal asym.

By similar reasoning as on page 31 and using theorems concerning $\lim_{x \to -\infty}$ similar to those on page 16 and 18 one could prove $\lim_{x \to -\infty} \frac{x^2+4}{x^2-x-6} = 1$

II VERTICAL ASYMPTOTES

$$f(x) = \frac{x^2+4}{x^2-x-6} = \frac{x^2+4}{(x+2)(x-3)} = \frac{x^2+4}{(x-3)(x-\{-2\})}$$

A. Check $x=3$ to see if it is a vertical asymptote

1. Let $T(x) = x^2+4$
2. T is continuous at 3 — Theorem 29-1
3. $T(3) = 9+4 = 13 > 0$
4. Let $B(x) = x+2$
5. B is continuous at 3 — Theorem 29-1 with $a=0$
6. $B(3) = 5 > 0$
7. Let $D(x) = x-3$
8. $\lim_{x \to 3^+} D(x) = 0^+$ — Th^{m} 26-1
9. $\lim_{x \to 3^+} \frac{T(x)}{B(x)D(x)} = +\infty$ — Th^{m} 26-2-a, 2,3,5,6,8
10. $\lim_{x \to 3^+} \frac{x^2+4}{(x+2)(x-3)} = +\infty$ — 9, 1, 4, 7
11. $x=3$ is a vertical asymptote for $f(x)$ since $\lim_{x \to 3^+} f(x) = +\infty$
12. $\lim_{x \to 3^-} D(x) = 0^-$ — Th^{m} 28-4, 7
13. $\lim_{x \to 3^-} \frac{T(x)}{B(x)D(x)} = -\infty$ — Th^{m} 28-5-a 2,3,5,6,12
14. $\lim_{x \to 3^-} f(x) = -\infty$ since $\frac{T(x)}{B(x)D(x)} = f(x)$, 13

B. check $x=-2$ to see if it is a vertical asymptote.

1. Let $T(x) = x^2+4$
2. T is continuous at -2 — Th^{m} 29-1
3. $T(-2) = 4+4 = 8 > 0$
4. Let $B(x) = x-3$
5. B is continuous at $x=-2$ — Th^{m} 29-1 with $a=0$

6. $B(-2) = -2-3 = -5 < 0$

7. Let $D(x) = x-(-2)$

8. $\lim\limits_{x \to -2^+} D(x) = 0^+$ Th^m 26-1 , 7

9. $\lim\limits_{x \to -2^+} \frac{T(x)}{B(x)D(x)} = -\infty$ Th^m 26-2-b, 2, 3, 5, 6, 8

10. $\lim\limits_{x \to -2^+} f(x) = -\infty$ 9, since $f(x) = \frac{T(x)}{B(x)D(x)}$

11. $x = -2$ is a vertical asymp. 10

12. $\lim\limits_{x \to -2^-} D(x) = 0^-$ Th^m 28-4, 7

13. $\lim\limits_{x \to -2^-} \frac{T(x)}{B(x)D(x)} = +\infty$ Th^m 28-5-b, 2, 3, 5, 6, 12

14. $\lim\limits_{x \to -2^-} f(x) = +\infty$ since $f(x) = \frac{T(x)}{B(x)D(x)}$, 13

III ANALYSIS

A. $\boxed{\text{See if the function crosses the horizontal asymptote } y=1}$

1. $f(x) = \dfrac{x^2+4}{x^2-x-6} = 1$
2. $x^2+4 = x^2-x-6$ 1
3. $4 = -x-6$ 2
4. $x = -10$
5. At $x=-10$, the graph crosses $y=1$

B. $\boxed{x<-10}$

1. $-x > 10$ B
2. $-x-10 > 0$ 1
3. $(x^2+4)+(-x-10) \overset{2}{>} x^2+4 > 0$
4. $1 \overset{3}{>} \dfrac{x^2+4}{x^2+4+(-x-10)} = \dfrac{x^2+4}{x^2-x-6} = f(x)$
5. for $x<-10$, $f(x)<1$ B, 4

C. $\boxed{x<-2}$

1. $x<3$ C
2. $x+2<0$ and $x-3<0$ C, 1
3. $x^2+4>0$
4. $\dfrac{x^2+4}{(x+2)(x-3)} = \dfrac{x^2+4}{x^2-x-6} = f(x) \overset{2,3}{>} 0$
5. for $x<-2$, $f(x)>0$

D. $\boxed{-2<x<3}$

1. $-2<x$ and $x<3$ D
2. $0<x+2$ and $x-3<0$ 1
3. $(x+2)(x-3)<0$ 2
4. $\frac{x^2+4}{(x+2)(x-3)}<0$; $\frac{x^2+4}{x^2-x-6}<0$ 3
5. For $-2<x<3$, the function is below the x-axis D,4

E. $\boxed{x>3}$

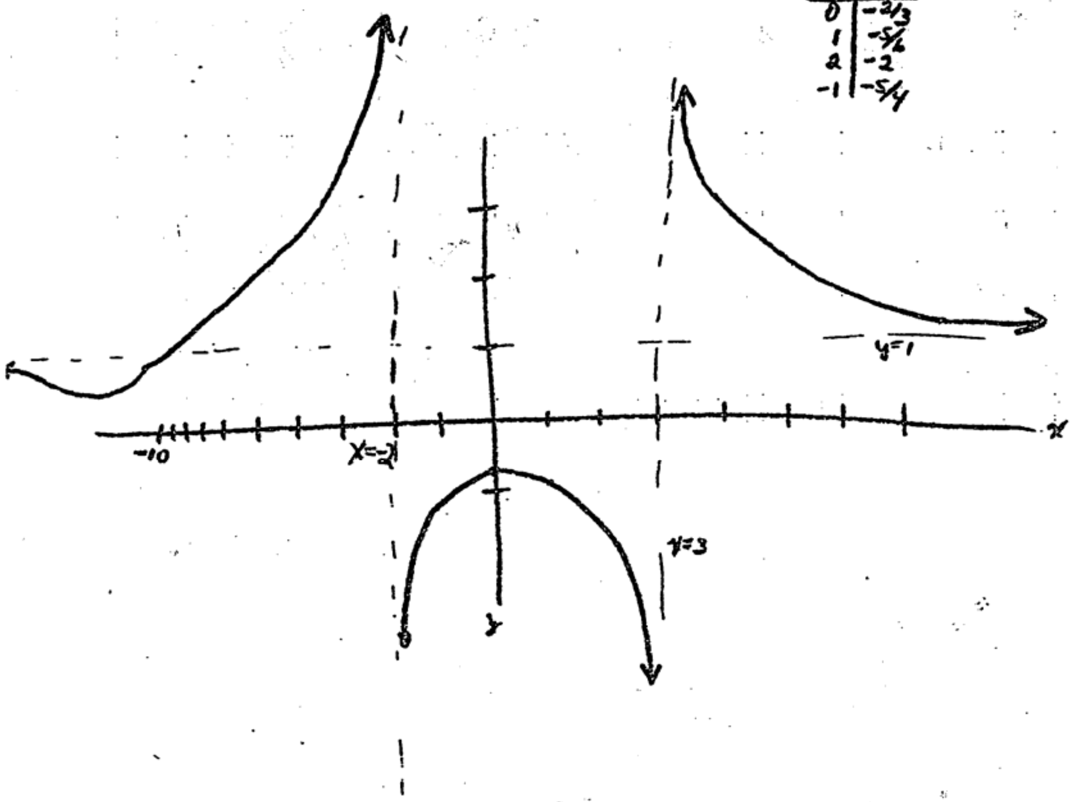
1. $x>-2$ E
2. $x-3>0$ and $x+2>0$ E, 1
3. $(x-3)(x+2)>0$ 2
4. $x^2-x-6>0$ 3
5. $(x^2+4)+(-x-10)>0$ 4
6. $x>-10$ E
7. $0>-x-10$ 6
8. $(x^2+4)\overset{7}{>}(x^2+4)+(-x-10)\overset{5}{>}0$
9. $\frac{x^2+4}{(x^2+4)+(-x-10)}\overset{8}{>}1$
10. $f(x)\overset{9}{>}1$
11. For $x>3$, $f(x)>1$ E,10

F. $\boxed{-10 < x < -2}$

1.	$x < -2$ and $x < 3$	F
2.	$x+2<0$ and $x-3<0$	1
3.	$(x+2)(x-3) > 0$	2
4.	$x^2 - x - 6 > 0$	3
5.	$(x^2+4)+(-x-10) > 0$	4
6.	$x > -10$	F
7.	$0 > -x-10$	6
8.	$(x^2+4) \overset{7}{>} (x^2+4)+(-x-10) \overset{5}{>} 0$	
9.	$\dfrac{x^2+4}{(x^2+4)+(-x-10)} > 1$	
10.	$f(x) > 1$	9
11.	For $-10<x<-2$, $f(x) > 1$	F, 10

B, C		F		D		E
$0 < f(x) < 1$	-10	$f(x) > 1$	-2	$f(x) < 0$	3	$f(x) > 1$

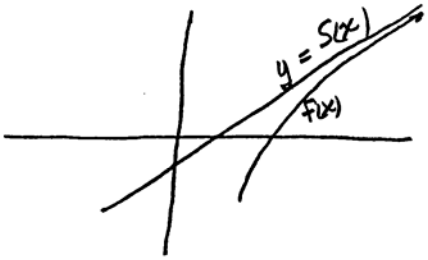
x	y
0	-2/3
1	-5/6
2	-2
-1	-5/4



$$f(x) = \frac{x^2+9}{x^2-x-12}$$

DO THE SAME RIGOROUS DETAILED PROOF ANALYSIS AND SKETCH OF THIS PROBLEM AS WAS DONE ON PAGES 31–34

SLANT ASYMPTOTES (OBLIQUE)



$S(x)$ is a slant asymptote for $f(x)$.

Does $f(x) = \dfrac{2x^2+7x+6}{x-1}$ have any slant asymptotes

Scratch thinking process:

$$\begin{array}{r|l} & \;\; 2x + 9 \\ \hline x-1 & 2x^2 + 7x + 6 \\ & \ominus 2x^2 \oplus 2x \\ \hline & \qquad 9x + 6 \\ & \quad \ominus 9x \oplus 9 \\ \hline & \qquad\quad 15 \end{array}$$

$$\frac{2x^2+7x+6}{x-1} = 2x+9+\frac{15}{x-1}$$

Note: as $x \to +\infty$ this goes to 0.

__Claim__: $S(x) = 2x+9$ is a slant asymptote for $f(x) = \dfrac{2x^2+7x+6}{x-1}$

Before we prove the claim the following definition is needed.

__Definition 36-1__. Let $S(x) = ax+b$ with $a \neq 0$.
$S(x)$ is a slant asymptote for $f(x)$ if and only if either $\lim\limits_{x \to +\infty} f(x) - S(x) = 0$ or

$\lim\limits_{x \to -\infty} f(x) - S(x) = 0$

LET $f(x) = \dfrac{2x^2+7x+6}{x-1}$. PROVE $S(x) = 2x+9$ IS A SLANT ASYMPTOTE FOR $f(x)$. FIND ALL VERTICAL ASYMPTOTES FOR $f(x)$. DO ENOUGH ANALSIS WITH PLOTTING TO SKETCH

A. PROVE $S(x)$ IS A SLANT ASYMPTOTE FOR $f(x)$ (SHOW $\lim\limits_{x\to\infty} f(x) - S(x) = 0$)

1. FOR $x \neq 1$, $2x+9 + \dfrac{15}{x-1} = \dfrac{(2x+9)(x-1)+15}{x-1}$
$= \dfrac{2x^2-2x+9x-9+15}{x-1} = \dfrac{2x^2+7x+6}{x-1}$

2. FOR $x \neq 1$, $f(x) - S(x) =$
$2x+9+\dfrac{15}{x-1} - (2x+9) = \dfrac{15}{x-1}$ — 1, DEF OF f, S

3. $\lim\limits_{x\to\infty} f(x) - S(x) = \lim\limits_{x\to\infty} \dfrac{15}{x-1} = \dfrac{0}{1} = 0$ — 2, T20A

Note: Similarly it could be proven $\lim\limits_{x\to-\infty} f(x) - S(x) = 0$

B. PROVE $\lim_{x \to 1^+} \frac{2x^2+7x+6}{x-1} = +\infty$ AND

$\lim_{x \to 1^-} \frac{2x^2+7x+6}{x-1} = -\infty$

1. FOR ALL $x \neq 1$, $\frac{2x^2+7x+6}{x-1} = \frac{2x^2+7x+6}{1 \cdot (x-1)}$
2. LET $T(x) = 2x^2+7x+6$
3. T IS CONTINUOUS AT 1   2, T29-1
4. $T(1) = 2(1^2)+7(1)+6 > 0$
5. LET $B(x) = 1$
6. B IS CONTINUOUS AT 1   5, T29-1 $a=b=0$
7. $B(1) = 1 > 0$ 5
8. LET $D(x) = x-1$
9. $\lim_{x \to 1^+} D(x) \overset{8}{=} \lim_{x \to 1^+} x-1 \overset{T26-1}{=} 0^+$
10. $\lim_{x \to 1^+} \frac{T(x)}{B(x)D(x)} = +\infty$ 3, 4, 6, 7, 9, T26-2a

11. $+\infty \overset{10}{=} \lim_{x \to 1^+} \frac{T(x)}{B(x)D(x)} \overset{2,5,8}{=} \lim_{x \to 1^+} \frac{2x^2+7x+6}{1 \cdot (x-1)}$

$= \lim_{x \to 1^+} \frac{2x^2+7x+6}{x-1} = \lim_{x \to 1^+} f(x)$

12. $x=1$ IS A VERTICAL ASYMPTOTE FOR $f(x)$ 11

13. $\lim_{x \to 1^-} D(x) = \lim_{x \to 1^-} x-1 = 0^-$ T28-4

14. $\lim_{x \to 1^-} \frac{T(x)}{B(x)D(x)} = -\infty$ 3,4,6,7,13,T28-5a

15. $\lim_{x \to 1^-} \frac{2x^2+7x+6}{x-1} = \lim_{x \to 1^-} \frac{2x^2+7x+6}{1(x-1)} \overset{2,5,8}{=}$

$\lim_{x \to 1^-} \frac{T(x)}{B(x)D(x)} \overset{14}{=} -\infty$

C. $f(x) = \frac{2x^2+7x+6}{x-1} = \frac{(2x+3)(x+2)}{x-1}$

$= \frac{2(x+\frac{3}{2})(x+2)}{x-1} = \frac{2(x-[-\frac{3}{2}])(x-[-2])}{x-1}$

38 B

$\boxed{\text{a. SHOW FOR } x<-2,\ f(x)<0}$

1. $x<-2$ SO $x<-\frac{3}{2}$ AND $x<1$
2. $x-[-2]<0$, $x-[-\frac{3}{2}]<0$, AND $x-1<0$, 1
3. $$\frac{\overset{pos}{2}\cdot\overset{neg}{\left(x-[-\frac{3}{2}]\right)}\cdot\overset{neg}{\left(x-[-2]\right)}}{\underset{neg}{x-1}}<0 \qquad 2$$
4. $f(x)<0$ C, 3

$\boxed{\text{b. SHOW FOR } -\frac{3}{2}<x<1,\ f(x)<0}$

1. $-\frac{3}{2}<x$ AND $x<1$, SO $x>-2$
2. $0<x-[-\frac{3}{2}]$, $x-1<0$, AND $x-[-2]>0$, 1
3. $$\frac{\overset{pos}{2}\cdot\overset{pos}{\left(x-[-\frac{3}{2}]\right)}\cdot\overset{pos}{\left(x-[-2]\right)}}{\underset{neg}{x-1}}<0 \qquad 2$$
4. $f(x)<0$ C, 3

$\boxed{\text{C. SHOW FOR } -2 < x < -\tfrac{3}{2},\ f(x) > 0}$

1. $-2 < x$ AND $x < -3/2$ SO $x < 1$
2. $0 < x - [-2]$, $x - [-\tfrac{3}{2}] < 0$ AND $x - 1 < 0$, 1
3. $\dfrac{\overset{\text{pos}}{2}\,\cdot\,\overset{\text{neg}}{\left(x - [-\tfrac{3}{2}]\right)}\,\cdot\,\overset{\text{pos}}{\left(x - [-2]\right)}}{\underset{\text{neg}}{x - 1}} > 0$ 2
4. $f(x) > 0$ C, 3

$\boxed{\text{D. SHOW FOR } x > 1,\ f(x) > S(x)}$

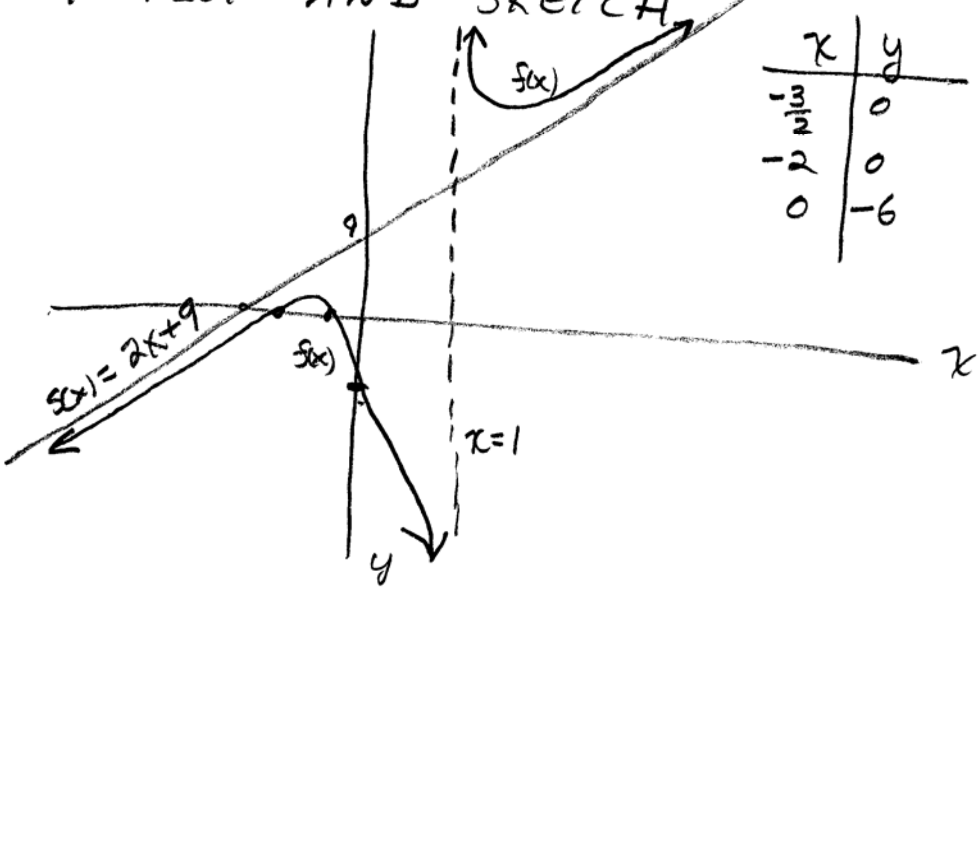
1. $x > 1$
2. $x - 1 > 0$ 1
3. $\dfrac{15}{x-1} > 0$ 2
4. $2x + 9 + \dfrac{15}{x-1} > 2x + 9$ 3, ADD $2x+9$
5. $\dfrac{2x^2 + 7x + 6}{x-1} > 2x + 9$ 4
6. $f(x) > S(x)$ 5

38D

E. SHOW FOR $x<1$, $f(x) < S(x)$

1. $x < 1$
2. $x - 1 < 0$
3. $\frac{15}{x-1} < 0$ 2
4. $2x+9 + \frac{15}{x-1} < 2x+9$ 3, ADD $2x+9$
5. $\frac{2x^2+7x+6}{x-1} < 2x+9$ 4
6. $f(x) < S(x)$

F. PLOT AND SKETCH



x	y
$-\frac{3}{2}$	0
-2	0
0	-6

Note: A slant asymptote is just a generalization of a horizontal asymptote.

Suppose $y=b$ is a horizontal asymptote for $f(x)$ and suppose $\lim\limits_{x\to+\infty} f(x)=b$. Intuitively we should see that $\lim\limits_{x\to+\infty} f(x)-b=0$. Let $a=0$. Then $\lim\limits_{x\to+\infty} f(x)-(ax+b)=0$. Let $S(x)=ax+b$.

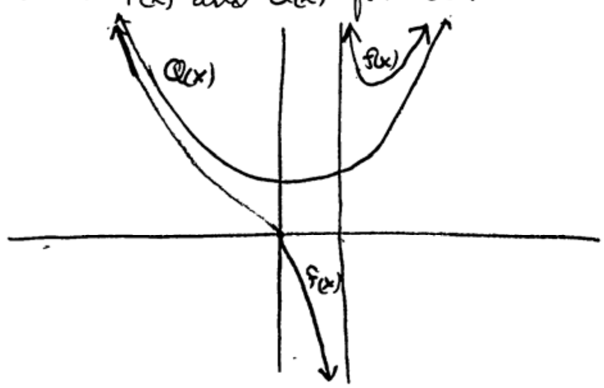
So $\lim\limits_{x\to+\infty} f(x)-S(x)=0$, which looks like part of the definition of $S(x)$ being a "slant" asymptote. (the restriction $a\neq 0$ was relaxed)

QUADRATIC ASYMPTOTES

Let $f(x)=\dfrac{x^3-x^2+x}{x-1}$. Long division gives the result $f(x)=x^2+1+\dfrac{1}{x-1}$.

Let $Q(x)=x^2+1$. $Q(x)$ will be a "quadratic" asymptote since $\lim\limits_{x\to\pm\infty} f(x)-Q(x)$ will equal 0

The graphs of $f(x)$ and $Q(x)$ follow.



GENERAL ASYMPTOTES

DEFINITION 40-1 $f(x)$ asymptotically approaches $G(x)$ if and only if either $\lim\limits_{x \to +\infty} f(x) - G(x) = 0$ or $\lim\limits_{x \to -\infty} f(x) - G(x) = 0$

What does $f(x) = \dfrac{x^4 + x - x^3}{x-1}$ asymptotically approach?

TOPICS WE MAY STUDY RIGOROUSLY

$\lim\limits_{x \to +\infty} f(x) = +\infty$

Faires & Faires approach to asymptotes

BUILDUP TO FAIRES & FAIRES IDEA

Theorem 41-1. If $b(x)$ is continuous at p and $b(p) \neq 0$, then there are a $B > 0$ and a $\delta > 0$ such that for all real numbers x, if $|x-p| < \delta$, then $|b(x)| > B$.

Proof: 1. Assume $b(x)$ is continuous at p and $b(p) \neq 0$. (Show there are a $B>0$ and a $\delta > 0$ such that for all real numbers x, if $|x-p|<\delta$, then $|b(x)| > B$)

2. For every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $|b(x) - b(p)| < \varepsilon$

3. $|b(p)| > 0$ 1, Thm A-1, Thm A-4a

4. $\dfrac{|b(p)|}{2} > 0$ 3

5. There is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $|b(x) - b(p)| < \dfrac{|b(p)|}{2}$

6. Let $B = \dfrac{|b(p)|}{2}$ (Show for all $x \in R$, if $|x-p|<\delta$, then $|b(x)| > B$)

7. Assume $x \in R$ (Show if $|x-p| < \delta$, then $|b(x)| > B$)

8. Assume $|x-p| < \delta$ (Show $|b(x)| > B$)

9. $|b(x) - b(p)| < \dfrac{|b(p)|}{2}$ 7, 8, 5

10 $|b(p)| - |b(x)| \overset{6.17\text{-}2}{\leq} |b(p) - b(x)| \overset{9}{<} \dfrac{|b(p)|}{2}$

11. $|b(p)| - \dfrac{|b(p)|}{2} \leq |b(x)|$ 10, Subt. $\dfrac{|b(p)|}{2}$, add $|b(x)|$

12. $B \overset{6}{=} \dfrac{|b(p)|}{2} \overset{11}{<} |b(x)|$

13 $|b(x)| > B$ 12

Theorem 42: If $t(x)$ is continuous at p, $b(x)$ is continuous at p and $b(p) \neq 0$, then $\frac{t(x)}{b(x)}$ is continuous at p.

Proof: 1. Assume $t(x)$ is continuous at p, $b(x)$ is continuous at p and $b(p) \neq 0$ (Show $\frac{t(x)}{b(x)}$ is continuous at p)
(Show for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $\left|\frac{t(x)}{b(x)} - \frac{t(p)}{b(p)}\right| < \varepsilon$)

2. Assume $\varepsilon > 0$ (Show there is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $\left|\frac{t(x)}{b(x)} - \frac{t(p)}{b(p)}\right| < \varepsilon$)

3. For every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $|t(x) - t(p)| < \varepsilon$ 1, defn of cont.

4. For every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in R$, if $|x-p| < \delta$, then $|b(x) - b(p)| < \varepsilon$ 1, defn of cont.

5. There are a $B > 0$ and $\delta_1 > 0$ such that for all $x \in R$, if $|x-p| < \delta_1$, then $|b(x)| > B$ 1, Thm 41-1

6. $\frac{B\varepsilon}{2} > 0$ 2,5

7. There is a $\delta_2 > 0$ such that for all $x \in R$, if $|x-p| < \delta_2$, then $|t(x) - t(p)| < \frac{B\varepsilon}{2}$ 6,3

8. $\frac{B|b(p)|\frac{\varepsilon}{2}}{|t(p)|+1} > 0$ 1, 5, 2, Thm A-1, Thm A-4a

9. There is a $\delta_3 > 0$ such that for all $x \in R$, if $|x-p| < \delta_3$, then $|b(x) - b(p)| < \frac{B|b(p)|\varepsilon/2}{|t(p)|+1}$

10. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Thus $\delta > 0$ and $\delta \le \delta_1$, $\delta \le \delta_2$, $\delta \le \delta_3$ (Show for all $x \in R$ if $|x-p| < \delta$, then $\left|\frac{t(x)}{b(x)} - \frac{t(p)}{b(p)}\right| < \varepsilon$) 5,7,9

$\boxed{43}$

11. Assume $x \in R$ and $|x-p|<\delta$ $\left(\text{Show } \left|\frac{t(x)}{b(x)} - \frac{t(p)}{b(p)}\right| < \varepsilon\right)$

12. $\left|\frac{t(x)}{b(x)} - \frac{t(p)}{b(p)}\right| = \left|\frac{t(x)b(p) - b(x)t(p)}{b(x)\,b(p)}\right| =$

$$\left|\frac{t(x)b(p) - t(p)b(p) + t(p)b(p) - b(x)\,t(p)}{b(x)\,b(p)}\right| \overset{A-5}{\underset{A-6}{=}}$$

$$\frac{|b(p)(t(x)-t(p)) + t(p)(b(p)-b(x))|}{|b(x)||b(p)|} \overset{L13-1}{\leq} 5, 11, \delta \leq \delta_1$$

$$\frac{|b(p)(t(x)-t(p))| + |t(p)(b(p)-b(x))|}{B\cdot|b(p)|} \overset{A-5}{=}$$

$$\frac{|b(p)||t(x)-t(p)| + |t(p)||b(p)-b(x)|}{B\cdot|b(p)|} =$$

$$\frac{\cancel{|b(p)|}|t(x)-t(p)|}{B\cancel{|b(p)|}} + \frac{|t(p)||b(x)-b(p)|}{B|b(p)|} < \begin{matrix} 7, 11, \delta \leq \delta_2 \\ 9, 11, \delta \leq \delta_3 \end{matrix}$$

$$\frac{1}{B}\left(\frac{B\varepsilon}{2}\right) + \frac{|t(p)|}{B|b(p)|}\left(\frac{B|b(p)|\,\varepsilon/2}{|t(p)|+1}\right) =$$

$$\frac{\varepsilon}{2} + \frac{|t(p)|}{|t(p)|+1}\,\frac{\varepsilon}{2} \overset{L17-1}{<} \frac{\varepsilon}{2} + (1)\frac{\varepsilon}{2} = \varepsilon$$

Theorem 44-1 If $M(x) = \frac{ax^2+bx+c}{dx^2+ex+f}$ and $dp^2+ep+f \neq 0$, then M is continuous at p.

Proof: 1. Assume $M(x) = \frac{ax^2+bx+c}{dx^2+ex+f}$ and $dp^2+ep+f \neq 0$
(Show M is continuous at p)

2. Let $t(x) = ax^2+bx+c$ and $b(x) = dx^2+ex+f$
3. t is cont. at p and b is cont. at p. 2, Th^m 29-1
4. $b(p) \neq 0$ 2, 1, definition of $b(p)$
5. $M(x) = \frac{t(x)}{b(x)}$ is continuous at p 3, 4, Th^m 42

Theorem 45-1 (Faires & Faires) Given M is continuous at p and $\lim_{x \to p^+} V(x) = +\infty$.

a) If $M(p) > 0$, then $\lim_{x \to p^+} M(x)V(x) = +\infty$
b) If $M(p) < 0$, then $\lim_{x \to p^+} M(x)V(x) = -\infty$

Proof of Th^m 45-1 a):

1. Assume $M(p) > 0$ (Show $\lim_{x \to p^+} M(x)V(x) = +\infty$)
(Show for all $H > 0$, there is a $\delta > 0$ such that for all $p < x < p+\delta$, $M(x)V(x) > H$)
2. Assume $H > 0$ (Show there is a $\delta > 0$ such that for all $p < x < p+\delta$, $M(x)V(x) > H$)
3. M is continuous at p GIVEN
4. There are a $\delta_1 > 0$, $L > 0$, $U > 0$ such that for all $x \in R$, if $|x-p| < \delta_1$, then $0 < L < M(x) < U$. 1, 3, Th^m 25-1

5. $\frac{H}{L} > 0$ 2, 4

6. $\lim\limits_{x \to p^+} V(x) = +\infty$ GIVEN

7. For all $H > 0$, there is a $\delta > 0$ such that for all $p < x < p+\delta$, $V(x) > H$ 6, def^n

8. There is a $\delta_2 > 0$ such that for all $p < x < p+\delta_2$, $V(x) > \frac{H}{L}$ 5, 7

9. Let $\delta = \min\{\delta_1, \delta_2\}$, $\delta > 0$, $\delta \le \delta_1$, $\delta \le \delta_2$ 4, 8
(Show for all $p < x < p+\delta$, $M(x)V(x) > H$)

10. Assume $p < x < p+\delta$ (Show $M(x)V(x) > H$)

11. $0 \overset{10}{<} x - p = |x-p| \overset{10}{<} \delta \overset{9}{\le} \delta_1$

12. $p < x < p + \delta \overset{9}{\le} p + \delta_2$ 10

13. $M(x) \cdot V(x) \overset{4,11}{>} L \cdot V(x) \overset{8,12}{>} L\left(\frac{H}{L}\right) = H$

Application 45 (compare with page 32)

Let $f(x) = \frac{x^2+4}{x^2-x-6} = \frac{x^2+4}{(x+2)(x-3)}$

1. Let $M(x) = \frac{x^2+4}{x+2}$ and $V(x) = \frac{1}{x-3}$

2. $3+2 \ne 0$

3. M is continuous at 3 1, 2, Th^m 44-1

4. $M(3) = \frac{3^2+4}{3+2} > 0$ 1

5. $\lim\limits_{x \to 3^+} \frac{1}{x-3} = +\infty$ Th^m 22-1

6. $\lim\limits_{x \to 3^+} M(x) \cdot V(x) = \lim\limits_{x \to 3^+} \left(\frac{x^2+4}{x+2}\right)\left(\frac{1}{x-3}\right) = \lim\limits_{x \to 3^+} \frac{x^2+4}{x^2-x-6} = +\infty$ 1, 3, 4, 5 Th^m 45-1a

Theorem 46-1 : (Faires & Faires) Given M is continuous at p and $\lim\limits_{x \to p^+} V(x) = -\infty$.

a) If $M(p) > 0$, then $\lim\limits_{x \to p^-} M(x)V(x) = -\infty$.

b) If $M(p) < 0$, then $\lim\limits_{x \to p^-} M(x)V(x) = +\infty$

Proof of Theorem 46-1 a) :

1. Assume $M(p) > 0$ (Show $\lim\limits_{x \to p^-} M(x)V(x) = -\infty$)

(Show for all $\mathcal{D} < 0$ there is a $\delta > 0$ such that for all $p-\delta < x < p$, $M(x)V(x) < \mathcal{D}$)

2. Assume $\mathcal{D} < 0$ (Show there is a $\delta > 0$ such that for all $p-\delta < x < p$, $M(x)V(x) < \mathcal{D}$)
3. M is continuous at p GIVEN
4. There are a $\delta_1 > 0$, $L, U > 0$ such that for all $x \in \mathbb{R}$, if $|x-p| < \delta_1$, then $0 < L < M(x) < U$. 3, 1, Th^m 25-1
5. $\lim\limits_{x \to p^-} V(x) = -\infty$ Given
6. For each $\mathcal{D} < 0$, there is a $\delta > 0$ such that for all $p-\delta < x < p$, $V(x) < \mathcal{D}$. 5, def^n
7. $\dfrac{\mathcal{D}}{L} < 0$
8. There is a $\delta_2 > 0$ such that for all $p-\delta_2 < x < p$, $V(x) < \dfrac{\mathcal{D}}{L}$ 6,7
9. Let $\delta = \min\{\delta_1, \delta_2\}$, $\delta > 0$, $\delta \le \delta_1$, $\delta \le \delta_2$ 4,8

(Show for all $p-\delta < x < p$, $M(x)V(x) < \mathcal{D}$)

10. Assume $p-\delta < x < p$ (Show $M(x)V(x) < \mathcal{D}$; $|M(x)V(x)| > |\mathcal{D}|$)

11. $-\delta \overset{10}{<} x-p \overset{10}{<} 0 \overset{9}{<} \delta$

12. $|x-p| \overset{A-3}{<} \delta \le \delta_1$

13. $p-\delta_2 \overset{9}{\le} p-\delta \overset{10}{<} x \overset{10}{<} p$ $\delta \overset{9}{\le} \delta_2$ so $-\delta_2 \le -\delta$, add p

14. $V(x) \overset{8,13}{<} \frac{\mathcal{D}}{L} \overset{7}{<} 0$

15. $|V(x)| \overset{14}{=} -V(x) \overset{14}{>} -\frac{\mathcal{D}}{L} \overset{14}{=} \left|\frac{\mathcal{D}}{L}\right|$

16. $|M(x)V(x)| \overset{A-5}{=} |M(x)||V(x)| \overset{4,12}{>} L|V(x)| \overset{15}{>} L\left|\frac{\mathcal{D}}{L}\right|$
$\overset{A-6}{=} \frac{L|\mathcal{D}|}{|L|} \overset{4,\text{def a.v.}}{=} \frac{L|\mathcal{D}|}{L} = |\mathcal{D}|$

17. $M(x)V(x) < 0$ 4, 14

18. $|M(x)V(x)| = -M(x)V(x)$ and $|\mathcal{D}| = -\mathcal{D}$ 17, 2, def^n ab. val.

19 $-M(x)V(x) > -\mathcal{D}$ 16, 18, Substitution

20 $M(x)V(x) < \mathcal{D}$ 19, multiply by -1

APPLICATION 47 (compare with page 32)

Let $f(x) = \frac{x^2+4}{x^2-x-6} = \frac{x^2+4}{(x+2)(x-3)}$

1. Let $M(x) = \frac{x^2+4}{x+2}$ and $V(x) = \frac{1}{x-3}$
2. $3+2 \ne 0$
3. M is continuous at 3
4. $M(3) = \frac{3^2+4}{3+2} > 0$ 1
5. Let $t(x) = 1 = b(x)$ and $D(x) = x-3$
6. t is cont at 3, b is cont at 3 Th^m 24-1
7. $\lim_{x\to 3^-} x-3 = 0^-$ Th^m 28-4
8. $t(3) = 1 > 0$ and $b(3) = 1 > 0$ 5
9. $\lim_{x\to 3^-} \frac{t(x)}{b(x)D(x)} = \lim_{x\to 3^-} \frac{1}{x-3} = \lim_{x\to 3^-} V(x) = -\infty$ Th^m 28-5a
10. $\lim_{x\to 3^-} M(x)V(x) = \lim_{x\to 3^-} \frac{x^2+4}{(x+2)(x-3)} = -\infty$ 3, 4, 9, Th^m 46-1a

<u>DEFINITION 48-1</u>: $\lim_{x\to\infty} f(x) = +\infty$ iff for each $H>0$, there is a $P>0$ such that for all $x>P$, $f(x)>H$.

<u>Theorem 48-2</u>: If $\lim_{x\to\infty} f(x)=+\infty$ and $\lim_{x\to\infty} g(x)=+\infty$ then $\lim_{x\to\infty} f(x)\cdot g(x) = +\infty$.

1. Assume $\lim_{x\to\infty} f(x)=+\infty$ and $\lim_{x\to\infty} g(x)=+\infty$. (Show $\lim_{x\to\infty} f(x)\cdot g(x)=+\infty$)
2. For all $H>0$, there is a $P>0$ such that for all $x>P$, $f(x)>H$. 1, def
3. For all $H>0$, there is a $P>0$ such that for all $x>P$, $g(x)>H$. 1, def

(Show for all $H>0$, there is a $P>0$ such that for all $x>P$, $f(x)\cdot g(x)>H$)

4. Assume $H>0$ (Show there is a $P>0$ such that for all $x>P$, $f(x)\cdot g(x)>H$)
5. $\sqrt{H}>0$ 4, def^n of $\sqrt{\ }$
6. There is a $P_1>0$ such that for all $x>P_1$, $f(x)>\sqrt{H}$ 2,5
7. There is a $P_2>0$ such that for all $x>P_2$, $g(x)>\sqrt{H}$ 3,5
8. Let $P=\max\{P_1,P_2\}$. $P>0$, $P\ge P_1$, $P\ge P_2$ 6,7

(Show for all $x>P$, $f(x)\cdot g(x)>H$)

9. Assume $x>P$ (Show $f(x)\cdot g(x)>H$)
10. $f(x)>\sqrt{H}$ 6, 9, $P\ge P_1$, 8
11. $g(x)>\sqrt{H}$ 7, 9, $P\ge P_2$, 8
12. $f(x)\cdot g(x)>g(x)\sqrt{H}$ 10, mult. by $g(x)>0$, 11, 5
13. $g(x)\sqrt{H}>\sqrt{H}\sqrt{H}=H$ 11, mult. by $\sqrt{H}>0$
14. $f(x)\cdot g(x)>H$ 12, 13, transitivity

<u>Lemma 48-3</u> For every $T\in R$, there is a $B\in R$ such that $B>T$ and $B>0$

1. Assume $T\in R$ (Show there is a $B\in R$ such that $B>T$ and $B>0$
2. <u>case 1</u> $1>T$
3. Let $B=1$
4. $B>T$ and $B>0$ 2,3
5. <u>case 2</u>: $1\le T$
6. Let $B=T+1$
7. $T+1>T\ge 1>0$
8. $B>T$ and $B>0$ 6,7

THM 49-1 ($K \in R$) IF $\lim_{x\to\infty} f(x) = +\infty$ AND $\lim_{x\to\infty} g(x) = K$, THEN $\lim_{x\to\infty} f(x)+g(x) = +\infty$

1. ASSUME $\lim_{x\to\infty} f(x) = +\infty$ AND $\lim_{x\to\infty} g(x) = K$
(SHOW $\lim_{x\to\infty} f(x)+g(x) = +\infty$) (SHOW FOR ALL $H>0$ THERE IS A $P>0$ SUCH THAT FOR ALL $x>P$, $f(x)+g(x)>H$)
2. ASSUME $H>0$ (SHOW THERE IS A $P>0$ SUCH THAT FOR ALL $x>P$, $f(x)+g(x)>H$)
3. FOR EVERY $\varepsilon>0$, THERE IS A $P>0$ SUCH THAT FOR ALL $x>P$, $|g(x)-K|<\varepsilon$ 1
4. THERE IS A $P_1>0$ SUCH THAT FOR ALL $x>P_1$, $|g(x)-K|<1$ 3, $1>0$
5. $H-(K-1) \in R$
6. THERE IS AN $H' \in R$ SUCH THAT $H' > H-(K-1)$ AND $H'>0$ 5, L 48-3
7. FOR ALL $H>0$, THERE IS A $P>0$ SUCH THAT FOR ALL $x>P$, $f(x)>H$
8. THERE IS A $P_2>0$ SUCH THAT FOR ALL $x>P_2$, $f(x)>H'$ 7, 6

49-A

9. LET $P = \max\{P_1, P_2\}$ SO $P > 0$
$P \geq P_1$, $P \geq P_2$ 4, 8

(SHOW FOR ALL $x > P$, $f(x) + g(x) > H$)

10. ASSUME $x > P$ (SHOW $f(x)+g(x) > H$)

11. $f(x) > H' > H - (K-1)$ 10, 8, 9, 6

12. $|g(x) - K| < 1$ 4, 9, 10

13. $-1 < g(x) - K < 1$ 12, A-3

14. $K - 1 < g(x)$ 13, ADD K

15. $g(x) > K - 1$ 14

16. $f(x) + g(x) > H - (K-1) + (K-1) = H$
11, 15, ADD

THEOREM 50 IF $\lim_{x\to\infty} f(x) = +\infty$ AND $\lim_{x\to\infty} g(x) = +\infty$, THEN $\lim_{x\to\infty} f(x)+g(x) = +\infty$

1. ASSUME $\lim_{x\to\infty} f(x) = +\infty$ AND $\lim_{x\to\infty} g(x) = +\infty$
   (SHOW $\lim_{x\to\infty} f(x)+g(x) = +\infty$, I.E. SHOW FOR ALL $H>0$, THERE IS A $P>0$ SUCH THAT FOR ALL $x>P$, $f(x)+g(x)>H$)
2. ASSUME $H_1>0$ (SHOW THERE IS A $P>0$ SUCH THAT FOR FOR ALL $x>P$, $f(x)+g(x)>H_1$)
3. FOR ALL $H>0$, THERE IS A $P>0$ SUCH THAT FOR ALL $x>P$, $f(x)>H$. 1, DEF
4. $\frac{H_1}{2}>0$ 2
5. THERE IS A $P_1>0$ SUCH THAT FOR ALL $x>P_1$, $f(x)>H_1/2$. 3, 4
6. FOR ALL $H>0$, THERE IS A $P>0$ SUCH THAT FOR ALL $x>P$, $g(x)>H$ 1, DEF

7. THERE IS A $P_2 > 0$ SUCH THAT FOR ALL $x > P_2$, $g(x) > H_1/2$ 6, 4

8. LET $P = MAX\{P_1, P_2\}$, SO $P > 0$, $P \geq P_1$, AND $P \geq P_2$ 5, 7

(SHOW FOR ALL $x > P$, $f(x) + g(x) > H_1$)

9. ASSUME $x > P$ (SHOW $f(x) + g(x) > H_1$)

10. $x > P \geq P_1$ 8, 9

11. $x > P \geq P_2$ 8, 9

12. $f(x) > H_1/2$ 5, 10

13. $g(x) > H_1/2$ 7, 11

14. $f(x) + g(x) > \frac{H_1}{2} + \frac{H_1}{2} = H_1$ 12, 13

THEOREM 52-1 $\lim_{x \to \infty} x = +\infty$

(SHOW FOR ALL $H > 0$, THERE IS A $P > 0$ SUCH THAT FOR ALL $x > P$, $x > H$)

1. ASSUME $H > 0$ (SHOW THERE IS A $P > 0$ SUCH THAT FOR ALL $x > P$, $x > H$)
2. LET $P = H$ (SHOW FOR ALL $x > P$, $x > H$)
3. ASSUME $x > P$ (SHOW $x > H$)
4. $x > H$ 2, 3

THEOREM 52-2 $\lim_{x \to \infty} 2x + 9 = +\infty$

1. $\lim_{x \to \infty} x = +\infty$ T52-1
2. $\lim_{x \to \infty} 2x = \lim_{x \to \infty} x + x = +\infty$ 1, T50
3. $\lim_{x \to \infty} 9 = 9$ T16-7
4. $\lim_{x \to \infty} 2x + 9 = +\infty$ 2, 3, T49-1

THEOREM 53 $\lim\limits_{x \to \infty} \dfrac{2x^2+7x+6}{x-1} =$

$$\lim_{x \to \infty} 2x+9+\frac{15}{x-1} = +\infty$$

1. $\lim\limits_{x \to +\infty} 2x+9 = +\infty$ T52-2

2. $\lim\limits_{x \to +\infty} \dfrac{15}{x-1} = \dfrac{0}{1} = 0$ T20A

3. $\lim\limits_{x \to +\infty} 2x+9+\dfrac{15}{x-1} = +\infty$ 1,2,T49-1

4. $\lim\limits_{x \to +\infty} \dfrac{2x^2+7x+6}{x-1} = +\infty$ 3

TRUTH GEMS

THE ANSWER IS IN THE BACK OF THE BOOK!

TRUTH GEM

BE IN THE WILL OF GOD FOR WHAT YOU DO

A. ROM 15:32 ... that I may come to you with joy by the WILL OF GOD, and may be refreshed together with you.

B. I come to you in the WILL OF GOD with joy and we will have refreshing math.

C. Being in the WILL OF GOD taking this course and faithfully, wisely studying you will flourish.

D. HEB. 10:36 For you have need of endurance, so that after you have done the WILL OF GOD you may receive the promise.

TG-2

TRUTH GEM

WISDOM

A. PR 1:7 WISDOM IS THE PRINCIPAL THING; THEREFORE GET WISDOM. AND IN ALL YOUR GETTING, GET UNDERSTANDING.

B. DEFINITIONS:

1. KNOWLEDGE: FACTS, GAINED INFORMATION
2. UNDERSTANDING: WHY A FACT IS A FACT.
3. WISDOM: BEING LED BY THE SPIRIT. KNOWING WHAT TO DO AT ANY MOMENT.

C. PRAY FOR WISDOM IN FAITH: JAMES 1:5 IF ANY OF YOU LACKS WISDOM, LET HIM ASK OF GOD, WHO GIVES TO ALL LIBERALLY AND WITHOUT REPROACH, AND IT WILL BE GIVEN HIM.

ONE
NOT DIS-INTEGRATED

A. I THESS 5:23 NOW MAY THE GOD OF PEACE HIMSELF SANCTIFY YOU COMPLETELY; AND MAY YOUR WHOLE SPIRIT, SOUL, AND BODY BE PRESERVED BLAMELESS AT THE COMING OF OUR LORD JESUS CHRIST.

B. MK 12:29 "... THE FIRST OF ALL THE COMMANDMENT IS: HEAR, O ISRAEL, THE LORD OUR GOD, THE LORD IS ONE.

C. YOU ARE ONE WHEN OUT OF LOVE OF GOD YOU FOCUS SPIRIT, SOUL, AND BODY ON THE GOD-LED TASK AT HAND ... THIS CLASS, IN THIS COURSE, NOW

NOT: BODY HERE AND MIND ELSEWHERE
NOT: BODY AND MIND HERE, BUT HEART NOT IN IT
NOT: ABSENT IN THE BODY BUT WITH US IN SPIRIT

} DIS-INTEGRATED

D. DESIRE AND ADMIRE BEING ONE.

TRUTH GEM

BEGIN

A. ACTS 1:1 "... of all that Jesus BEGAN both to do and teach."

B. MK 4:1 "And again He BEGAN to teach by the sea."

C. For a task to be accomplished, you must BEGIN.

D. BEGINNINGS:

1. BEGIN to see yourself as a faithful, good math student
2. See changes you need to make, and BEGIN on those changes.
3. See and learn BEGINNINGS of different problem types.

TRUTH GEM

PRESS ON

A. PLP 3:12 NOT THAT I HAVE ALREADY ATTAINED... BUT I <u>PRESS</u> <u>ON</u>, THAT I MAY LAY HOLD OF THAT FOR WHICH CHRIST JESUS HAS ALSO LAID HOLD OF ME.

B. LIKE A DISTANCE RUNNER OR CRUISE CONTROL

1. CONSTANCY, STEADFASTNESS
2. PATIENCE POWER

GOOD WORDS

C. <u>PRESS</u> <u>ON</u> CYCLE DONE WITH A GOOD ATTITUDE (NOT AT LAST MINUTE)

HEAR THE NEW

STUDY

DO HOMEWORK

REVIEW

D. HEB 6:12 ... THAT YOU DO NOT BECOME SLUGGISH (LAZY), BUT IMITATE THOSE WHO THROUGH FAITH AND PATIENCE INHERIT THE PROMISES.

TG-6

TRUTH GEM

COMPLETE IT

A. PLP 1:6 "... being confident of this very thing, that He who has BEGUN a good work in you will COMPLETE IT until the day of Jesus Christ.

B. THINGS GOD BEGAN, IN HIS WILL, GOD GIVES ABILITY AND PROVISION TO COMPLETE

1. SEE THESE THINGS THROUGH TO THE END.
2. VICTORY IS SWEET

C. THERE CAN BE BARRIERS TO BREAK THROUGH AT THE END.

1. LIKE A TAPE AT THE END OF A RACE.
2. LIKE THE SOUND BARRIER.

TRUTH GEM

BE ESTABLISHED

A. PS 90:17 And let the beauty of the Lord our God be upon us, and ESTABLISH the work of our hands for us;

B. Be established in the BEGIN - PRESS ON - COMPLETE IT cycle for working problems

C. Be established in knowing how to work certain problem types

D. Hebrew: Established = koon : things brought into incontrovertible existence like:

1. Your nature to faithfully study
2. Your ability to work certain problem types

E. To learn better how to be ESTABLISHED

1. Is 54:14a In righteousness you shall be ESTABLISHED
2. We will learn of righteousness

TRUTH GEM

MEDITATE

A. PS 1:2 BUT HIS DELIGHT IS IN THE LAW OF THE LORD AND IN HIS LAW HE MEDITATES* DAY AND NIGHT.

*PONDERS BY TALKING TO HIMSELF

B. WHAT YOU ARE IN THE WILL OF GOD TO LEARN CAN BE MEDITATED.

C. PICK A DEFINITION, THEOREM, OR PROBLEM DERIVATION.

1. CHEW ON IT WORD FOR WORD SEEKING UNDERSTADING
2. SPEAK IT OUT SO YOU CAN HEAR IT
3. WRITE IT DOWN OVER AND OVER
4. REVIEW IT

TG-9

TRUTH GEM

RIGHTEOUSNESS

A. (HEB 9:28a) For He will finish the work and cut it short in RIGHTEOUSNESS;

B. RIGHTEOUSNESS = RIGHT STANDING WITH GOD BY FAITH.

(PLP 3:9) and be found in Him, not having my own righteousness, which is from the law, but that which is through faith in Christ, the righteousness which is from God by faith.

C. BEING IN THE WILL OF GOD FOR WHAT YOU DO, IN RIGHT STANDING WITH GOD, THERE IS GREAT LIBERTY AND SPEEDUP IN WHAT YOU DO. (RIGHTEOUSNESS ENHANCED ACCELERATED LEARNING)

D. IT WOULD SLOW DOWN LEARNING IF A TEACHER WERE MAD AT A STUDENT (I.E. THE STUDENT WAS NOT IN RIGHT STANDING WITH THE TEACHER) AND STOOD OVER THEM SCOWLING AT THEM AS THEY TOOK A TEST (VS. "HIS FACE SHINE UPON YOU" FROM (NUM 6:25)).

TG-10

TRUTH GEM

RIGHTEOUSNESS - BOLD

A. PR. 28:1 THE WICKED FLEE WHEN NO ONE PURSUES, BUT THE RIGHTEOUS ARE AS BOLD AS A LION

B. PR. 30:30 A LION, WHICH IS MIGHTY AMONG BEASTS AND DOES NOT TURN AWAY FROM ANY;

C. LEARNING IS SLOWED DOWN BY WIMPILY, TIMIDLY TURNING AWAY FROM SOME MATH PROBLEMS.

D. LEARNING SPEEDUP BY BOLDLY (IN ACCORDANCE WITH YOUR RIGHTEOUS NATURE) TAKING ON THINGS ON YOUR PATH AND OVERCOMING.

E. NEXT, BOLDNESS AND HUMILITY, NOT CONTRADICTARY.

TG-11

TRUTH GEM

MAGNIFY THE SOLUTION AND NOT THE PROBLEM

A. PS 34:3 OH, MAGNIFY THE LORD WITH ME AND LET US EXALT HIS NAME TOGETHER

B. FOCUS ON WHAT YOU SEE IS TRUE AND GOOD AND CAN BE DONE. DO THAT. THE PROBLEM SHRINKS. SEE SOMETHING ELSE THAT IS TRUE AND GOOD AND CAN BE DONE. DO THAT. THE PROBLEM SHRINKS. PRESS ON DOING THIS UNTIL THE PROBLEM IS GONE!

C. PR 3:27 DO NOT WITHHOLD GOOD FROM THOSE TO WHOM IT IS DUE, WHEN IT IS IN THE POWER OF YOUR HAND TO DO SO.

D. JAMES 4:17 THEREFORE TO HIM WHO KNOWS TO DO GOOD AND DOES NOT DO IT, TO HIM IT IS SIN.

TG-12

TRUTH GEM

HOLY SPIRIT BAPTISM IMPLIES BOLDNESS

A. ACTS 4:31b ... THEY WERE ALL FILLED WITH THE HOLY SPIRIT AND SPOKE THE WORD OF GOD WITH BOLDNESS

B. RECALL PR 28:1b .. THE RIGHTEOUS ARE AS BOLD AS A LION.

C. YOU NEED GREAT BOLDNESS TO WORK SOME MATH PROBLEMS.
RIGHTEOUSNESS BOLDNESS + HOLY SPIRIT POWER BOLDNESS = SOLVED PROBLEMS IN THIS COURSE.

D. JESUS TOLD HIS DISCIPLES TO WAIT UNTIL THE RECEIVED POWER FROM ON HIGH UNTIL THEY WENT OUT TO WITNESS

E. WHATEVER WE DO IN HIS WILL THIS BOLDNESS IS AVAILABLE.

F. LK 11:13b ... HOW MUCH MORE WILL YOUR HEAVENLY FATHER GIVE THE HOLY SPIRIT TO THOSE WHO ASK HIM.

TG-13

TRUTH GEM

GRACE

A. (I Pet 4:10-11 part) As each one has received a GIFT, minister it to one another as good stewards of the manifold GRACE of God... If anyone ministers let him do it as with the ABILITY which God supplies...

B. DEFINITION: GRACE - God's ability gift to live and function in the gifts and callings.

C. Grace is part of God's supernatural provision to do excellently what God has called us to do

D. (2 Cor 9:8) And God is able to make all GRACE abound toward you, that you, always having all sufficiency in all things, may have an abundance for every good work.

TRUTH GEM

GRACE - WORKS EFFECTIVELY

A. Parts of Gal 2:7-9 ...when they SAW that the gospel for the uncircumcised had been committed to me.. for He who... WORKED EFFECTIVELY in me toward the Gentiles, ... when James, Cephas, and John,... perceived the GRACE that had been given to me.

B. A GRACEFUL PERSON WORKS EFFECTIVELY.

C. GRACE: GOD'S ABILITY GIFT TO LIVE AND FUNCTION IN THE GIFTS AND CALLINGS

D. BEING IN GOD'S WILL FOR TAKING THIS COURSE, THERE IS GRACE (SUPERNATURAL ABILITY TO WORK EFFECTIVELY) FOR YOU TO DO SO WELL IT CAN BE SEEN.

INMOST BEING COMPATIBILITY TEACHING AND LEARNING

A. ACTS 17:28A " FOR IN HIM WE LIVE AND MOVE AND HAVE OUR BEING.

B. SEEK AND FIND TEACHING\LEARNING METHODS IN ACCORDANCE TO THE WAY YOU ARE TO BE.

C. THINGS ARE TO BE RIGHT IN\WITH YOUR INMOST BEING.

D. MATT. 7:7 "... SEEK, AND YOU WILL FIND."

CALLING & DESTINY

A. EPH 1:18 ... the eyes of your understanding enlightened that you may know what is the hope (i.e. destiny) of His calling

B. (Jack Shoup) The calling is the office. The destiny is to be fulfilled in that office. Grace is the provision to fulfill your destiny within and by being in your calling

C. EXAMPLE: Part of my calling is to be a math teacher. Part of my destiny is to "make it plain". Since I have answered the call - there is grace to fulfill it.

Similar example: call: college student
destiny - x=major grade=y

D. You can answer the call and not fulfill your destiny (Jack Shoup)

E. PLP 3:14 I press toward the goal for the prize (destiny) of the upward call of God in Christ Jesus

TG-17

TRUTH GEM

LEARN FROM THE CLEAR
CLEARLY DO

A. JOSH. 11:15 a AS THE LORD COMMANDED MOSES HIS SERVANT, SO MOSES COMMANDED JOSHUA, AND SO JOSHUA DID.

B. WHEN HIRED, THE FIRST THING DONE USUALLY IS TRAINING.

C. THERE IS A MAJOR NEED FOR PEOPLE TO BE TAUGHT CLEARLY FROM THOSE WHO SEE CLEARLY AND FOR THE TAUGHT ONES TO FAITHFULLY DO IT IN TUNE AND IN FOCUS.

D. ROM 13:10 a LOVE DOES NO HARM... (HARM GENERALLY COMES FROM NOT DOING A JOB WELL, THE WAY YOU HAVE BEEN TRAINED)

E. LIFE IS NOT ALL SUBJECTIVE WORD PROBLEMS. MUCH OF IT IS LEARNING FROM THOSE WHO SEE CLEARLY AND DOING IT.

F. SEEK THE CLEAR ONES AND LEARN.

TG-18

TRUTH GEM

PRIORITIZE

A. NEH 6:3 I AM DOING A GREAT WORK, SO THAT I CANNOT COME <u>DOWN</u>. WHY SHOULD THE WORK CEASE WHILE I LEAVE IT AND GO <u>DOWN</u> TO YOU?

B. HAVE GOD'S PRIORITIES FOR YOUR LIFE CLEAR, DECIDED, SET, ESTABLISHED... A DIVINE ORDER. WHEN SOMETHING COMES UP FOR A DECISION, DISCERN WHAT CATEGORY IT IS IN AND THE DECISION HAS ALREADY BEEN MADE.

C. BEING IN GOD'S WILL FOR TAKING THIS COURSE MEANS THIS COURSE IS A HIGH PRIORITY, SO REGULAR, NONDISTRACTED STUDY TIME IS A <u>HIGH</u> PRIORITY, SO... DO NOT LEAVE IT AND GO <u>DOWN</u> TO DO A LESSER PRIORITY.

TG-19

TRUTH GEM

OVERCOME

A. I JN 5:4 FOR WHATEVER IS BORN OF GOD OVERCOMES THE WORLD. AND THIS IS THE VICTORY THAT HAS OVERCOME THE WORLD— OUR FAITH.

B. BEING IN THE WILL OF GOD FOR TAKING THIS COURSE, AND HENCE DOING A GREAT WORK, YOU WILL BE COME AGAINST TO TRY TO STOP, HINDER, OR HARASS THE WORK OF REGULAR STUDY — OVERCOME IT WITH SUPERNATURAL HELP

C. WITH A BOLD, NONTIMID, STRONG AND COURAGEOUS INNER MAN — SAY NO AND RESIST AND OVERCOME THE OPPOSITION

D. EPH 3:16 ... THAT HE WOULD GRANT YOU, ACCORDING TO THE RICHES OF HIS GLORY, TO BE STRENGTHENED WITH MIGHT THROUGH HIS SPIRIT IN THE INNER MAN

TG-20

<u>GOD</u>-PACED LEARNING
NOT <u>SELF</u>-PACED LEARNING

A. (Mt 16:24) IF ANYONE DESIRES TO COME AFTER ME, LET HIM DENY HIM<u>SELF</u>, AND TAKE UP HIS CROSS AND <u>FOLLOW ME</u>.

B. <u>SELF</u> WILL WANT TO STOP AND HAVE PIZZA

C. <u>SELF</u> WILL DECEIVE IT<u>SELF</u> DUE TO HUNGER FOR <u>SELF</u>-ESTEEM

IN ONE STUDY THE U.S. CAME IN LAST IN MATH SCORES, BUT FIRST IN HOW THEY <u>FELT</u> ABOUT MATH

D. HUNGER TO BE <u>GOD</u>-ESTEEMED, <u>GOD</u>-PACED EMPOWERED BY GRACE.